Chapter 2

§ 2.1 Introduction to Limits of Functions

Let f be a function of real numbers and let $a \in \mathbb{R}$ and $L \in \mathbb{R}$.

Consider now the meaning of $\lim_{x \to a} f(x) = L$.

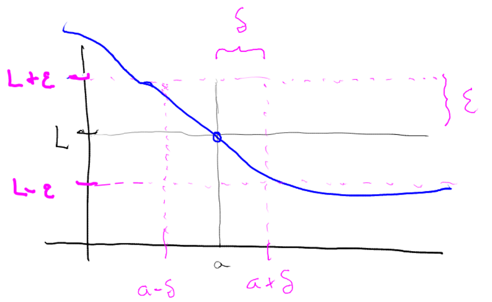
"As x gets closer and closer to a, $f(x)$ gets closer and closer to L"

Def: Let $f: \mathbb{R} \to \mathbb{R}$ be a function and let $a \in \mathbb{R}$ and $L \in \mathbb{R}$. We say that the limit of f at a is L if:

for every $\varepsilon > 0$ there exists a choice of $\delta > 0$ such that, for every x satisfying $0 < |x-a| < \delta$ we have $|f(x) - L| < \varepsilon$.

δ = "delta"
lowercase Greek "d"

Picture

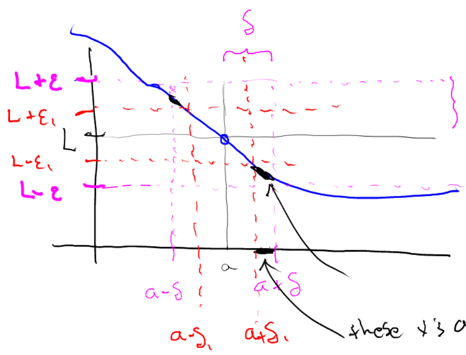


Choose some tolerance level $\varepsilon > 0$ for $f(x)$.
(This is how close we want to get to L)

For this $\varepsilon > 0$, is there a small enough $\delta > 0$ such that, when zooming in around a, all x's have $f(x)$ within ε of L?

(For all x's in δ-window, $f(x)$ is within ε-window of L)

If we choose a smaller ε tolerance, we must choose a smaller δ.

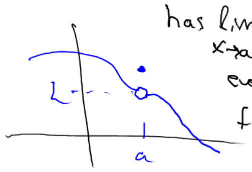


these x's are δ-close to L, but $f(x)$ not in ε_1 window

To prove a limit: Find a strategy for picking δ depending on what ε-tolerance is given. If your strategy works for every possible ε, the we prove the limit!

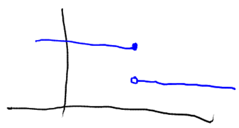
Notes:

- The limit does not depend on what happens at $x = a$



has $\lim_{x \to a} f(x) = L$ even though $f(a) \neq L$

- For the limit to exist, the values of the function must approach L from both sides.



No limit!

Examples

1) Prove $\lim_{x\to 2}(5x+1) = 11$

<u>Proof</u>. Let $\varepsilon > 0$. Choose $\delta > 0$ s.t. $\delta < \frac{\varepsilon}{5}$. Let x be such that $0 < |x-2| < \delta$. Now

$$|5x+1-11| = 5|x-2| < 5\delta < \varepsilon. \quad \square$$

Scratch

If. $0 < |x-2| < \delta$,

want $|(5x+1) - 11| < \varepsilon$.

$$|5x+1-11| = |5x-10|$$
$$= 5|x-2| < \varepsilon$$
$$\Rightarrow \quad |x-2| < \frac{\varepsilon}{5}$$

so choose $0 < \delta < \frac{\varepsilon}{5}$!

