§3.12 Implicit Differentiation

Given a function f:

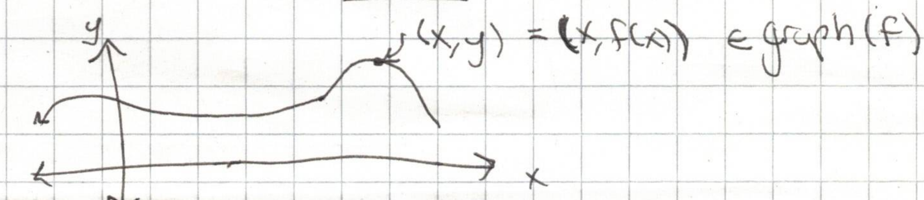
- the **domain** of f (denoted $\mathrm{dom}(f)$) is the set of all valid inputs into f. (I.e., all x such that $f(x)$ is defined).

$\mathrm{dom}(f) \subseteq \mathbb{R}$

- the **graph** of f is the set
$$\{(x,y) \in \mathbb{R}^2 \mid y = f(x),\ x \in \mathrm{dom}(f)\}$$
(i.e. collection of all pairs of points (x,y) in $\mathbb{R}^2$ such that $y = f(x)$).

$\mathrm{graph}(f) \subseteq \mathbb{R}^2$

The graph of f is a continuous function a **curve** that we can sketch



Here, the variable y is **explicitly** dependent on x.

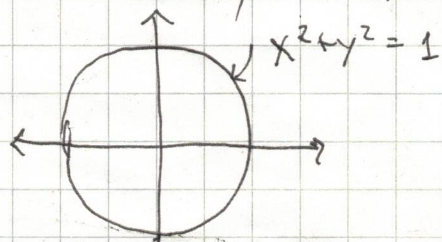
[Another way to think about it: The variables x and y are related by the equation $y = f(x)$.]

But not all **curves** are graphs of functions!

Consider the set of points:

$$\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

This is the unit circle. (Set of all points in $\mathbb{R}^2$ that have distance 1 away from origin



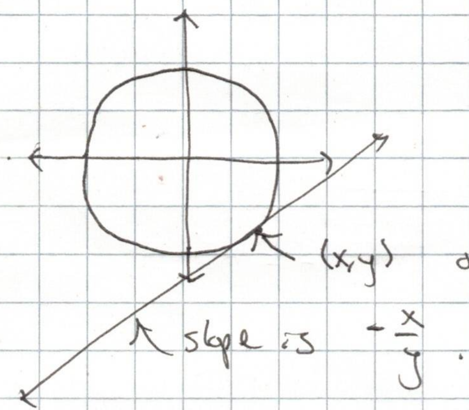
We say: This set is defined by the equation $x^2+y^2=1$.

What about for points on bottom half? $y = g(x) = -\sqrt{1-x^2}$

$$g'(x) = -\frac{1}{2}\frac{1}{\sqrt{1-x^2}}(-2x)$$

$$= \frac{x}{\sqrt{1-x^2}} = -\frac{x}{g(x)} = -\frac{x}{y}.$$

And slope of tangent line at (x,y) is $m = -\frac{x}{y}$.



We can get this answer <u>without</u> explicitly defining f and g.

Simply differentiate both sides of the equation

$$x^2 + y^2 = 1$$

with respect to x!

$$\underbrace{\frac{d}{dx}(x^2+y^2)} = \underbrace{\frac{d}{dx}(1)}_{0}$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow y\frac{dy}{dx} = -x$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

Regardless of what x and y are, if the point (x,y) is on the curve defined by $x^2+y^2=1$, the slope of tangent line is $-\frac{x}{y}$!

