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UNIVERSITY OF WATERLOO

Examination
Quiz 5
Fall 2019
ECE 206/MATH 212

Times: Wednesday 2019-11-13 at 17:30 to 18:30 (5:30 to 6:30PM)
Duration: 1 hour (60 minutes)
Exam ID: 4310034
Sections: ECE 206 LEC 001
Instructors: Mark Girard

Solutions

Notes:

1. Fill in your name (first and last) and student ID number.
2. This quiz contains 6 pages (including this cover page) and 4 problems. Check to see if any pages are missing.
3. Answer all questions in the space provided. Extra space is provided on the last page. If you want the overflow page marked, be sure to clearly indicate that your solution continues.
4. Show all of your work on each problem.
5. **Your grade will be influenced by how clearly you express your ideas, and how well you organize your solutions.**
6. You may **not** use your books, notes, calculator, or any other aids on this quiz. The use of personal electronic or communication devices is prohibited.

Problem	Points
1	3
2	4
3	4
4	4
Total:	15

Please initial:

[3] 1. Find all complex numbers z satisfying $e^z = -\sqrt{3} + j$. What is $\mathrm{Log}\,(-\sqrt{3} + j)$?

$z = x + jy$

$e^z = e^x \cos y + j e^x \sin y$

$= -\sqrt{3} + j$

$\Rightarrow \begin{cases} e^x \cos y = -\sqrt{3} \\ e^x \sin y = 1 \end{cases}$

$\left.\begin{array}{l} e^x = \sqrt{(-\sqrt{3})^2 + 1^2} \\ = \sqrt{3+1} = \sqrt{4} = 2 \end{array}\right\} \Rightarrow x = \ln 2$

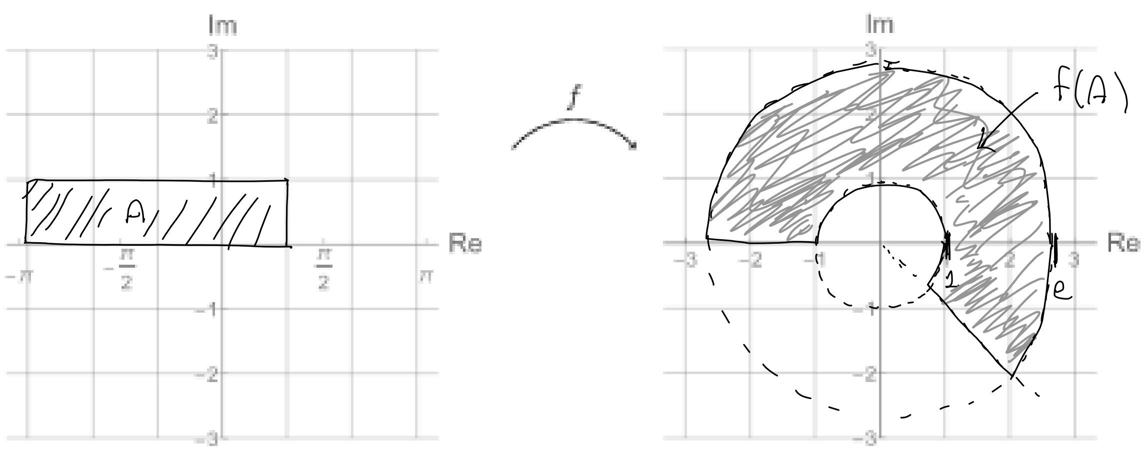
$\left.\begin{array}{l} \cos y = \frac{-\sqrt{3}}{2} \\ \sin y = \frac{1}{2} \end{array}\right\} \Rightarrow y = \frac{5\pi}{6} + 2k\pi$ for $k \in \mathbb{Z}$

Solutions: $\boxed{z = \ln 2 + j\frac{5\pi}{6} + j2k\pi}$

$\mathrm{Log}(-\sqrt{3} + j) = \mathrm{Log}\left(2e^{j\frac{5\pi}{6}}\right)$

$= \boxed{\ln 2 + j\frac{5\pi}{6}}$

[4] 2. Consider the set A of complex numbers defined by $A = \{x + jy \mid x \in [-\pi, \frac{\pi}{4}],\ y \in [0, 1]\}$. Sketch the set A, then find and sketch the image of A under the mapping $f(z) = e^{-jz}$.



