§3 Derivatives and Differentiability

Definition Given a function f, we say that f is differentiable at a point a if the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists.

In this case, we write $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ and call $f'(a)$ the derivative of f at a.

There is another useful way to express the derivative of f at a. Since any x near a can be written as $x = a + h$ for some small h, we get

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

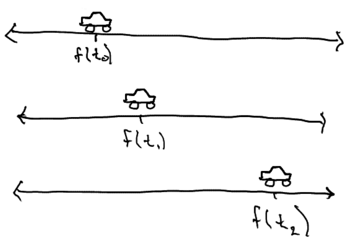
provided that the limit exists.

§3.1 Derivative as instantaneous velocity

To motivate our understanding of the derivative of a function, let's explore how it can be interpreted as an "instantaneous rate of change."

Suppose the position of an object (e.g. a car) along a track as a function of time is given by a function $f(t)$.

E.g.

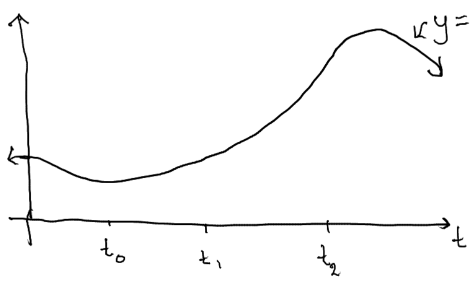


at time $t = t_0$

at time $t = t_1$

at time $t = t_2$

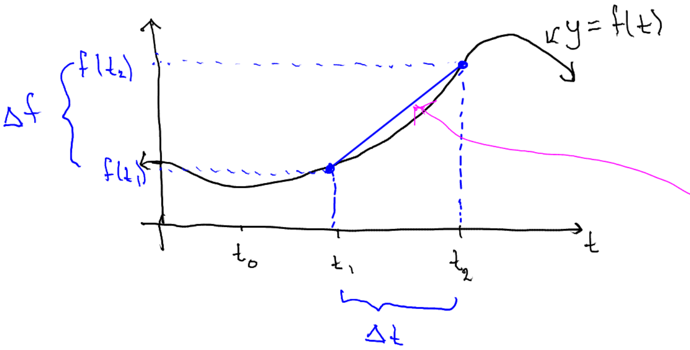
We can plot the position as a function of time as a graph



What is the average velocity of the object from time t_1 to t_2?

$$(\text{average vel. from } t_1 \text{ to } t_2) = v_{ave} = \frac{\text{total displacement}}{\text{elapsed time}}$$

$$= \frac{f(t_2) - f(t_1)}{t_2 - t_1} = \frac{\Delta f}{\Delta t}$$



where $\Delta f = f(t_2) - f(t_1)$

and $\Delta t = t_2 - t_1$

This kind of straight line (that intersects the curve $y = f(t)$ at two points) is called a secant.

Here, v_{ave} is the slope of the straight line segment connecting the points $(t_1, f(t_1))$ and $(t_2, f(t_2))$.

To compute the average velocity of the object between time $t = t_0$ and time $t = t_0 + h$, we get:

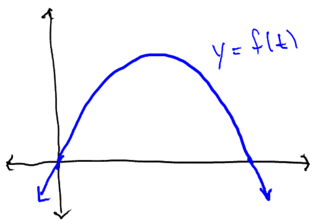
$$v_{ave} = \frac{f(t_0+h) - f(t_0)}{t_0+h-t_0} = \frac{f(t_0+h) - f(t_0)}{h}.$$

Taking the limit of this value as $h \to 0$ yields the instantaneous velocity of the object at time $t = t_0$:

$$v(t_0) = \lim_{h\to 0} \frac{f(t_0+h) - f(t_0)}{h} = f'(t_0).$$

Thus the derivative is the slope of the tangent line to the curve $y = f(t)$ at $t = t_0$!

Example Suppose the height of a stone trown into the air as a function of time is given by $f(t) = 2t - t^2$.



