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ECE 206 Spring 2018

Midterm Solutions

Name: ____________________

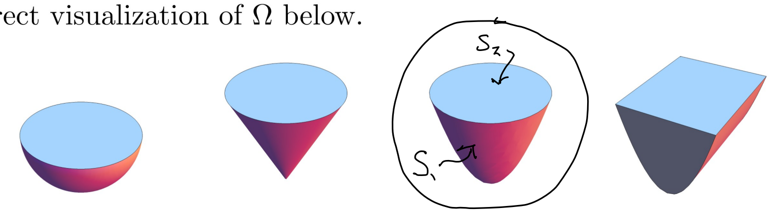
Problem	Points	Score
1	11	
2	9	
3	9	
4	9	
5	3	
6	4	
Total:	45	

Notes:

1. Fill in your name (first and last) and student ID number in the space above.
2. This midterm contains 10 pages (including this cover page) and 6 problems. Check to see if any pages are missing.
3. Answer all questions in the space provided. Extra space is provided at the end. If you want the overflow page marked, be sure to clearly indicate that your solution continues.
4. Show all of your work on each problem.
5. **Your grade will be influenced by how clearly you express your ideas, and how well you organize your solutions.**

1. Consider the region Ω in $\mathbb{R}^3$ that is above the surface $x^2 + y^2 - z = 4$ and below the xy-plane. Also consider the vector field $\vec{\boldsymbol{F}}(x, y, z) = (0, 0, z)$.

[2] (a) Circle the the correct visualization of Ω below.



$\partial\Omega = S_1 \cup S_2$

[7] (b) Use a surface integral to directly compute the flux of $\vec{\boldsymbol{F}}$ through the entire surface $\partial\Omega$, with respect to the outward pointing normals.

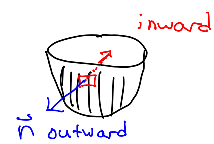
[2] (c) Use the Divergence Theorem and your answer in (b) to determine the volume of Ω.

(b) Boundary of Ω consits of two parts $\partial\Omega = S_1 \cup S_2$, which must be considered separately.

$\underline{S_1}$: Parameterize as $x = r\cos\theta$, $y = r\sin\theta$, $z = x^2+y^2-4 = r^2-4$, $r \in [0,2]$ $\theta \in [0,2\pi]$

or $\Phi(r,\theta) = (r\cos\theta, r\sin\theta, r^2-4)$ so that $\frac{\partial\Phi}{\partial r} = (\cos\theta, \sin\theta, 2r)$ and $\frac{\partial\Phi}{\partial\theta} = (-r\sin\theta, r\cos\theta, 0)$. Hence

$$\frac{\partial\Phi}{\partial r} \times \frac{\partial\Phi}{\partial\theta} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-2r^2\cos\theta, -2r^2\sin\theta, r\cos^2\theta + r\sin^2\theta)$$
$$= (-2r^2\cos\theta, -2r^2\sin\theta, \underbrace{r}_{\geq 0}). \leftarrow \text{points } \underline{\text{up}} \text{ and } \underline{\text{inward}}$$



Need $\underline{\text{outward}}$ facing normal, so $\vec{n} = (2r^2\cos\theta, 2r^2\sin\theta, \underline{-r})$. Now,

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,dA = \iint_{S_1} (0,0,z)\cdot\frac{\vec{n}}{\|\vec{n}\|}\,dA = \iint_{S_1} -zr\,\frac{1}{\|\vec{n}\|}\,dA = -\iint_{D_{r\theta}} (r^2-4)\,r\,dr\,d\theta$$
$$= -\left(\int_0^{2\pi} d\theta\right)\left(\int_0^2 r^3 - 4r\,dr\right) = -(2\pi)\left(\tfrac{1}{4}r^4 - 2r^2\right)\Big|_{r=0}^{2}$$
$$= -2\pi(4-8) = \boxed{16\pi}.$$

$\underline{S_2}$: On xy-plane, $z=0$, so $\vec{F}(x,y,0) = (0,0,0) = \vec{0}$ on S_2. Hence $\iint_{S_2} \vec{F}\cdot\hat{n}\,dA = 0$.