Let $z = y^2$

<u>Chain</u> $\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$

$$\frac{dz}{dy} = \frac{d}{dy}(y^2) = 2y$$

so $\frac{dz}{dx} = 2y\frac{dy}{dx}$

$$\Rightarrow \frac{d}{dx}[y^2] = 2y\frac{dy}{dx}$$

Implicit differentiation

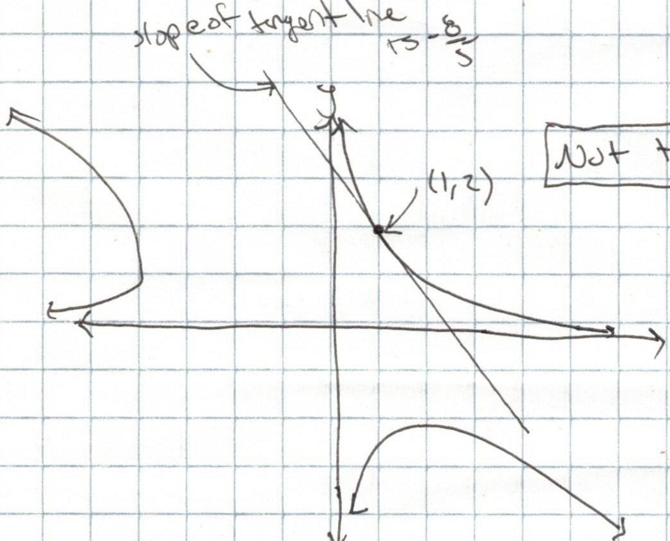
If x and y are related implicitly by an equation, we may differentiate and solve for $\frac{dy}{dx}$ to find slope of tangent line to the curve at (x,y).

Ex

Find the slope of the tangent line to the curve defined by

$$x^2y + xy^2 = 6$$

at the point $(1,2)$ [Note $x=1$ and $y=2$ is a solution to the equation so this point is on the curve]



Not the graph of any function

$$\underbrace{\frac{d}{dx}\left[x^2y + xy^2\right]}\ = \underbrace{\frac{d}{dx}[6]}_{0}$$

$$\Rightarrow \left[\frac{d}{dx}x^2\right]y + x^2\left[\frac{d}{dx}y\right] + \left[\frac{d}{dx}x\right]y^2 + x\left[\frac{d}{dx}y^2\right] = 0$$

$$\Rightarrow 2xy + x^2\frac{dy}{dx} + y^2 + x\,2y\frac{dy}{dx} = 0$$

$$\Rightarrow \left[\frac{dy}{dx}\right](x^2 + 2xy) = -y^2 - 2xy$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2xy + y^2}{x^2 + 2xy}$$

Now $$\left.\frac{dy}{dx}\right|_{(x,y)=(1,2)} = -\frac{2(1)(2) + (2)^2}{(1)^2 + 2(1)(2)}$$

$$= -\frac{8}{5}$$

Logarithmic Differentiation

Suppose we had functions f and g and define h as

$$h(x) = f(x)^{g(x)}.$$

How to compute h'? Set $y = h(x)$ and take logarithm

$$\ln y = \ln h(x) = \ln\left[f(x)^{g(x)}\right] = g(x)\ln(f(x))$$

Now differentiate: $\frac{d}{dx}\ln(y) = \frac{d}{dx}\left\{g(x)\ln(f(x))\right\}$

$\underline{Ex}$: Compute h' if

- $h(x) = (\ln x)^{\sin x}$ for $x > 1$.

$$\boxed{\begin{aligned} z &= \ln y \\ \frac{dz}{dx} &= \frac{dz}{dy}\frac{dy}{dx} \\ &= \frac{1}{y}\frac{dy}{dx} \end{aligned}}$$

$y = (\ln x)^{\sin x}$ $\quad y = h(x) \Rightarrow h'(x) = \frac{dy}{dx}$

$\ln y = (\sin x)\ln(\ln(x))$

$\frac{d}{dx}\ln y = \frac{d}{dx}[\quad]$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \cos x \ln(\ln x) + \sin x \frac{1}{\ln x}\frac{1}{x}$

$\Rightarrow h'(x) = \frac{dy}{dx} = y\left\{\cos x \ln(\ln x) + \frac{\sin x}{x\ln x}\right\}$

