Math 137 F18 Final.

Solutions

Notes:

1. Answer all questions in the space provided. You may use the last page as additional space for solutions. Clearly mark this if you do.
2. Your grade will be influenced by how clearly you express your ideas and how well you organize your solutions. Show all details to get full marks. Numerical answers should be in exact values (no approximations). For example, $\frac{\sqrt{3}}{2}$ is acceptable, 0.8660 is not.
3. There are a total of 103 possible points, plus 1 possible bonus point.
4. Check that your exam has 16 pages, including the cover page.
5. DO NOT write on the Crowdmark QR code at the top of the pages or your exam will not be scanned (and will receive a grade of zero).
6. Use a dark pen or pencil.

(MC) Answer the following multiple choice questions by writing either a, b, c, or d in the box to the right of the question. Note there is only one correct answer for each question.

[2] 1. $\lim_{x\to a} |f(x)| = 0$ if and only if $\boxed{A}$

(a) $\lim_{x\to a} f(x) = 0$

(b) $f(x)$ is continuous at $x = a$.

(c) $f(x)$ is differentiable at $x = a$.

(d) None of the above.

[2] 2. If f is a continuous function on $[a, b]$, which of the following must be true? $\boxed{C}$

(a) f is differentiable on (a, b).

(b) $f(a) \leq f(x) \leq f(b)$ for all $x \in [a, b]$.

(c) There exists a point $c \in [a, b]$ such that $f(c) \geq f(x)$ for all $x \in [a, b]$.

(d) None of the above.

[2] 3. $\lim_{x\to\infty} \dfrac{\ln(2x)}{\ln(3x)} =$ $\boxed{B}$

(a) 2/3.

(b) 1.

(c) 3/2.

(d) None of the above.

4. The statement "For all $\epsilon > 0$, there is a $\delta > 0$ such that when $0 < x - a < \delta$ we get

[2] $|f(x) - L| < \epsilon$" is equivalent to $\boxed{C}$

(a) $\lim_{x\to a} f(x) = L$

(b) $\lim_{x\to a^-} f(x) = L$

(c) $\lim_{x\to a^+} f(x) = L$

(d) None of the above.

[2] 5. If you drive 200km in 2 hours, then which of the following is certain? $\boxed{D}$

(a) You were traveling less than 100km/h for at most one hour.

(b) You must have been going 100km/h for at least ten minutes.

(c) You may not have been traveling at 100km/h at any point during your trip.

(d) None of the above.

(TF) True/False, answer in the box below the question by writing TRUE or FALSE.

[1] 6. TRUE or FALSE: If $x < c$ then $|x - a| \leq |c - a|$ for all $x, c, a \in \mathbb{R}$.

$\boxed{\text{FALSE}}$

[1] 7. TRUE or FALSE: $f(x) = \dfrac{1}{\ln(x)}$ satisfies the conditions of the MVT on $[2, 4]$.

$\boxed{\text{TRUE}}$

[1] 8. TRUE or FALSE: If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function with exactly 4 distinct roots, then it must have exactly 3 local extrema.

$\boxed{\text{FALSE}}$

[1] 9. TRUE or FALSE: Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that $f(2) = 3$ and $f(4) = 13$. Then, there must exist $c \in (2, 4)$ such that $f'(c) = 5$.

$\boxed{\text{FALSE}}$

[1] 10. TRUE or FALSE: If f and g are continuous functions on $\mathbb{R}$ that are not differentiable at 0, then fg is not differentiable at 0.

$\boxed{\text{FALSE}}$

(SA) Short answer questions. marks only awarded for a correct final answer, you do not need to show any work. **Clearly indicate your final answer.**

[2] 1. If $a_1 = 1$ and $a_{n+1} = \sqrt{2a_n + 35}$ for $n \geq 1$, write down the possible candidates for the limit given that $a_n \geq 0$ for all $n \in \mathbb{N}$.