At a given time t_0, what is the instantaneous velocity?

$$v(t_0) = \lim_{h\to 0} \frac{f(t_0+h) - f(t_0)}{h}$$

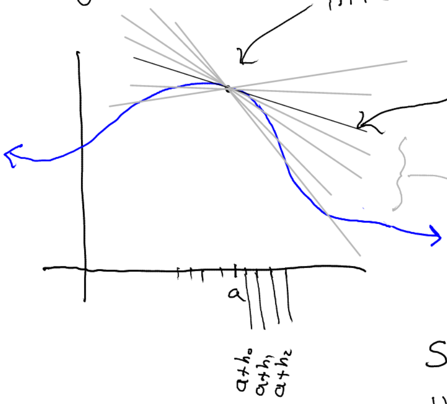
$$= \lim_{h\to 0} \frac{2(\cancel{t_0}+h) - (t_0+h)^2 - (2\cancel{t_0} - t_0^2)}{h}$$

$$= \lim_{h\to 0} \frac{2h \cancel{-t_0^2} - 2t_0 h - h^2 \cancel{+t_0^2}}{h} = \lim_{h\to 0} \frac{2h(1-t_0) - h^2}{h}$$

$$= \lim_{h\to 0} (2 - 2t_0 - h) = 2(1-t_0).$$

The tangent line

intersects graph at $x=a$, $y=f(a)$.



tangent line (has slope $f(a)$)

Secants of the curve $y=f(x)$ through points $(a, f(a))$ and $(a+h, f(a+h))$ for values of $h\to 0$.

Slope of secant line approaches slope of tangent line in the limit as $h \to 0$.

Tangent line

Suppose f is differentiable at a point a. The tangent line to the graph of f at a is the line defined by the equation

$$y = f(a) + (x-a)f'(a).$$

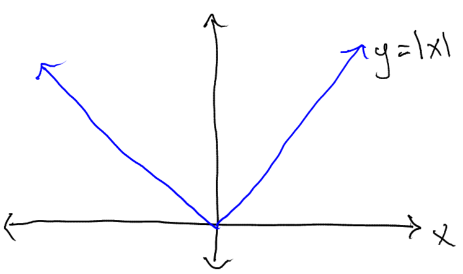
It is the line tangent to the curve of the graph of f that passes through the point $(a, f(a))$.

§3.2.2 Differentiability vs Continuity

Q: If a function is continuous at a point a, must it also be differentiable there?

Ans: No!

Example: Consider $f(x) = |x|$. It is continuous everywhere. But it is not differentiable at $x=0$!



Consider

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|}{h}.$$

This limit does not exist!

Indeed, $\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$ but $\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = +1$.

So $f(x) = |x|$ is not differentiable at $x=0$ even though it is continuous there.

However, if f is differentiable at a point, then it <u>must</u> be continuous there as well!

Theorem 3.1 (Differentiability implies continuity).
If a function f is differentiable at a point a, then it is also continuous at a.

<u>Proof</u>: Let f be a function and suppose f is differentiable at a point a. Then $f(a)$ is defined and $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists.

Now
$$\lim_{x\to a} f(x) = \lim_{x\to a}\left(f(x)-f(a)+f(a)\right)$$
$$= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}(x-a)\right) + f(a)$$
$$= \underbrace{\lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)}_{f'(a)} \underbrace{\lim_{x\to a}(x-a)}_{0} + f(a)$$
$$= 0 f'(a) + f(a) = f(a).$$

Thus $\lim_{x\to a} f(x) = f(a)$, so f is continuous at a $\square$.

<u>Note</u>: The contrapositive of this theorem is the implication:

f not continuous at a $\Rightarrow$ f not differentiable at a

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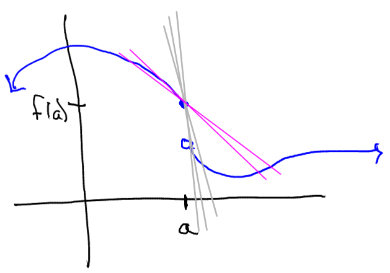
<u>Ex</u>



Slopes of secant lines do not approach one value as $h \to 0$.

The slope of the secant lines don't aproach <u>one</u> value.

In fact, for this case,
$$\lim_{x\to a^-} \frac{f(x)-f(a)}{x-a} \text{ exists}$$
but
$$\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} = -\infty \text{ (i.e. does not exist)}$$