§3.3 The derivative function

Definition

We say that a function f is differentiable on an interval I if

$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists for every $a \in I$.

The function f' that takes numbers $x \in I$ as inputs and outputs $f'(x)$ is the derivative function of f.

Example: From last time, the function f defined by $f(x) = 2x - x^2$ has derivative function f' given by $f'(x) = 2 - 2x$ for all x.

Notation: Leibniz vs Newton

Historically, two mathematicians, Gottfried Leibniz (1646-1716) and Isaac Newton (1642 - 1726), independently "discovered" the rules of what we now call "Differential Calculus". (That is, the system of rules that allow us to compute derivatives from the limit definition of a derivative.) They each developed their own notational systems for symbolically representing derivatives.

	Derivative as a function	Derivative evaluated at $x=a$.
Newton:	f', or y' if $y=f(x)$ (or $\dot{y}$)	$f'(a)$
Leibniz:	$\frac{df}{dx}$ or $\frac{dy}{dx}$	$\left.\frac{df}{dx}\right\rvert_{x=a}$

In general,

$\frac{d}{dx} f = \frac{df}{dx} = f'$ (these are different notations for the same function)

and we call "$\frac{d}{dx}$" a differential operator.

(You apply it on a function and it turns it into a different function.)

Higher derivatives

Suppose f is a differentiable function, then its derivative f' is also a function. If f' is also differentiable as a function, we denote its derivative as

$$\frac{d}{dx} f' = (f')' = f'' = f^{(2)}$$

or

$$\frac{d}{dx} f' = \frac{d}{dx}\left(\frac{d}{dx} f\right) = \left(\frac{d}{dx}\right)^2 f = \frac{d^2}{dx^2} f$$

which we call the second derivative of f.

We can continue this process (assuming f'' is also differentiable) to get

$$\left(\frac{d}{dx}\right)^3 f = (f'')' = f''' = f^{(3)}$$

and for any $n \in \mathbb{N}$, the n^{th} derivative is denoted

$$\left(\frac{d}{dx}\right)^n f = f^{\overbrace{\prime\prime\cdots\prime}^{n}} = f^{(n)}$$

if it exists.

[Later in the course we'll get an understanding of what f' and f'' mean in terms of the "shape" of the graph of f.]

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§3.4 Derivatives of Elementary Functions

- Constant functions

Consider a constant function defined by $f(x) = c$ for every $x \in \mathbb{R}$ (where c is some constant). Then for every $x \in \mathbb{R}$,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h}$$

$$= \lim_{h \to 0} 0 = 0.$$

Thus f is differentiable everywhere with derivative function given by $f'(x) = 0$.

In Leibniz notation we write $\boxed{\frac{d}{dx} c = 0.}$

- Linear functions

Consider a function defined by $f(x) = mx + b$ for some constants m and b. Then for every x

$$f'(x) = \lim_{h \to 0} \frac{m(\cancel{x}+h) \cancel{+b} - (\cancel{mx} \cancel{+b})}{h} = \lim_{h \to 0} \frac{mh}{h} = m.$$

This makes sense — we know from experience that m represents the slope of the line defined by $y = mx + b$.

In Leibniz notation: $\boxed{\frac{d}{dx}(mx+b) = m.}$

- Powers of x

Consider now $f(x) = x^2$. Then

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{\cancel{x^2} + 2hx - h^2 \cancel{-x^2}}{h}$$

$$= \lim_{h \to 0} \left[\frac{h}{h} (2x - h) \right] = \lim_{h \to 0} [2x - h]$$

$$= 2x.$$

or $\boxed{\frac{d}{dx} x^2 = 2x}$

We'll show later that $\frac{d}{dx} x^n = n x^{n-1}$ for every $n \in \mathbb{N}$.

- Trigonometric functions

There are a few facts about sin and cos that we'll need before we can determine their derivatives.