$L = \sqrt{2L+35} \to L^2 - 2L - 35 = 0$

$\to (L-7)(L+5) = 0$

$\to L = 7, -5$, but $a_n \geq 0 \to$ $L = 7$

[2] 2. For $f(x) = x^{\frac{2}{3}}$, write the equation for $L^f_{125}(x)$.

$L^f_{125} = 25 + \frac{2}{15}(X - 125)$

[2] 3. Given $f(4) = 5$ and $f'(4) = \frac{2}{3}$, determine the value of $(f^{-1})'(5)$.

$(f^{-1})'(5) = \frac{1}{f'(f^{-1}(5))} = \frac{1}{f'(4)} = \frac{1}{(2/3)} = 3/2$

[2] 4. State the Mean Value Theorem for f on $[a, b]$.

If f is continuous on $[a,b]$ and differentiable on (a,b), then there exists a point $c \in (a,b)$ so that $f'(c) = \frac{f(b) - f(a)}{b-a}$

[2] 5. Write down the third-order Taylor polynomial centred at $x = 0$. that is. $T_{3,0}(x)$ for $f(x) = 4x^4 + 3x^3 + 2x^2 + x + 1$.

$T_{3,0}(X) = 3X^3 + 2X^2 + X + 1$

(LA) The remaining questions are long answer questions. please show all of your work.

1. Let $f(x) = \sin(x)$

[4] (a) Prove that $\lim_{x\to\infty} f(x)$ does not exist by finding two sequences x_n and y_n, where both $x_n \to \infty$ and $y_n \to \infty$ as $n \to \infty$, but $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$.

Let $x_n = n\pi$, $y_n = 2n\pi + \frac{\pi}{2}$. Clearly $x_n \to \infty$ & $y_n \to \infty$ but $f(x_n) = 0$ for all $n \in \mathbb{N}$ while $f(y_n) = 1$.

So $\lim_{n\to\infty} f(x_n) = 0 \neq 1 = \lim_{n\to\infty} f(y_n)$.

Therefore, by Sequential Characterization, $\lim_{x\to\infty} f(x)$ does not exist.

[3] (b) Compute $\lim_{x\to\infty} \frac{f(x)}{x}$.

We know $-1 \leq f(x) \leq 1$, so for $x > 0$,

$-\frac{1}{x} \leq \frac{f(x)}{x} \leq \frac{1}{x}$ and $\lim_{x\to\infty} \pm\frac{1}{x} = 0$,

so by the Squeeze Theorem, $\lim_{x\to\infty} \frac{f(x)}{x} = 0$ as well.

[4] 2. Suppose $A, B \in \mathbb{R}$, $A > 0$, $B > 0$ and $f : \mathbb{R} \to \mathbb{R}$ is a function such that if $|x - y| < A$ then $|f(x) - f(y)| < B|x - y|$ for all $x, y \in \mathbb{R}$. Use an $\epsilon - \delta$ argument to prove that f is continuous on $\mathbb{R}$.

Let $\epsilon > 0$ be given. Let $a \in \mathbb{R}$ be given.

Pick $\delta = \min\{A, \epsilon/B\}$. Then, if $|x-a| < \delta$

we get $|f(x) - f(a)| < B|x-a| < B\delta \leq \frac{B\epsilon}{B} = \epsilon$

(Since $|x-a| < \delta \leq A$)

QED.

3. For each of the following functions. compute $f'(x)$ using any method. You do not need to simplify your answers.

[3] (a) $f(x) = \dfrac{\sin(x) + \cos(x)}{1 + x^2}$

$$f'(x) = \frac{(1+x^2)(\cos x - \sin x) - (\sin x + \cos x)(2x)}{(1+x^2)^2}$$

[3] (b) $f(x) = \ln(\arctan(e^x))$

$$f'(x) = \frac{1}{\arctan(e^x)} \cdot \frac{1}{1+e^{2x}} \cdot e^x.$$

[1] (BONUS) $f(x) = (2018)^x + x^{2018} + \ln(2018)$

$$f'(x) = (2018)^x \ln(2018) + 2018x^{2017}$$

[3] 4. (a) Find y' if $\sin(xy) = \sin(x) + \sin(y)$. Implicit Differentiation:

$$\cos(xy)(y+xy') = \cos x + (\cos y)y'$$

$$\Rightarrow xy'\cos(xy) - y'\cos(y) = \cos x - y\cos(xy)$$

$$\Rightarrow y'(x\cos(xy) - \cos(y)) = \cos x - y\cos(xy)$$

$$\Rightarrow y' = \frac{\cos(x) - y\cos(xy)}{x\cos(xy) - \cos(y)}$$

[3] (b) Find all critical points for $f(x) = x^x$ for $x > 0$. Logarithmic differentiation.

$$\ln(f(x)) = x\ln x$$

$$\Rightarrow \frac{f'(x)}{f(x)} = \ln x + 1$$

$$\Rightarrow f'(x) = x^x(\ln x + 1)$$

So $f'(x) = 0$ if and only if $x = e^{-1} = \frac{1}{e}$

So the only critical point is at

$x = \frac{1}{e}$ $\left(\frac{1}{e}, \left(\frac{1}{e}\right)^{1/e}\right)$.

5. Let $f(x) = x^3 + 2x - 2$.

[3] (a) Prove that f has at least one root on $[0, 1]$.

$f(0) = -2 < 0$

$f(1) = 1 > 0$

so since f is continuous (it's a polynomial!) f has a root on $[0,1]$, by the IVT.

[3] (b) Prove that f has exactly one root on $\mathbb{R}$.

$f'(x) = 3x^2 + 2 > 0$ for all $x \in \mathbb{R}$, so f has no critical points. By a problem done on an assignment, f has at most one root.

$\therefore$ f has exactly one root.

Alternate solution: $f'(x) = 3x^2 + 2 > 0$ for all $x \in \mathbb{R}$, so f is increasing and therefore one-to-one on $\mathbb{R}$. Therefore, there exists at most one $c \in \mathbb{R}$ such that $f(c) = 0$.

Together with (a), f has exactly one root.

Can also use Rolle's Theorem.

QED.

[3] (c) Using $x_1 = 0$, perform two iterations of Newton's Method to find x_2 and x_3 to approximate the root of f.

$$x_{n+1} = x_n - \frac{(x_n)^3 + 2(x_n) - 2}{3(x_n)^2 + 2}$$

$$x_1 = 0, \quad x_2 = 0 - \frac{0^3 + 2(0) - 2}{3(0)^2 + 2} = 1$$

$$x_3 = 1 - \frac{1 + 2 - 2}{3 + 2} = 1 - \frac{1}{5} = \frac{4}{5}$$

6. Suppose f is continuous on $[5,6]$ and differentiable on $(5,6)$, and $-4 \le f'(x) \le -2$ for all $x \in (5,6)$.

[2] (a) Write down the inequalities you would get by applying the Bounded Derivative Theorem to f on $[5,6]$.

$$f(5)-4(x-5) \le f(x) \le f(5)-2(x-5)$$

[2] (b) If $f(6) = -2$, determine an interval that $f(5)$ must lie in.

Sub in $X=6$ to the inequalities in Part (a):

$$f(5)-4(1) \le f(6) \le f(5)-2(1)$$

$$\Rightarrow -4 \le -2-f(5) \le -2 \quad (\text{mult. by } -1)$$

$$\Rightarrow 4 \ge 2+f(5) \ge 2$$

$$\Rightarrow 2 \ge f(5) \ge 0, \text{ so } f(5) \in [0,2]$$

[4] 7. Use the Mean Value Theorem to prove that $|\sin(a)-\sin(b)| \le |a-b|$ for all $a, b \in \mathbb{R}$.

Sin(X) is continuous and differentiable on $\mathbb{R}$.

Fix $a,b \in \mathbb{R}$, and WLOG say $a<b$.

