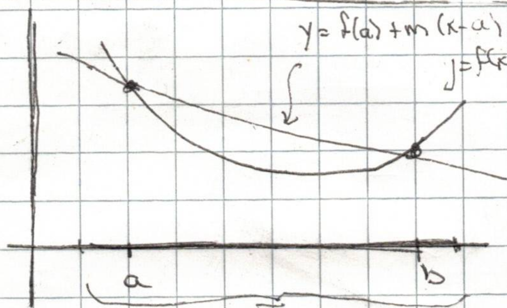
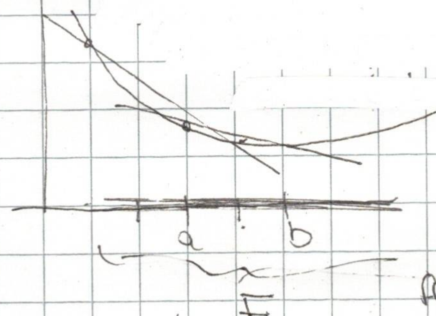
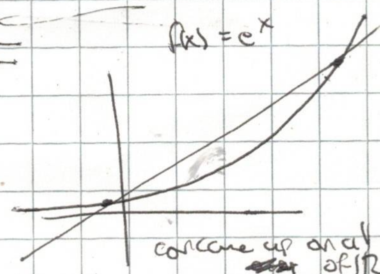
Def A function f is said to be concave upwards on an interval I, if for every pair of points $a, b \in I$ and every $x \in [a,b]$ it holds that

$$f(x) \leq f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$





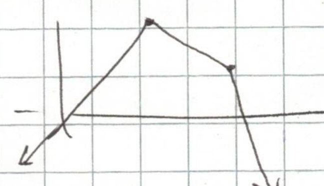
"For every $a, b \in I$, the secant line segment connecting points $(a, f(a))$ and $(b, f(b))$ lies above the graph of f".

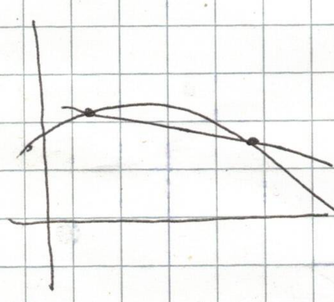


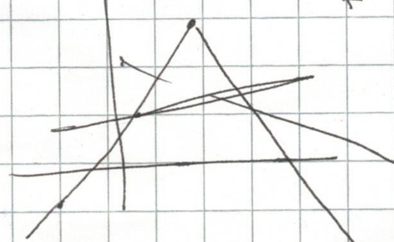
Analogous def for concave downwards,

For every $a, b \in I$ and every $x \in [a,b]$,

$$f(x) \geq f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

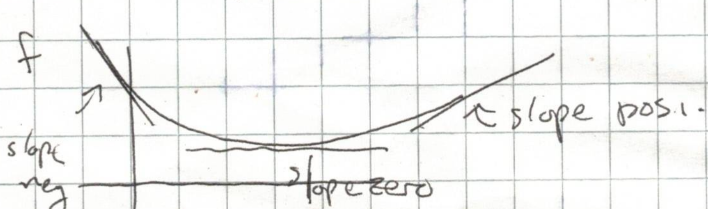


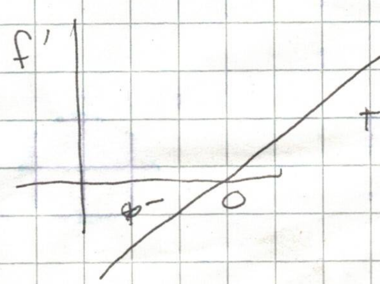




f doesn't have to be diff'ble on I. to be concave up/down

Suppose f is twice differentiable. (i.e. graphs of f and f' are "smooth".)