$= \ln x^{\sin x}\left[\cos x \ln(\ln x) + \frac{\sin x}{x \ln x}\right]$

- $h(x) = x^x$ $\quad y = x^x$ $\quad \ln y = x\ln x$

$h'(x) = \frac{dy}{dx}$

$\frac{d}{dx}\ln y = \frac{d}{dx}\{x\ln x\}$

$\frac{1}{y}\frac{dy}{dx} = \ln x + \frac{x}{x} \Rightarrow h'(x) = \frac{dy}{dx} = y(\ln x + 1)$

$= x^x(\ln x + 1)$

§3.13 Local Extrema

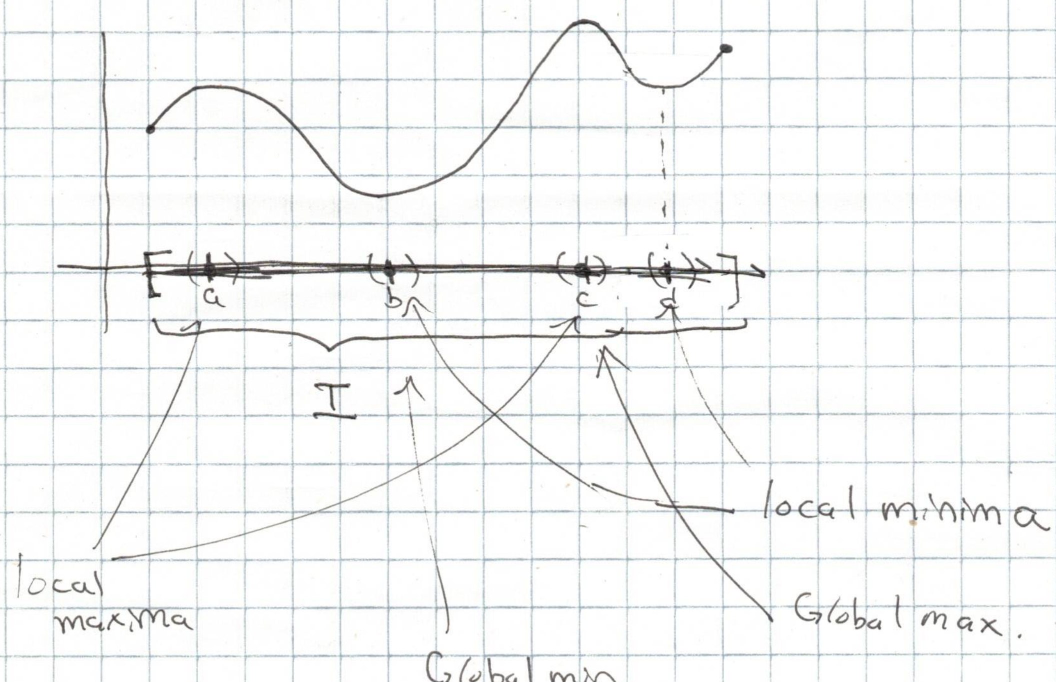
<u>Definition</u>: Given a function f, we say a point c is a <u>local maximum</u>, iff if there is an open interval I containing c such that

$$f(x) \leq f(c) \quad \text{for all } x \in I.$$

(Analogous definition for local minimum).

<u>Definition</u>:
Given a function f over an interval I, we say a point $c \in I$ is a <u>global maximum of f over I</u> if

$$f(x) \leq f(c) \quad \text{for all } x \in I.$$

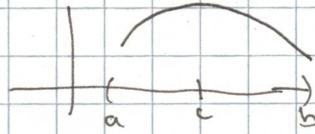


> <u>Theorem</u>: If c is a local max or min of f and $f'(c)$ exists then $f'(c) = 0$

<u>Proof idea</u>: Suppose c is a local maximum. Then there is some open interval $I = (a, b)$ such that *

$$f(x) \leq f(c) \quad \text{for every } x \in (a,b).$$

~~Also~~, Since $f'(c)$ exists, the limit $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ exists.