Putting it together, $$\oiint_{\partial\Omega} \vec{F}\cdot\hat{n}\,dA = \iint_{S_1} \vec{F}\cdot\hat{n}\,dA + \iint_{S_2} \vec{F}\cdot\hat{n}\,dA = 16\pi + 0 = \boxed{16\pi}$$

(c) Note that $\nabla\cdot\vec{F} = 0 + 0 + \frac{\partial z}{\partial z} = 1$, so

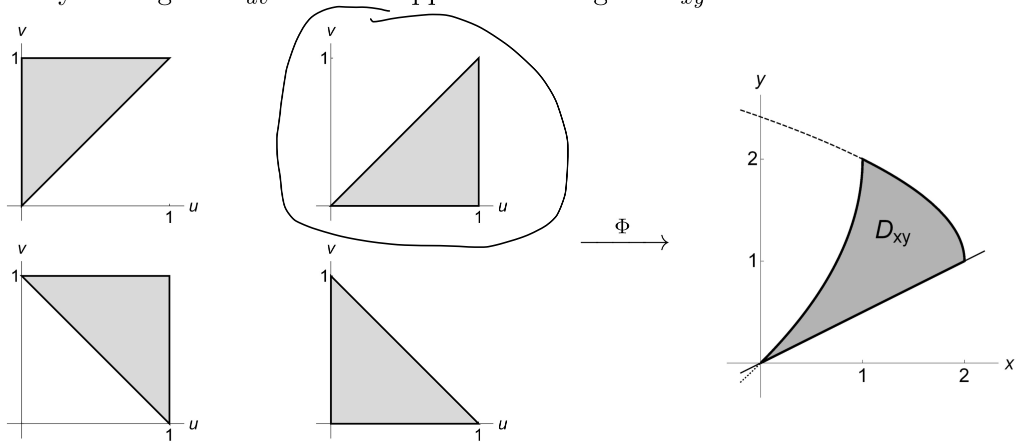
$$\text{vol}(\Omega) = \iiint_\Omega 1\,dV = \iiint_\Omega \nabla\cdot\vec{F}\,dV \underset{\text{by Divergence Theorem}}{=} \oiint_{\partial\Omega} \vec{F}\cdot\hat{n}\,dA = \boxed{16\pi}.$$

2. Consider the region D_{xy} (depicted below) that is bound by the curves $y = \frac{1}{2}x$, $y = 1 + \sqrt{2-x}$, and $y = 2(1 - \sqrt{1-x})$, and let Φ be the transformation defined by

$$\Phi(u,v) = \big(x(u,v),\, y(u,v)\big) = \big(2u - v^2,\, u + v\big)$$

that transforms a region D_{uv} into D_{xy}.

[2] (a) Identify the region D_{uv} that is mapped to the region D_{xy} under the transformation Φ.



[2] (b) Determine the Jacobian of the transformation Φ.

[5] (c) Let $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by $\vec{F}(x,y) = (y^2 + x, 3xy)$. Compute the circulation of $\vec{F}$ around the boundary of D_{xy} oriented counterclockwise.
(Hint: Use a theorem to set up and evaluate a double integral over the region D_{uv}.)

(b) $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ -2v & 1 \end{vmatrix} = 2 + 2v = \boxed{2(v+1)}.$$

(c) Note that $\frac{\partial F_2}{\partial x} = 3y$, $\frac{\partial F_1}{\partial y} = 2y$, and $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 3y - 2y = y$.

By Green's theorem,

$$\oint_{\partial D_{xy}} \vec{F} \cdot d\vec{r} = \iint_{D_{xy}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \iint_{D_{xy}} y \, dA$$

$$= \iint_{D_{uv}} (u+v) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv \qquad \text{since } y = u+v$$

$$= \int_0^1 \int_0^u (u+v)\, 2(v+1)\, dv\, du = 2\int_0^1 \int_0^u (uv + u + v^2 + v)\, dv\, du$$

$$= 2\int_0^1 \left(\tfrac{1}{2}uv^2 + uv + \tfrac{1}{3}v^3 + \tfrac{1}{2}v^2\right)\Big|_{v=0}^{u} du$$

$$= 2\int_0^1 \left(\tfrac{1}{2}u^3 + u^2 + \tfrac{1}{3}u^3 + \tfrac{1}{2}u^2\right) du = 2\int_0^1 \left(\tfrac{5}{6}u^3 + \tfrac{3}{2}u^2\right) du$$

$$= 2\left(\tfrac{5}{24}u^4 + \tfrac{1}{2}u^3\right)\Big|_{u=0}^{1} = 2\left(\tfrac{5}{24} + \tfrac{1}{2}\right) = \boxed{\tfrac{17}{12}}.$$

3. Consider the two vector fields $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ and $\vec{G}: \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\vec{F}(x,y,z) = (ye^{xy} + z,\ xe^{xy} + z,\ x + y) \qquad \text{and} \qquad \vec{G}(x,y,z) = (xy + yz,\ xy + xz,\ e^z).$$

[5] (a) One of the two vector fields is conservative and the other is not. Determine which one is conservative and find a scalar potential. For the non-conservative field, show that it is not conservative.