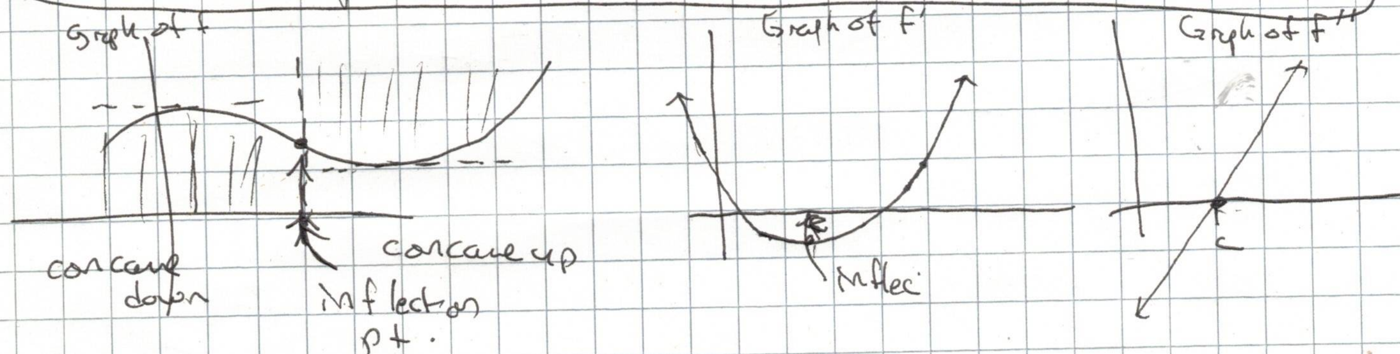
must be
So f' ~~is~~ increasing!

Thm Suppose f is twice diff'ble on I.

1) If $f''(x) > 0$ for every $x \in I$ then f is concave up on I
2) If $f''(x) < 0$ " " " " " down "

Idea: $f''(x) > 0 \Rightarrow f'$ is increasing $\Rightarrow f$ is concave up.

Def An *inflection point* of a function f is a point c such that f is continuous at c and the concavity of f changes at c.



Thm Suppose f'' is continuous at c. If c is an inflection point of f then $f''(c) = 0$.

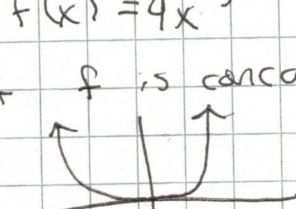
f'' continuous
c inflection $\Rightarrow f''(c) = 0$

Note Converse is false! (That is, $f''(c) = 0$ does not necessarily imply c is inflection pt)

Ex f defined by $f(x) = x^4$ for all $x \in \mathbb{R}$.

$f'(x) = 4x^3 \quad f''(x) = 12x^2 \quad$ so $f''(0) = 0$

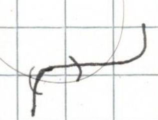
but f is concave up everywhere. so zero not infl. pt



Solving for $f''(x) = 0$ finds candidates for infl. points

<u>Problem</u> Same as above but for $f(x) = \frac{1}{x}$.

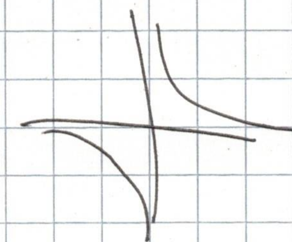
<u>sol</u> $f'(x) = -\frac{1}{x^2}$ $f''(x) = \frac{2}{x^3}$ f'' DNE at $x=0$.



If $x<0$ then $x^3<0$ (ie $(-1)^3 = (-1)(-1)(-1) = (+1)(-1) = -1 < 0$)
If $x>0$ then $x^3>0$.

$\Rightarrow$

	$x<0$	$x>0$
x^3	$-$	$+$
$f''(x) = \frac{2}{x^3}$	$-$	$+$



$\Rightarrow$ f is concave down on $(-\infty, 0)$
" up on $(0, +\infty)$

$x=0$ is <u>NOT</u> an inflection pt since f not cts there

<u>§4.2.7</u> Classifying critical pts using 2nd deriv

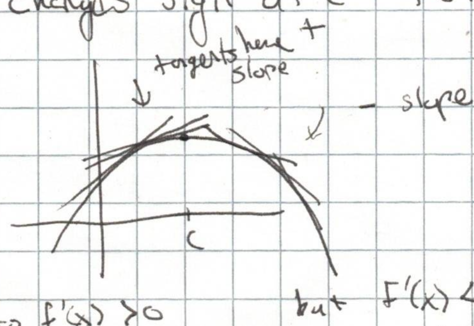
Recall:

f has a local min or max at c $\Rightarrow$ c is a crit point of f (i.e. $f'(c) = 0$ or $f'(c)$ DNE.

But not all crit points are local extrema!
How can we determine which crit pts are / aren't?