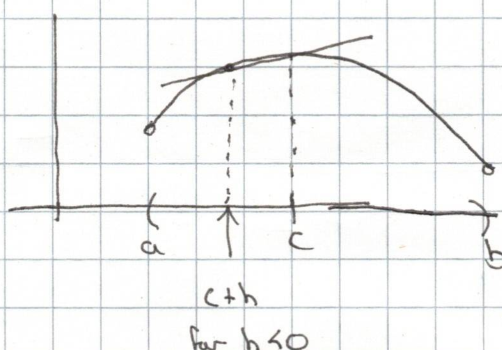


This means

$$\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h} = \lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}.$$

First consider $h \to 0^-$. For h small enough, we have $a < c+h < c$ and thus

$$f(c+h) \leq f(c).$$



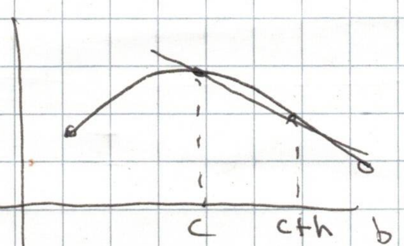
Hence $f(c+h) - f(c) \leq 0$

but $h<0$

so $\dfrac{f(c+h)-f(c)}{h} \geq 0.$

Thus $\displaystyle\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h} \geq 0.$

Now consider $h \to 0^+$. For h small enough $c < c+h < b$ and thus $f(c+h) \leq f(c)$.



Hence $f(c+h) - f(c) \leq 0$

and $\dfrac{f(c+h)-f(c)}{h} \leq 0$

as $h > 0$.

Thus $\displaystyle\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h} \leq 0.$

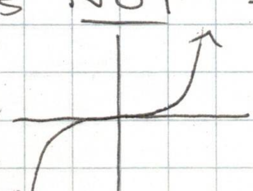
Now $\displaystyle 0 \leq \lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h} = \lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h} \leq 0.$

So both one-sided limits equal zero and thus $f'(c)=0$

$\square$

<u>Note</u>: Converse is NOT true.

<u>Ex</u> For $f(x) = x^3$,



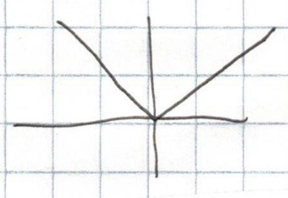
$f'(x) = 3x^2$

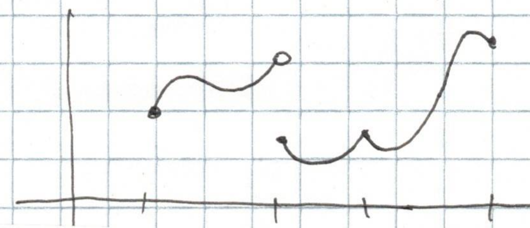
So $f(0) = 0$

but 0 <u>NOT</u> a local extremum

<u>Note</u>: Could be that c is local extremum but $f'(c)$ DNE.

<u>Ex</u> $f(x) = |x|$. Has local min at $x=0$ but $f'(0)$ DNE





<u>Def</u>: A point c is called a <u>critical point</u> of f if either $f'(c)=0$ or $f'(c)$ DNE.

<u>Finding Global maxima/minima</u> (§4.2-8)

<u>Theorem</u> Let f be a function on an interval I. If a point $c \in I$ is a global extremum of f then c is a critical point of f.

Critical points
- end points
- points of discontinuity
- "cusps" (where $f'(c)$ DNE)

$\}$ $f'(c)$ DNE

- local min/max
- saddle points (where $f'(c)=0$ but c not local min/max)

$\}$ $f'(c)=0$

Note: Not all critical points are global extrema.

Method to find global extrema of a continuous function f on a closed interval $[a,b]$.