- Recall the sum rules for sin and cos:

  For every $a, b \in \mathbb{R}$:

$$\sin(a+b) = \cos a \sin b + \sin a \cos b$$

and $$\cos(a+b) = \cos a \cos b - \sin a \sin b.$$

- For every $a \in \mathbb{R}$, $\boxed{\sin^2 a + \cos^2 a = 1.}$

- Earlier, we proved that $\boxed{\lim_{a \to 0} \frac{\sin a}{a} = 1.}$ $(*)$

Let's use this to compute

$$\lim_{a \to 0} \left(\frac{\cos a - 1}{a} \right) = \lim_{a \to 0} \left[\frac{\cos a - 1}{a} \cdot \frac{\cos a + 1}{\cos a + 1} \right]$$

$$= \lim_{a \to 0} \left[\frac{\cos^2 a - 1}{a(\cos a + 1)} \right]$$

$$= \lim_{a \to 0} \left[\frac{-\sin^2 a}{a(\cos a + 1)} \right] \quad \text{since } \cos^2 a - 1 = -\sin^2 a$$

$$= -\lim_{a \to 0} \left[\frac{\sin a}{a} \cdot \frac{\sin a}{\cos a + 1} \right]$$

$$= -\underbrace{\left[\lim_{a \to 0} \frac{\sin a}{a} \right]}_{=1} \cdot \underbrace{\left[\lim_{a \to 0} \frac{\sin a}{\cos a + 1} \right]}_{= \frac{\sin 0}{\cos 0 + 1} = \frac{0}{1+1} = 0.}$$

$$= -1 \cdot 0$$

$$= 0.$$

Thus $\boxed{\lim_{a \to 0} \frac{\cos a - 1}{a} = 0}$ (**)

We are now ready to determine the derivatives of sin and cos.

For every $x \in \mathbb{R}$,

$$\sin'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos(x)\sin(h) + \sin(x)\cos(h) - \sin(x)}{h} \quad \text{Using sum rule.}$$

$$= \lim_{h \to 0} \left[\cos(x) \frac{\sin(h)}{h} + \sin(x) \frac{\cos(h) - 1}{h} \right]$$

$$= \cos(x) \underbrace{\lim_{h \to 0} \frac{\sin(h)}{h}}_{1 \text{ by } (*)} + \sin(x) \underbrace{\lim_{h \to 0} \frac{\cos h - 1}{h}}_{0 \text{ by } (**)}$$

$$= \cos(x).$$

So $\boxed{\sin'(x) = \cos(x)}$

And

$$\cos'(x) = \lim_{h \to 0} \frac{\cos(x+h) - \cos(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} \quad \text{using sum rule}$$

$$= \lim_{h \to 0} \left[\cos(x) \frac{\cos(h) - 1}{h} - \sin(x) \frac{\sin(h)}{h} \right]$$

$$= \cos(x) \underbrace{\lim_{h \to 0} \frac{\cos(h) - 1}{h}}_{0 \text{ by } (**)} - \sin(x) \underbrace{\lim_{h \to 0} \frac{\sin(h)}{h}}_{1 \text{ by } (*)}$$

$$= -\sin(x)$$

So $\boxed{\cos'(x) = -\sin(x)}$

- ## Exponential functions

For each real number $a > 0$ consider the function f_a defined by

$$f_a(x) = a^x$$

for every $x \in \mathbb{R}$. What is f_a' ?

For $x \in \mathbb{R}$,

$$f_a'(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

since $a^{x+h} = a^x a^h$

$$= \lim_{h \to 0} \frac{a^x \cdot a^h - a^x}{h} = \lim_{h \to 0} \left[a^x \frac{a^h - 1}{h} \right]$$

$$= a^x \underbrace{\lim_{h \to 0} \frac{a^h - 1}{h}}_{= \lim_{h \to 0} \frac{f_a(h) - f_a(0)}{h} = f_a'(0)}.$$

Since a^x is constant with respect to h

definition of $f_a'(0)$
(where we note $f_a(0) = a^0 = 1$)

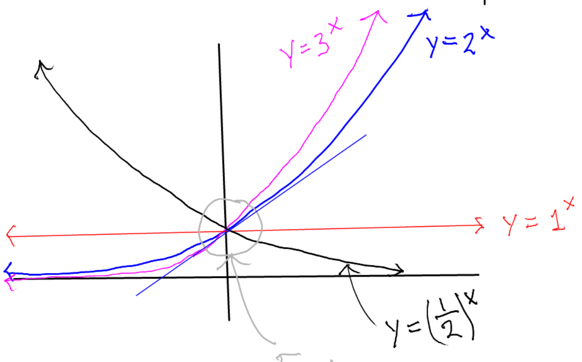
$$= f_a(x) f_a'(0).$$

Here $f_a'(0) = C_a$ is some constant. It is the slope of the tangent line to the curve defined by $y = a^x$ at $x = 0$.

That is $\boxed{f_a'(x) = C_a f_a(x)}$ for every $x \in \mathbb{R}$.

If we can determine what $C_a = f_a'(0)$ is, then we know what $f_a'(x)$ is for every $x \in \mathbb{R}$.