$\vec{F}$: $\nabla \times \vec{F} = (1-1,\ 1-1,\ e^{xy} + xye^{xy} - e^{xy} - xye^{xy}) = (0,0,0) = \vec{0}$ conservative!

Find $\Psi(x,y,z)$ so that $\nabla\Psi = \vec{F}$.

- $\frac{\partial \Psi}{\partial x} = ye^{xy} + z \Rightarrow \Psi(x,y,z) = e^{xy} + xz + f(y,z)$

- $\frac{\partial}{\partial y}\left(e^{xy} + xz + f(y,z)\right) = xe^{xy} + 0 + \frac{\partial f}{\partial y}(y,z) = xe^{xy} + z \Rightarrow \frac{\partial f}{\partial y}(y,z) = z \Rightarrow f(y,z) = yz + g(z)$

- $\frac{\partial}{\partial z}\left(e^{xy} + xz + yz + g(z)\right) = x + y + g'(z) = x + y \Rightarrow g'(z) = 0 \Rightarrow g(z) = c$ constant.

So $\boxed{\Psi(x,y,z) = e^{xy} + xz + yz + c}$ is a potential.

$\vec{G}$: $\nabla \times \vec{G} = (0 - x,\ 0 - y,\ y + z - x - z) = (-x, -y, y - x) \neq \vec{0}$

not conservative!

[4] (b) Let Γ be the part of the curve $y = x^2$ on the plane $z = 0$ from $x = 0$ to $x = 1$. Evaluate the line integrals $\int_\Gamma \vec{F} \cdot d\vec{r}$ and $\int_\Gamma \vec{G} \cdot d\vec{r}$. (Hint: only one integral must be evaluated directly.)

Parameterize Γ by $\vec{\gamma}(t) = (t, t^2, 0)$ for $t \in [0,1]$. Then $\vec{\gamma}(0) = (0,0,0)$, $\vec{\gamma}(1) = (1,1,0)$.

$$\int_\Gamma \vec{F} \cdot d\vec{r} = \int_\Gamma \nabla\Psi \cdot d\vec{r} = \Psi(\vec{\gamma}(1)) - \Psi(\vec{\gamma}(0)) = \Psi(1,1,0) - \Psi(0,0,0) = \boxed{e - 1}$$

$\vec{\gamma}'(t) = (1, 2t, 0)$ and $\vec{G}(\vec{\gamma}(t)) = (t^3, t^3, 1)$, so

$$\int_\Gamma \vec{G} \cdot d\vec{r} = \int_0^1 \vec{G}(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t)\, dt = \int_0^1 (t^3 + 2t^4)\, dt$$

$$= \left(\frac{1}{4}t^4 + \frac{2}{5}t^5\right)\Big|_{t=0}^{1} = \frac{1}{4} + \frac{2}{5} = \frac{5+8}{20} = \boxed{\frac{13}{20}}$$

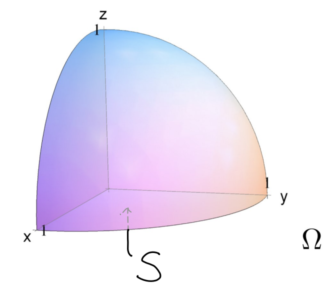
4. Let the region Ω (depicted below) be the part of the interior sphere $x^2+y^2+z^2=1$ in the quadrant $x, y, z \geq 0$. Suppose the electric field $\vec{\boldsymbol{E}}$ in the region Ω is given by $\vec{\boldsymbol{E}}(x,y,z) = (y - xz)\,\hat{\boldsymbol{\imath}} + x^2 z\,\hat{\boldsymbol{\jmath}} + z^2\,\hat{\boldsymbol{k}}$.