[Note: Extreme Value Thm (EVT) guerentees existance of Global min / max]

1. Evaluate $f(a)$ and $f(b)$
2. Find all critical points $c \in (a,b)$.
3. Evaluate $f(c)$ for all crit points c.
4.

Ex

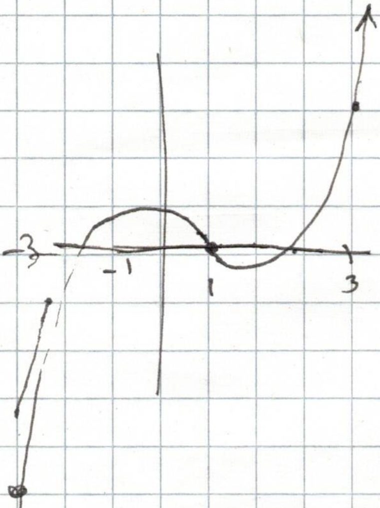
Find global min and max of f on $[-3,3]$ where $f(x) = x^3 - 3x + 2$.

Solution

Endpoints

- $f(-3) = (-3)^3 - 3(-3) + 2 = -27 + 9 + 2 = -16$

  $f(3) = 27 - 9 + 2 = 20$



- $f'(x) = 3x^2 - 3$

  solve $f'(x) = 0$

  $\Rightarrow x^2 - 1 = 0$

  $\Rightarrow x = \pm 1$.

crit points

c	$f(c)$
-3	-16
-1	4
1	0
3	20

$f(1) = 1 - 3 + 2 = 0$

$f(-1) = -1 + 3 + 2 = 4$

Global min at -3 (min value -16)

Global max at 3 (max value 20)

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§ 4.1 Mean Value Theorem (MVT)

Theorem (MVT)

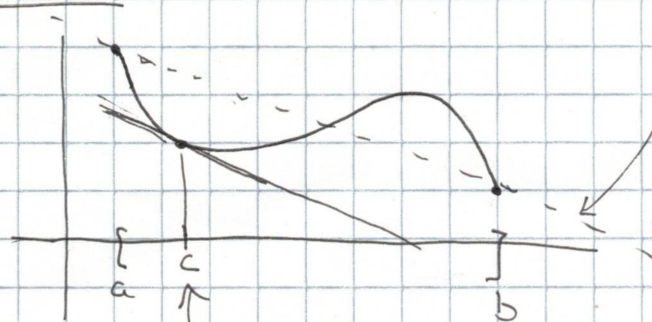
Suppose f is a continuous function on $[a,b]$ and suppose f is differentiable on (a,b).

There exists a point $c \in (a,b)$ such that

$$f'(c) = \underbrace{\frac{f(b) - f(a)}{b - a}}_{\text{Average rate of change of } f \text{ over } [a,b]}$$

"There is a point $c \in (a,b)$ where the derivative is equal to the average rate of change of f over $[a,b]$"

Picture



slope of line through $(a, f(a))$ and $(b, f(b))$

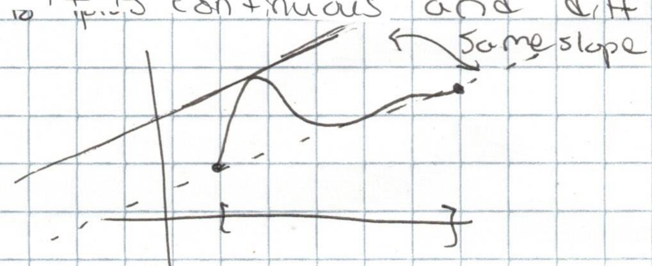
is

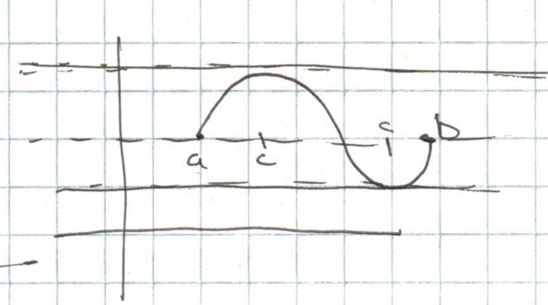
$$m = \frac{f(b) - f(a)}{b - a} \leftarrow \frac{\text{change in y}}{\text{change in x}}$$

Slope of tangent at c

is $f'(c) = m$!