Let's look at a few examples:



Each a^x has a different slope at $x = 0$.

Slope at $x = 0$ depends on base a of exponent.

Here's why $e = 2.71828....$ is important

Fact: e is the <u>unique</u> number with the property that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 0.$$

There are other ways to define e mathematically:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$
$$= \sum_{n=0}^{\infty}\frac{1}{n!}$$

but proving these are equivalent is beyond the scope of this course.

$\boxed{\text{Corollary: If } f(x)=e^x \text{ then } f'(x)=e^x.}$

More generally, we can use $\lim_{h\to 0}\frac{e^h-1}{h}=1$ to compute

$$f_a'(0) = \lim_{h\to 0}\frac{a^h-1}{h}$$

$$= \lim_{h\to 0}\frac{e^{h\ln(a)}-1}{h} \qquad \text{since } a^h = e^{h\cdot\ln(a)}$$

$$= \lim_{h\to 0}\left[\ln(a)\,\frac{e^{h\ln(a)}-1}{h\ln(a)}\right]$$

$$= \ln(a)\underbrace{\lim_{k\to 0}\frac{e^k-1}{k}}_{1} \qquad \text{using the substitution } k=h\ln(a)$$

$$= \ln(a).$$

$\boxed{\text{Fact: For } a>0, \ \lim_{h\to 0}\frac{a^h-1}{h}=\ln(a)}$

$\boxed{\text{Corollary: If } f_a(x)=a^x \text{ then } f_a'(x)=\ln(a)\cdot a^x}$

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§3.7 Arithmetic Rules for Differentiation

Now that we know how to find the derivatives of some elementary functions, let's look at rules that tell us how to differentiate combinations of these functions.

In Leibniz notation

Theorem (Arithmetic Rules for Differentiation)

Suppose functions f and g are differentiable at a.

- (Constant multiple rule)
  If $c \in \mathbb{R}$ and $h(x) = cf(x)$, then h is differentiable at a and
  $$h'(a) = cf(a)$$

$$\frac{d}{dx}\left[c f(x)\right] = c \frac{d}{dx} f(x)$$

- (Sum rule)
  If $h(x) = f(x) + g(x)$ then h is differentiable at a and
  $$h'(a) = f'(a) + g'(a)$$

$$\frac{d}{dx}\left[f(x) + g(x)\right] = \left(\frac{d}{dx} f(x)\right) g(x) + f(x)\left(\frac{d}{dx} g(x)\right)$$

- (Product rule)
  If $h(x) = f(x)g(x)$ then h is differentiable at a and
  $$h'(a) = f'(a)g(a) + f(a)g'(a)$$

- (Reciprocal rule)
  If $f(a) \neq 0$ and $h(x) = \frac{1}{f(x)}$ then h is differentiable at $x = a$ and
  $$h'(a) = -\frac{f'(a)}{f(a)^2}.$$

$$\frac{d}{dx}\frac{1}{f(x)} = -\frac{1}{f(x)^2}\frac{d}{dx} f(x)$$

- (Quotient rule)
  If $h(x) = \frac{f(x)}{g(x)}$ and $g(a) \neq 0$ then h is differentiable at $x = a$ and
  $$h'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$$

Proof:

By the assumption that f and g are differentiable at a, we know that the limits $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$ and $\lim_{h \to 0} \frac{g(a+h) - g(a)}{h} = g'(a)$ exist.

- Constant rule:

$$h'(a) = \lim_{h \to 0} \frac{cf(a+h) - cf(a)}{h} = c \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = cf'(a).$$

- Sum rule

$$h'(a) = \lim_{h\to 0} \frac{(f(a+h)+g(a+h)) - (f(a)-g(a))}{h}$$

$$= \lim_{h\to 0} \frac{f(a+h)-f(a) + g(a+h)-g(a)}{h}$$

$$= \left[\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}\right] + \left[\lim_{h\to 0} \frac{g(a+h)-g(a)}{h}\right]$$

$$= f'(a) + g'(a).$$

- Product rule

$$h'(a) = \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h}$$

$$= \lim_{h\to 0} \frac{f(a+h)g(a+h) + f(a)g(a+h) - f(a)g(a+h) - f(a)g(a)}{h}$$

$$= \lim_{h\to 0} \left[\frac{f(a+h)-f(a)}{h} g(a+h) + f(a) \frac{g(a+h)-g(a)}{h}\right]$$

$$= \left[\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}\right] \cdot \underbrace{\left[\lim_{h\to 0} g(a+h)\right]}_{g(a)} + f(a)\left[\lim_{h\to 0} \frac{g(a+h)-g(a)}{h}\right]$$

$$= f'(a)\, g(a) + f(a)\, g'(a)$$

where $\lim_{h\to 0} g(a+h) = g(a)$ since g is continuous at a (because it is differentiable there).