Then, there exists $c \in (a,b)$ so that

$$\cos(c) = \frac{\sin(a)-\sin(b)}{a-b} \quad \text{by the MVT}$$

Then $$\frac{|\sin(a)-\sin(b)|}{|a-b|} = |\cos(c)| \le 1 \quad (\text{Since } |\cos(c)| \le 1)$$

$$\Rightarrow |\sin(a)-\sin(b)| \le |a-b|$$

as desired.

QED

8. In each case, compute the limit using any method.

[3] (a) $\lim_{x\to\infty} x^{1/\sqrt{x}}$ (type ∞^0)

$$= e^{\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}}} \quad (\text{type } \infty/\infty)$$

$$\underset{=}{LHR} e^{\lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{2\sqrt{x}}\right)}}$$

$$= e^{\lim_{x\to\infty} \frac{2\sqrt{x}}{x}}$$

$$= e^{\lim_{x\to\infty} \frac{2}{\sqrt{x}}}$$

$$= e^0 = 1$$

[3] (b) $\lim_{x\to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln(x)}\right)$ (type $\infty - \infty$)

$$= \lim_{x\to 1^+} \frac{\ln x - (x-1)}{\ln x (x-1)} \quad (\text{type } 0/0)$$

$$\underset{=}{LHR} \lim_{x\to 1^+} \frac{\frac{1}{x} - 1}{\frac{1}{x}(x-1) + \ln(x)}$$

$$= \lim_{x\to 1^+} \frac{\left(\frac{1-x}{x}\right)}{\left(\frac{x-1}{x}\right) + \ln(x)}$$

$$= \lim_{x\to 1^+} \frac{1-x}{x-1+x\ln x} \quad \text{type } 0/0$$

$$\underset{=}{LHR} \lim_{x\to 1^+} \frac{-1}{1+\ln x+1}$$

$$= \lim_{x\to 1^+} \frac{-1}{2+\ln x} = \frac{-1}{2}$$

[4] 9. Find all values of $a, b \in \mathbb{R}$ so that $\lim_{x\to 3} \frac{ax^2+bx}{\ln(x-2)} = 3$.

First, $\lim_{x\to 3} \ln(x-2) = 0$, so for the limit to exist, we need $\lim_{x\to 3} ax^2+bx=0$ too.

So $9a+3b=0 \to b=-3a$.

So, we get $\lim_{x\to 3} \frac{ax^2-3ax}{\ln(x-2)}$ (type 0/0)

$$\underline{\underline{LHR}} \lim_{x\to 3} \frac{2ax-3a}{(1/x-2)}$$

$$= \lim_{x\to 3} (2ax-3a)(x-2)$$

$$= (6a-3a)(1) = 3a.$$

But, we want the limit to be 3, so $3a=3$

or $a=1$

so $b=-3$

[4] 10. If f has derivatives of all orders on $\mathbb{R}$, determine, for $a \in \mathbb{R}$:

$$\lim_{h\to 0} \frac{f(a+h)-2f(a)+f(a-h)}{h^2}.$$ (type 0/0)

$$\underline{\underline{LHR}} \lim_{h\to 0} \frac{f'(a+h)-f'(a-h)}{2h}$$ (type 0/0)

$$\underline{\underline{LHR}} \lim_{h\to 0} \frac{f''(a+h)+f'(a-h)}{2}$$

$$= f''(a)$$

11. Consider the function $f(x) = (1+x)^{\frac{1}{5}}$.

[2] (a) Find the second-degree Taylor polynomial for f centred at $x = 0$, $T_{2,0}(x)$.

$f(x) = (1+x)^{1/5} \to f(0) = 1$

$f'(x) = \frac{1}{5}(1+x)^{-4/5} \to f'(0) = \frac{1}{5}$

$f''(x) = \frac{-4}{25}(1+x)^{-9/5} \to f''(0) = \frac{-4}{25}$

So $T_{2,0}(x) = 1 + \frac{x}{5} - \frac{4x^2}{50}$ $\left(= 1 + \frac{x}{5} - \frac{4}{25} \cdot \frac{x^2}{2!} \right)$

[2] (b) Use $T_{2,0}$ to approximate $\sqrt[5]{1.5}$.