No matter what our function looks like, as long as it is continuous and diff'ble we can do this





How to prove MVT? First let's prove a simpler version.

Suppose $f(a) = f(b)$.

> **Rolle's Theorem**
>
> Suppose f is continuous on $[a,b]$ and diff'ble on (a,b) such that $f(a) = f(b) = 0$.
> There is a point $c \in (a,b)$ such that $f'(c) = 0$.

Proof: Three cases:

1) If $f(x) = 0$ for all $x \in [a,b]$ then $f'(x) = 0$ and every $c \in (a,b)$ satisfies $f'(c) = 0$.

2) If there is a point $x \in (a,b)$ so that $f(x) > 0$, by EVT there is a point $c \in (a,b)$ that is a global maximum. Thus c is a critical point of f and $f'(c) = 0$.

3) Similar to case (2), except there is a c that's a global minimum.

$\square$

Now prove MVT

Proof (of MVT):

Define h on $[a,b]$ as

$$h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

Then h is continuous and diff'ble with

$$h(a) = f(a) - f(a) - m(a-a) = 0$$

$$h(b) = f(b) - f(a) - \frac{f(b)-f(a)}{b-a}(b-a) = 0.$$

By Rolle's theorem, there is a point $c \in (a,b)$ satisfying $h'(c) = 0$.

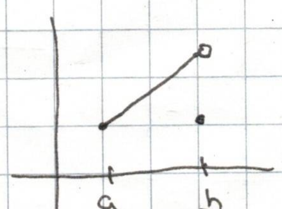
Now $h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$.

At $x=c$, $0 = h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a} \qquad \square$$

Note Need continuity at endpoints!

For example:

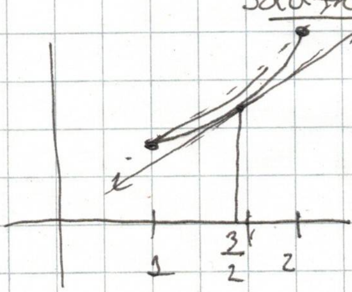


has $f(a) = f(b)$
but no point $c \in (a,b)$
with $f'(c) = 0$.

more next page

Ex Consider $f(x) = x^2 + 2x + 1$ on $[1,2]$.
Find all points $c \in (0,1)$ satisfying the MVT.

Solution $f(1) = 4$ $\quad f(2) = 9$ $\quad f'(x) = 2x+2$

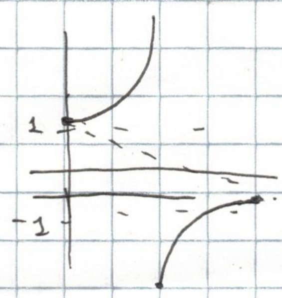


Need $2(c+1) = f'(c) = \frac{f(2)-f(1)}{2-1} = \frac{9-4}{1} = 5$

$$\Rightarrow 2(c+1) = 5 \Rightarrow c = \frac{5}{2} - 1 = \frac{3}{2}$$

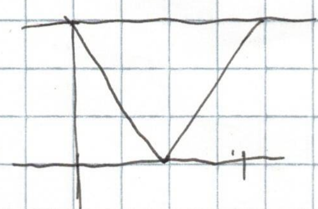
Need continuity in (a,b).

Ex $f(x) = \sec x$ on $[0, \pi]$



No point $c \in (0, \pi)$ with $f'(c) = -\frac{2}{\pi}$

Need differentiability in (a,b)



$f(x) = |x-1|$ on $[0,2]$.

No point $c \in (0,2)$ with $f'(c) = 0$

4.2 Applications of MVT

§ 4.2.1 Antiderivatives

How can we "undo" differentiation?

If we know $f(x)$ can we find a function F s.t. $F' = f$?

> **Definition** Given a function f, another function F is an <u>antiderivative</u> of f if $F' = f$.