- Reciprocal rule

$$h'(a) = \lim_{h\to 0} \frac{1/f(a+h) - 1/f(a)}{h} = \lim_{h\to 0} \frac{f(a)-f(a+h)}{h\, f(a) f(a+h)}$$

$$= -\lim_{h\to 0} \left[\frac{f(a+h)-f(a)}{h} \cdot \frac{1}{f(a)} \frac{1}{f(a+h)}\right]$$

$$= -\left[\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}\right] \cdot \frac{1}{f(a)} \cdot \left[\lim_{h\to 0} \frac{1}{f(a+h)}\right]$$

$$= -f'(a) \frac{1}{f(a)^2}.$$

- Quotient rule:

Combine the product rule and reciprocal rule! $\square$

Power Rule

- Now we can prove that $\boxed{\frac{d}{dx} x^n = n x^{n-1}}$ for every $n \in \mathbb{N}$.

Note that $\frac{d}{dx} x^1 = 1 = 1 x^0$. (Base case)

Let $k \in \mathbb{N}$ and suppose $\frac{d}{dx} x^k = k x^{k-1}$. (Induction step).

By the product rule,

$$\frac{d}{dx} x^{k+1} = \frac{d}{dx}(x^k \cdot x) = k x^{k-1} \cdot x + x^k \cdot 1$$

$$= k x^k + x^k = (k+1) x^k$$

- For every $\alpha \neq 0$, it is also the case that

$$\boxed{\frac{d}{dx} x^\alpha = \alpha x^{\alpha - 1}}$$

wherever x^α is defined.

The proof of this for non-integer α is much more difficult.

For $\alpha \in \mathbb{Q}$, its possible with more tools from later in the course (chain rule and inverse function theorem), but for $\alpha \notin \mathbb{Q}$ the proof is outside the scope of this course.

Examples

1) $f(x) = x^2 \sin(x)$ $\quad f'(x) = \frac{d}{dx} f(x) = \left(\frac{d}{dx} x^2\right) \sin(x) + x^2 \left(\frac{d}{dx} \sin x\right)$

$$= 2x \sin(x) + x^2 \cos(x).$$

2) $f(x) = \dfrac{x^4 + 1}{x - 7}$

Quotient rule

$$f'(x) = \frac{(x-7)\frac{d}{dx}(x^4+1) - (x^4+1)\frac{d}{dx}(x-7)}{(x-7)^2}$$

$$= \frac{(x-7)(4x) - (x^4+1)(1)}{(x-7)^2} = \frac{-x^4 + 4x^2 - 7x - 1}{(x-7)^2}.$$

3) $f(x) = 3x^5$

$f'(x) = 15x^4$

$f''(x) = 60x^3$

$f'''(x) = 180x^2$

$f^{(4)}(x) = 360x$

$f^{(5)}(x) = 360$

$f^{(6)}(x) = 0$.

§3.8 Chain Rule

Theorem (Chain Rule)

Assume f is differentiable at a and g is differentiable at $b = f(a)$. Then $h = g \circ f$ is differentiable at a with derivative given by $h'(a) = \boxed{(g \circ f)'(a) = f'(a)\, g'(f(a))}$.

Recall that $(g \circ f)(x) = g(f(x))$.

In Leibniz notation:

Suppose variables $x, y,$ and z are related by

$$y = f(x) \quad \text{and} \quad z = g(y).$$

Then $$\frac{dz}{dx} = \frac{dz}{dy}\,\frac{dy}{dx}.$$

or

$$\boxed{\frac{d}{dx} g(f(x)) = g'(f(x))\,\frac{d}{dx} f(x)}$$

Proof (idea):