$\sqrt[5]{1.5} = \sqrt[5]{1+0.5} \approx T_{2,0}(0.5) = 1 + \frac{0.5}{5} - \frac{4(0.5)^2}{50}$

$= 1 + \frac{1}{10} - \frac{1}{50} = \frac{54}{50} = \frac{27}{25}$

[2] (c) Use Taylor's Theorem to write down what $f(x) - T_{2,0}(x)$ is equal to (in terms of x and c) for $x > 0$.

$f(x) - T_{2,0}(x) = \frac{f'''(c)}{3!} x^3 = \frac{36}{125}(1+c)^{\frac{-14}{5}} \frac{x^3}{3!}$

for $c \in (0, x)$. $= \frac{6}{125}(1+c)^{-14/5} x^3$

[2] (d) Find an upper bound on the error in your approximation in part (b).

error $= |R_{2,0}(0.5)| = \frac{6}{125}(1+c)^{\frac{-14}{5}}(0.5)^3 \leq \frac{6}{125}(0.5)^3$

$= 3/500$

[2] (e) Is the estimate in part (b) an over or under estimate?

$f(0.5) - T_{2,0}(0.5) = \frac{6}{125}(1+c)^{\frac{-14}{5}}(0.5)^3 > 0$ for $c \in [0, 0.5]$, so $f(0.5) > T_{2,0}(0.5)$, so it is an underestimate.

[2] (f) Give an interval that $\sqrt[5]{1.5}$ must lie in, be as specific as possible.

$\sqrt[5]{1.5} \in \left[\frac{27}{25}, \frac{27}{25} + \frac{3}{500} \right]$

[12] 12. Sketch the graph of $f(x)$, where

$$f(x) = (x^2-1)^{\frac{2}{3}}, \qquad f'(x) = \frac{4x}{3(x^2-1)^{\frac{1}{3}}}, \qquad f''(x) = \frac{4(x^2-3)}{9(x^2-1)^{\frac{4}{3}}}.$$

Use this page for your work, on the next page you will summarize your findings and draw your graph. **Marks will be awarded to the next page only**. On your graph, label any intercepts, critical points, points of inflection, and asymptotes. In your summary, all points should include both x- and y-coordinates.

X-int $(y=0)$ $\quad 0 = (x^2-1)^{2/3} \rightarrow x = \pm 1$

y-int $(x=0)$: $y=1$

No Asymp.

f' DNE @ $\underbrace{x=\pm 1}_{C.P.}$, $f'=0$ $\underbrace{@\, x=0}_{C.P.}$ $\quad (0,1)$

f'' DNE @ $x=\pm 1$, $f''=0$ @ $x=\pm\sqrt{3}$. $\quad (\pm\sqrt{3}, 2^{2/3})$

		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
f''	$+$		$-$		$-$		$-$		$-$		$+$
f'	$-$		$-$		$+$		$-$		$+$		$+$
f	∪ ↘		∩ ↘		∩ ↗		∩ ↘		∩ ↗		∪ ↗
Shape											
		I.P.				Local max					

Summary:

Intercepts	Asymptotes	Critical Points	Inflection Points
$x=1$ $(1,0)$ $x=-1$ $(-1,0)$ $y=1$ $(0,1)$	None.	$x=0$ $x=\pm 1$ $(0,1)$ $(\pm 1,0)$	$x=-\sqrt{3}$ $x=\sqrt{3}$ $(\pm\sqrt{3}, 2^{2/3})$

The domain of f is: $\mathbb{R}$

Intervals where f is increasing: $[-1,0]$ and $[1,\infty)$

Intervals where f is decreasing: $(-\infty,-1]$ and $[0,1]$

Intervals where f is concave up: $(-\infty,-\sqrt{3}]$ and $[\sqrt{3},\infty)$

Intervals where f is concave down: $[-\sqrt{3},-1]$, $[-1,1]$, $[1,\sqrt{3}]$