<u>Method #1</u>
If f' changes sign at c then f has local extremum at c.

Idea:



so $f'(x)>0$ for $x<c$ but $f'(x)<0$ for $x>c$

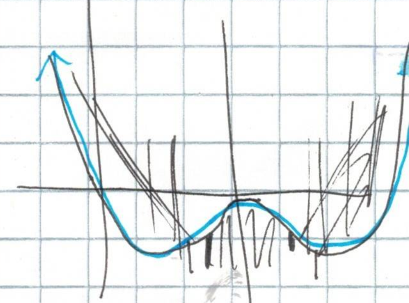
	$x<c$	$x>c$
$f'(x)$	$+$	$-$
slope of f	$\nearrow$	$\searrow$

$f'(x)$ changes sign at $x=c$

(in the same way solving $f'(x)=0$ only finds possible candidates for crit points).
Must do further analysis to confirm crit / infl. points

Problem: Find intervals of concavity and locations of inflection points of the function f defined by
$$f(x) = x^4 - 6x^2.$$

Sol
$$f'(x) = 4x^3 - 12x$$
$$f''(x) = 12x^2 - 12 = 12(x^2-1)$$



so possible infl pts are at $x = \pm 1$.

	$x < -1$	$-1 < x < +1$	$+1 < x$
$f''(x)$	$+$ $(x^2 > 1)$ $\Rightarrow x^2-1 > 0$	$-$ $0 < x^2 < 1$ $\Rightarrow x^2-1 < 0$	$+$ $x^2 > 1 \Rightarrow x^2-1 > 0$
shape of f	$\cup$	$\cap$	$\cup$

$\Rightarrow$ f is concave up on $(-\infty, -1]$ and $[1, \infty)$
f is concave down on $[-1, +1]$.

f, f' and f'' are continuous everywhere and f'' changes sign at $x = \pm 1$.
so $+1$ and -1 are infl. points

~~Curve Sketching~~
~~Give the general "shape" of the graph and identify important pieces of information~~
- ~~roots~~ ~~$f(x)=0$~~
- ~~locations of local min/max's~~
- ~~asymptotes~~
- ~~critical points~~ ~~$f'(x)=0$ or DNE~~
- ~~inflection points.~~ ~~$f''(x)=0$.~~

Thm: Assume c is a crit point of f and f is cts at c.

1) If there is an interval (a,b) containing c such that
$$f'(x) > 0 \quad \text{for all } x \in (a,c)$$
and
$$f'(x) < 0 \quad \text{for all } x \in (c,b)$$
then f has a local max at c.

2) Same as above, but if
$$f'(x) < 0 \quad \text{for all } x \in (a,c)$$
and
$$f'(x) > 0 \quad \text{" "} \; x \in (c,b)$$
" " " local min

Ex Find local min/max's of f defined by
$$f(x) = \frac{x^3}{3} + \frac{-3}{2}x^2 + 2x + 1$$

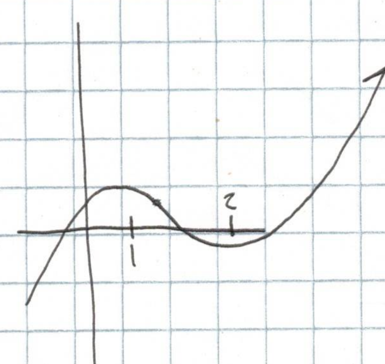
sol ~~$f'(x) = x^2 - 3x = x(x-3)$~~
$$f'(x) = x^2 - 3x + 2$$
$$= (x-2)(x-1)$$

So crit points at $x=2$ and $x=1$.

~~If $x < 2$~~

Make table:

	$x \in (-\infty, 1)$	$x \in (1,2)$	$x \in (2, \infty)$
$x-2$	$-$	$-$	$+$
$x-1$	$-$	$+$	$+$
$f'(x) = (x-1)(x-2)$	$+$	$-$	$+$
shape of f	$\nearrow$	$\searrow$	$\nearrow$



$\Rightarrow$ f has local max at $x=1$
and local min at $x=2$.

Sign changes for f' at $x=1$ and $x=2$.

Note! $f(x) = x^3$ has crit pt at $x=0$
$f'(x) = 3x^2$ $\quad f'(0) = 0$ but
f' does NOT change sign as $f'(x) > 0$