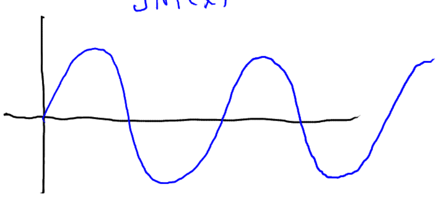
$$\begin{aligned}
(g \circ f)'(a) &= \lim_{h \to 0} \frac{g(f(a+h)) - g(f(a))}{h} \\
&= \lim_{h \to 0} \left[\frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} \cdot \frac{f(a+h) - f(a)}{h} \right] \\
&= \lim_{k \to 0} \left[\frac{g(f(a)+k) - g(f(a))}{k} \right] \left[\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \right] \\
&= g'(f(a))\, f'(a)
\end{aligned}$$

where we make the substitution $k = f(a+h) - f(a)$

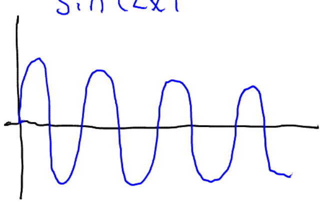
such that $k \to 0$ as $h \to 0$. $\square$

Idea: Consider for example the function $h(x) = \sin(2x) = f(g(x))$ where $f(y) = \sin y$ and $g(x) = 2x$. Then $h'(x) = 2\sin(2x)$

$\sin(x)$



$\sin(2x)$



changes in $\sin(2x)$ happen twice as fast as changes in $\sin(x)$.

[See geometric argument on pg. 170-173 of course notes.]

For deeper compositions: If $y = f(x)$, $z = g(y)$, $w = h(z)$, $\dfrac{dw}{dx} = \dfrac{dw}{dz}\,\dfrac{dz}{dy}\,\dfrac{dy}{dx}$

Or $(h\circ g\circ f)'(x) = \frac{d}{dx} h(g(f(x))) = h'(g(f(x))\frac{d}{dx}g(f(x)) = h'(g(f(x))\,g'(f(x))\frac{d}{dx}f(x)$
$= h'(g(f(x))\,g'(f(x))\,f'(x)$.

Corollary: Generalized Power Rule

If $g(x) = f(x)^\alpha$ for $\alpha\in\mathbb{R}$, $\alpha\neq 0$, then

$$g'(x) = \alpha\, f(x)^{\alpha-1}\, f'(x)$$

In Leibniz:

$$\boxed{\frac{d}{dx}(f(x)^\alpha) = \alpha f(x)^{\alpha-1}\frac{d}{dx}f(x)}$$

Examples

Find f' if f is given by

- $f(x) = (3x^2+2x+7)^{17}$

$$f'(x) = 17(3x^2+2x+7)^{16}\cdot(6x+2)$$

- $f(x) = \sin(e^x + x^e)$

$$f'(x) = \cos(e^x+x^e)\cdot(e^x + e\cdot x^{e-1})$$

- $f(x) = e^{\sin(x^2)}$

$$\frac{d}{dx}e^{\sin(x^2)} = e^{\sin(x^2)}\cdot\frac{d}{dx}\sin(x^2)$$
$$= e^{\sin(x^2)}\cdot\cos(x^2)\cdot\frac{d}{dx}x^2$$
$$= e^{\sin(x^2)}\cdot\cos(x^2)\cdot 2x \quad = 2x\cos(x^2)e^{\sin(x^2)}$$

We can use the Chain Rule to get a derivative rule for other exponential functions.

For $a>0$, if $f(x) = a^x$ then $a^x = e^{x\ln(a)}$

so $f'(x) = \frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln(a)} = \underbrace{e^{x\ln(a)}}_{a^x}\frac{d}{dx}(x\ln(a))$
$= a^x \cdot \ln(a)$
$= \ln(a)\,a^x$

$$\boxed{\frac{d}{dx}a^x = \ln(a)\,a^x}$$

Example: If $f(x) = 2^{3x} + 5^{\cos x}$ $= e^{3\ln(2)x} + e^{\ln(5)\cos x}$

then $f'(x) = 3\ln(2)\,2^{3x} + \ln(5)(-\sin x)\,5^{\cos x}$
$= \ln(8)\,8^x - \ln(5)\sin(x)\,5^{\cos x}$
since $2^3=8$

§3.9 Derivatives of other Trig functions

We know: $\frac{d}{dx}\sin(x) = \cos(x)$

$\frac{d}{dx}\cos(x) = -\sin(x)$

Now: • $\tan(x) = \frac{\sin(x)}{\cos(x)}$

by Quotient Rule:

$$\tan'(x) = \frac{\cos(x)\sin'(x) - \sin(x)\cos'(x)}{\cos^2(x)}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$

• $\sec(x) = \frac{1}{\cos(x)}$

By Reciprocal rule

$$\sec'(x) = -\frac{\cos'(x)}{\cos^2(x)} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}\frac{\sin x}{\cos x}$$

$$= \sec x \tan x$$

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\csc x$	$-\csc x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\csc^2 x$