Ex given $f(x) = x^2$, the function F defined by

$$F(x) = \frac{x^3}{3}$$

is antiderivative of f.

But so is $G(x) = \frac{x^3}{3} + 7$.

In fact, so is $H(x) = \frac{x^3}{3} + c$ for any constant $c \in \mathbb{R}$

- Antiderivatives are not unique
- If F is an antideriv of f then so is $G(x) = F(x) + c$.

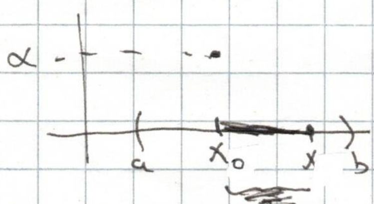
Q: Is this all the antiderivatives? Yes!

Theorem (constant function) Suppose f is diff'ble on some interval (a,b).
If $f'(x) = 0$ for all $x \in (a,b)$ then there is a constant $C \in \mathbb{R}$ s.t. $f(x) = C$ for all $x \in I$

Proof Pick $x_0 \in (a,b)$. And define $C = f(x_0)$.

Let $x \in (a,b)$ be any other point $x \neq x_0$.

Consider interval with endpoints x_0 and x.



By MVT, there is a point c between x and x_0 s.t. $f'(c) = \dfrac{f(x)-f(x_0)}{x-x_0}$.

But $f'(c)=0$, so $f(x)-f(x_0) = 0$

$\Rightarrow f(x) = f(x_0) = C$

Thus $f(x) = \alpha$ for all $x \neq x_0$ and $f(x_0) = C$ □

Note: All antiderivatives of $g(x) = 0$ are of the form $f(x) = \alpha$ for some constant $\alpha \in \mathbb{R}$.

Theorem (Antiderivative Theorem)
Suppose f and g are diff'ble functions on some interval I s.t.
$f'(x) = g'(x)$ for all $x \in I$.
Then there is a constant $C \in \mathbb{R}$ s.t.
$f(x) = g(x) + C$ for all $x \in I$

Proof Define h as $h(x) = f(x) - g(x)$ for all $x \in I$.
Then h is diff'ble and $h'(x) = f'(x) - g'(x) = 0$ for all $x \in I$.
By Constant Func theorem there is $C \in \mathbb{R}$ s.t. $h(x) = C$ for all $x \in I$.

That is $f(x) - g(x) = C$ and thus $f(x) = g(x) + C$ for all $x \in I$

□

Leibniz notation for Antiderivatives

For any function f, there are *infinitely many* Antiderivs.
But each is of the form $G(x) = F(x) + C$
for some antiderive F.
If we find one, we find them all!

We denote the family of antiderivs of f as

$$\int f(x)\,dx = \{ F : F' = f \}$$

set of all functions whose deriv is f

$$= \{ F + C : C \in \mathbb{R} \}$$ if F is one Antideriv.

(Also called *indefinite integral* of f)

Ex If $f(x) = x^2$

$$\int x^2\,dx = \frac{x^3}{3} + C$$

Every antideriv of f is of the form $F(x) = \frac{x^3}{3} + c$ for some $c \in \mathbb{R}$

Power rule for antiderivs:
If $\alpha \neq -1$,

$$\boxed{\int x^\alpha\,dx = \frac{x^{\alpha+1}}{\alpha+1} + C}$$

Check: If $c \in \mathbb{R}$, $\frac{d}{dx}\left[\frac{x^{\alpha+1}}{\alpha+1} + c\right] = \frac{\alpha+1}{\alpha+1} x^\alpha + 0 = x^\alpha$.

Constant multiples and sums
If F and G are antiderivs of f and g, then
$\alpha F + \beta G$ is antideriv of $\alpha f + \beta g$.

Idea: $\frac{d}{dx}(\alpha F(x) + \beta G(x)) = \alpha F'(x) + \beta G'(x) = \alpha f(x) + \beta g(x)$

Table of known Antiderivs

$$\int \frac{1}{x}\,dx = \ln(x) + C$$

$$\int e^x\,dx = e^x + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$