$f(z) = e^{-jz} = e^{-j(x+jy)} = e^y e^{-jx}$

$e \approx 2.7$

- $|f(z)| = e^y \quad 0 \le y \le 1 \Rightarrow 1 \le |f(z)| \le e$
- $\mathrm{Arg}(f(z)) = -x$

$-\pi \le x \le \frac{\pi}{4} \Rightarrow -\frac{\pi}{4} \le \mathrm{Arg}(f(z)) \le \pi$

[4] 3. Find all possible values of $z \in \mathbb{C}$ that satisfy $\sin z = 3$. $z = x + jy$

$$\sin(x+jy) = \sin(x)\cosh(jy) + \cos(x)\sinh(jy)$$

$$= \sin x \cosh y + j\cos x \sinh y = 3$$

$$\Rightarrow \begin{cases} \sin x \cosh y = 3 \\ \cos x \sinh y = 0 \end{cases}$$

$\cos x \sinh y = 0 \Rightarrow y = 0$ or $x = \frac{\pi}{2} + n\pi \quad n \in \mathbb{Z}$

- $y = 0 \Rightarrow \cosh y = 1$

  $\Rightarrow \sin x = 3 \leftarrow$ can't happen

- $x = \frac{\pi}{2} + n\pi \quad n \in \mathbb{Z}$

  $\Rightarrow \sin\left(\frac{\pi}{2} + n\pi\right) = \begin{cases} +1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$

  - n odd $\Rightarrow -\cosh y = 3 \leftarrow$ no solutions
  - n even $\Rightarrow \cosh y = 3$

$$\cosh y = \frac{e^y + e^{-y}}{2} = 3 \Rightarrow e^y - 6 + e^{-y} = 0$$

$$\Rightarrow e^{2y} - 6e^y + 1 = 0$$

$$\Rightarrow e^y = \frac{6 \pm \sqrt{36-4}}{2}$$

$$= 3 \pm \sqrt{9-1}$$

$$= 3 \pm \sqrt{8}$$

$$\Rightarrow y = \ln(3 \pm \sqrt{8})$$

solutions: $\boxed{z = \frac{\pi}{2} + 2k\pi + j\ln(3 \pm \sqrt{8}) \quad k \in \mathbb{Z}}$

Please initial:

[4] 4. Consider the mapping $f : \mathbb{C} \to \mathbb{C}$ defined by $f(x + jy) = x^2 + jy^2$. At which points is the mapping differentiable? What is the derivative at those points?

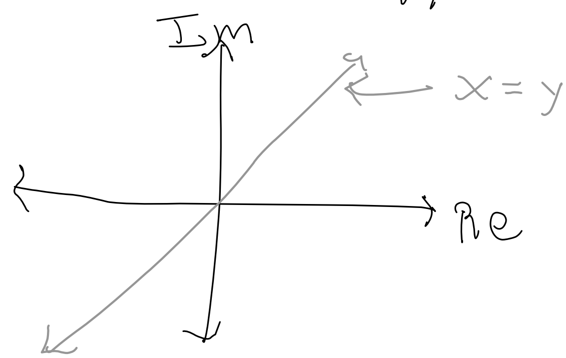
$$f(x+jy) = u(x,y) + jv(x,y)$$

$$u(x,y) = x^2 \qquad v(x,y) = y^2$$

CRE

$$\left.\begin{array}{l} u_x - v_y = 0 \\ v_x + u_y = 0 \end{array}\right. \Rightarrow \left.\begin{array}{l} 2x - 2y = 0 \\ 0 + 0 = 0 \end{array}\right\} \Rightarrow x = y$$

$\Rightarrow f$ differentiable at all points $z = x + jy$ with $x = y$



Derivative at these points is

$$f'(x+jy) = u_x(x,y) + j v_x(x,y)$$
$$= 2x + j0 = 2x \ (= 2y).$$

This space is for sketch work or overflow

(If you want something here marked, be sure to clearly indicate on the question page.)

Please initial:

Trigonometric identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta$$
$$\sin(A \pm B) = \sin A\cos B \pm \cos A \sin B \qquad \sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos(A \pm B) = \cos A\cos B \mp \sin A \sin B \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Complex trigonometric and hyperbolic identities

$$\cos z = \frac{e^{jz}+e^{-jz}}{2} \qquad \sin z = \frac{e^{jz}-e^{-jz}}{2j} \qquad \cosh z = \frac{e^{z}+e^{-z}}{2} \qquad \sinh z = \frac{e^{z}-e^{-z}}{2}$$

For real numbers $y \in \mathbb{R}$:

$$\cos jy = \cosh y \qquad \sin jy = j\sinh y$$

Cauchy-Riemann equations

$$\boxed{u_x = v_y \qquad u_y = -v_x}.$$

If $f = u + jv$ is differentiable, its derivative in Cartesian coordinates is given by

$$f'(z) = f(x+jy) = u_x(x,y) + jv_x(x,y).$$