Divergence

a) $\nabla \cdot \vec{E} = \frac{\partial}{\partial x}(y - xz) + \frac{\partial}{\partial y}(x^2 z) + \frac{\partial}{\partial z}(z^2)$

$= -z + 0 + 2z$

$= \boxed{z}$



Ω

[2] (a) Determine the divergence of $\vec{\boldsymbol{E}}$.

[5] (b) Use Gauss' law to determine the total amount of charge contained in Ω.

[2] (c) Let S be the part of the boundary of Ω that lines on the xy-plane (i.e. $z = 0$). What is the electric flux through S? Explain.

(b) $Q_{encl} = \varepsilon_0 \oiint_{\partial\Omega} \vec{E} \cdot \hat{n}\, dA \overset{\text{Divergence Theorem}}{=} \varepsilon_0 \iiint_{\Omega} \nabla \cdot \vec{E}\, dV = \varepsilon_0 \iiint_{\Omega} z\, dV$

Switch to spherical coordinates: $z = \rho\cos\varphi$

$= \varepsilon_0 \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{1} (\rho\cos\varphi)(\rho^2\sin\varphi)\, d\rho\, d\theta\, d\varphi$

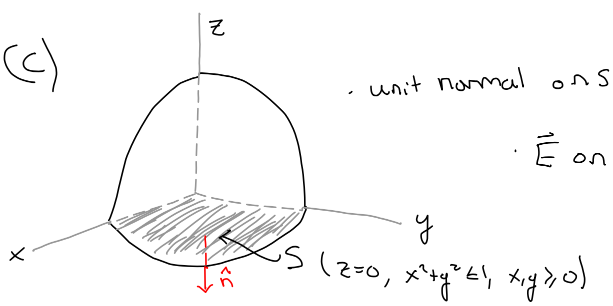
$= \varepsilon_0 \left(\int_0^{\pi/2} d\theta\right)\left(\int_0^{\pi/2} \cos\varphi \sin\varphi\, d\varphi\right)\left(\int_0^1 \rho^3\, d\rho\right)$

$= \varepsilon_0 \left(\frac{\pi}{2}\right)\left(\frac{1}{2}\sin^2(\varphi)\Big|_{\varphi=0}^{\pi/2}\right)\left(\frac{\rho^4}{4}\Big|_{\rho=0}^{1}\right)$

$= \varepsilon_0 \frac{\pi}{16}\left(\sin^2\frac{\pi}{2} - \sin^2 0\right)$

$= \boxed{\varepsilon_0 \frac{\pi}{16}}$

(c)



- unit normal on S is $\hat{n} = -\hat{k} = (0,0,-1)$
- $\vec{E}$ on S is $\vec{E}(x,y,0) = (y,0,0)$

Note that $\vec{E}$ and $\hat{n}$ are orthogonal ($\vec{E}\cdot\hat{n} = 0$) on all of S, so

$\boxed{\iint_S \vec{E}\cdot\hat{n}\, dA = 0}$.

[3] 5. Suppose a $\mathcal{C}^1$ vector field $\vec{\boldsymbol{F}} : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies $\nabla \cdot \vec{\boldsymbol{F}} = 0$ everywhere. Let S_1 and S_2 be two surfaces in $\mathbb{R}^3$ that share the boundary curve $\Gamma = \partial S_1 = \partial S_2$. Show that the flux of $\vec{\boldsymbol{F}}$ is independent of the surface chosen (S_1 or S_2).

(Your solution should reference an important theorem.)

Since $\nabla \cdot \vec{F} = 0$ everywhere (i.e. $\vec{F}$ is solenoidal), there is a vector field $\vec{A} : \mathbb{R}^3 \to \mathbb{R}^3$ so that $\vec{F} = \nabla \times \vec{A}$. By Stokes' theorem,

$$\iint_{S_1} \vec{F} \cdot \hat{n}\, dA = \iint_{S_1} (\nabla \times \vec{A}) \cdot \hat{n}\, dA = \oint_{\partial S_1} \vec{A} \cdot d\vec{r} = \oint_{\partial S_2} \vec{A} \cdot d\vec{r} = \iint_{S_2} (\nabla \times \vec{A}) \cdot \hat{n}\, dA = \iint_{S_2} \vec{F} \cdot \hat{n}\, dA.$$

Since $\partial S_1 = \partial S_2$

So $\iint_{S_1} \vec{F} \cdot \hat{n}\, dA = \iint_{S_2} \vec{F} \cdot \hat{n}\, dA$.