2) Prove $\lim_{x\to 5} x^2 = 25$

<u>Proof</u>: Let $\varepsilon > 0$ be given. Choose $\delta > 0$ such that $\delta < \min(5, \frac{\varepsilon}{15})$. Let $x \in \mathbb{R}$ and suppose $0 < |x-5| < \delta$. Then

$$|x^2-5| = |x-5||x+5| < \cdots \qquad < \varepsilon$$

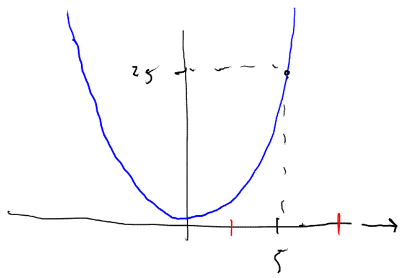
Scratch

$$|x^2-25| = |(x-5)(x+5)|$$
$$= |x-5||x+5| < \varepsilon$$

Want $|x-5| < \frac{\varepsilon}{|x+5|}$

depends on x!

Need to find way to bound $|x+5|$



Might as well assume we can choose δ small enough so that $x > 0$.

If $\delta = 10$ works, then so does $\delta = 5$.

So we can always choose $\delta < 5$

Thus, if $|x-5| < 5$ $\quad -5 < x-5 < 5$

then $\quad x+5 = x-5+10 < 15$

so $|x+5| < 15$.

$$\Rightarrow \quad \frac{1}{15} < \frac{1}{|x+5|} \Rightarrow \qquad \frac{\varepsilon}{15} < \frac{\varepsilon}{|x+5|}$$

Hence, if we can make $|x-5| < \frac{\varepsilon}{15}$ then also $|x-5| < \frac{\varepsilon}{|x+5|}$ and thus

$$|x-5||x+5| < \varepsilon$$

$$(5+\delta)^2 = 25+\varepsilon$$

$$\cancel{25} + 10\delta + \delta^2 = \cancel{25} + \varepsilon$$

$$\delta^2 + 10\delta - \varepsilon = 0$$

$$\delta = \frac{-10 \pm \sqrt{100+4\varepsilon}}{2}$$

$$= -5 \pm \sqrt{25+\varepsilon}$$

$$\sqrt{25+\varepsilon} - 5$$

$\lim_{x\to 4} \sqrt{x} = 2$ $\qquad$ $y = \sqrt{x}$

suppose $\varepsilon = 0.5$

want $2 - \frac{1}{2} < y < 2 + \frac{1}{2}$

$1.5 < \sqrt{x} < 2.5$

$(1.5)^2 < x < (2.5)^2$

$2.25 < x < 6.25$

$-1.75 < x - 4 < 2.25$

So, if we choose $\delta = 1.75$

then $0 < |x-4| < \delta$

implies

$-1.75 < x-4 < 1.75$ $\quad$ Since $1.75 < 2.25$

which implies

$-1.75 < x-4 < 1.75$

and thus $|\sqrt{x} - 2| < \frac{1}{2}$.

What if $\varepsilon = 0.01$? How close does x have to get to 4 to ensure $\sqrt{x}$ is within ε of 2?

$|y-2| < \varepsilon \iff -\varepsilon < y - 2 < \varepsilon$

$\iff -\varepsilon < \sqrt{x} - 2 < \varepsilon$

$\iff 2-\varepsilon < \sqrt{x} < 2+\varepsilon$

$(2-\varepsilon)^2 < x < (2+\varepsilon)^2$

$4 - 4\varepsilon + \varepsilon^2 < x < 4 + 4\varepsilon + \varepsilon^2$

$-(4\varepsilon - \varepsilon^2) < x-4 < 4\varepsilon + \varepsilon^2$

So choose $\delta = \min(\varepsilon(4-\varepsilon), \varepsilon(4+\varepsilon))$

~~What~~ Problem: if $\varepsilon \geq 4$, then $\delta \leq 0$!

solution, if $\varepsilon \geq 4$, might as well assume $\delta = 1$

Let's see why this works:

If $\varepsilon \geq 4$ and $\delta = 1$,

Suppose $|x-2| < 1$.

Then

$1 < x < 3$

$\Rightarrow \sqrt{1} < \sqrt{x} < \sqrt{3}$

$\Rightarrow 0 < \sqrt{x} < 4$

$\Rightarrow -2 < \sqrt{x} - 2 < 2$

$\Rightarrow |\sqrt{x} - 2| < 2 < \varepsilon$

since $\varepsilon \geq 4$.

If $\varepsilon < 4$, choose $\delta = 4\varepsilon - \varepsilon^2$.

If $\varepsilon \geq 4$ choose $\delta = 1$.