6. Suppose $\vec{\boldsymbol{F}}$ is a radial vector field of the form $\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}) = f(r)\vec{\boldsymbol{r}}$, where f is an arbitrary function, $\vec{\boldsymbol{r}} = (x, y, z)$, and $r = \|\vec{\boldsymbol{r}}\| = \sqrt{x^2 + y^2 + z^2}$.

[4] (a) Expand out $\nabla \cdot \vec{\boldsymbol{F}}$ and show that $\nabla \cdot \vec{\boldsymbol{F}} = 0$ only if f satisfies a certain differential equation. (Your differential equation should be only in terms of r, and not x, y, z.)

[2 bonus] (b) **Extra Credit**: Solve the separable differential equation from part (a) to find the form of $f(r)$ that satisfies $\nabla \cdot \vec{\boldsymbol{F}} = 0$.

$$\nabla \cdot \vec{F} = \nabla \cdot (f(r)\vec{r}) = (\nabla f(r)) \cdot \vec{r} + f(r)(\nabla \cdot \vec{r})$$

$$= \underbrace{\left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial x}, \frac{\partial f}{\partial r}\frac{\partial r}{\partial y}, \frac{\partial f}{\partial r}\frac{\partial r}{\partial z}\right)}_{= f'(r)\left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}\right)} \cdot (x, y, z) + f(r)\underbrace{\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right)}_{= 1+1+1 = 3}$$

$$= f'(r)\left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right) \cdot (x, y, z) + 3f(r)$$

$$= f'(r)\underbrace{\frac{x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}}}_{\frac{r^2}{r} = r} + 3f(r)$$

$$= \boxed{r f'(r) + 3f(r) = 0} \quad \text{or} \quad r\frac{df}{dr} = -3f$$

(b) Solve: $r\dfrac{df}{dr} = -3f \Rightarrow \dfrac{df}{f} = \dfrac{-3\,dr}{r}$

$$\Rightarrow \ln f = -3\ln r + C \qquad C \text{ constant}$$

$$\Rightarrow f = \underbrace{e^C}_{a \text{ constant}} r^{-3} \Rightarrow \boxed{f(r) = \frac{a}{r^3}}$$

Trigonometric identities

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B \qquad \sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$
$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Change of variable formula If $\Phi(u,v) = (x(u,v), y(u,v))$ is a transformation then

$$\iint_{D_{xy}} f(x,y)\,dx\,dy = \iint_{D_{uv}} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

where the region D_{xy} is mapped to D_{uv} under Φ. The Jacobian is defined as

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

The Jacobian satisfies the following property: $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{1}{\frac{\partial(x,y)}{\partial(u,v)}}$.

- **Polar coordinates:** $x = r\cos\theta$ and $y = r\sin\theta$. The Jacobian is $\dfrac{\partial(x,y)}{\partial(r,\theta)} = r$
- **Cylindrical coordinates:** $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. The Jacobian is $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$
- **Spherical coordinates:** $x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, $z = \rho\cos\varphi$. The Jacobian is $\dfrac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \rho^2\sin\varphi$

Vector calculus identities

$$\nabla \times (f\vec{\boldsymbol{F}}) = (\nabla f) \times \vec{\boldsymbol{F}} + f(\nabla \times \vec{\boldsymbol{F}})$$
$$\nabla \cdot (f\vec{\boldsymbol{F}}) = (\nabla f) \cdot \vec{\boldsymbol{F}} + f(\nabla \cdot \vec{\boldsymbol{F}})$$
$$\nabla \times (\nabla f) = \vec{\boldsymbol{0}}$$
$$\nabla \cdot (\nabla \times \vec{\boldsymbol{F}}) = 0$$
$$\nabla \times (\nabla \times \vec{\boldsymbol{F}}) = \nabla(\nabla \cdot \vec{\boldsymbol{F}}) - \nabla^2\vec{\boldsymbol{F}}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

Conservative and solenoidal vector fields. Let $\vec{\boldsymbol{F}}$ be a $\mathcal{C}^1$ vector field.

- If it holds that $\nabla \times \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$ everywhere, then there is a scalar field Ψ such that $\nabla\Psi = \vec{\boldsymbol{F}}$.
- If it holds that $\nabla \cdot \vec{\boldsymbol{F}} = 0$ everywhere, then there is a vector field $\vec{\boldsymbol{A}}$ such that $\nabla \times \vec{\boldsymbol{A}} = \vec{\boldsymbol{F}}$.

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