Algebra III

PMAT 611
University of Calgary
Winter semester 2014

Lecturer:
Clifton Cunningham
Notes taken by Mark Girard

February 13, 2016

Contents

1 Lecture 1

(9 January 2014)

The text for this course is [1]. The standard reference that has been used in the past is [2].

1.1 Rings

In this course, all rings will be **with** unity.

Theorem 1.1 (First isomorphism theorem (also called the *canonical decomposition* or *canonical factorization* theorem))**.** *Any ring homomorphism $\varphi\colon R \longrightarrow S$ factors as*

$$\begin{array}{ccc} R & \longrightarrow & S \\ {\scriptstyle\text{quotient}}\downarrow & & \uparrow{\scriptstyle\text{inclusion}} \\ R/\ker\varphi & \overset{\sim}{=\!=\!=} & \operatorname{im}\varphi = \phi(R) \end{array}$$

Recall: Given a (two-sided) ideal $I \subset R$ (often written as $I \lhd R$), the *quotient homomorphism* is

$$R \xrightarrow{\pi_I} R/I$$
$$r \longmapsto r + I.$$

Definition 1.2. Let R be a ring and fix an ideal $I \lhd R$. A map $\pi\colon R \longrightarrow R'$ is a ***quotient*** of R by I if $\pi(I) = 0$ and for all maps $R' \xrightarrow{\varphi} R''$ with $\varphi(I) = 0$ there exists a unique $\theta\colon R' \longrightarrow R''$ such that the diagram

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & R'' \\ & {\scriptstyle\pi}\searrow \quad \nearrow{\scriptstyle\exists!\theta} & \\ & R' & \end{array}$$

commutes. (This is a *universal property*).

Question: Is the map π_I from above a quotient of R by I?

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & R' \\ & {\scriptstyle\pi_I}\searrow \quad \nearrow & \\ & R/I & \end{array}$$

Yes, by the isomorphism theorem!

1.2 Categorical notions

Definitions 1.3. Let A and B be objects of any category and $A \xrightarrow{\varphi} B$ a map in between those two objects.

i) The map $A \xrightarrow{\varphi} B$ is an ***epimorphism*** if for all maps $B \underset{\beta}{\overset{\alpha}{\rightrightarrows}} C$ we have

$$\alpha \circ \varphi = \beta \circ \varphi$$

implies $\alpha = \beta$.

ii) The map $A \xrightarrow{\varphi} B$ is an ***monomorphism*** if for all maps $Z \underset{\beta}{\overset{\alpha}{\rightrightarrows}} A$ we have

$$\varphi \circ \alpha = \varphi \circ \beta$$

implies $\alpha = \beta$.

Note 1.4. In the category of sets (denoted Set), epimorphism and monomorphisms are just surjective and injective functions.

Note 1.5. In the category of rings, we can have epimorphisms that are not surjections (as set-theoretic functions). Indeed, the natural injective map

$$\mathbb{Z} \overset{\iota}{\hookrightarrow} \mathbb{Q}$$

is clearly injective, but not surjective. However, it is *both* an epimorphism and a monomorphism.

Consider another ring R and two maps $R \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \mathbb{Z}$ such that $\iota \circ \alpha = \iota \circ \beta$.

$$R \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \mathbb{Z} \overset{\iota}{\hookrightarrow} \mathbb{Q}$$
$$n \longmapsto \frac{n}{1}$$

For all $r \in R$, we have $(\iota \circ \alpha)(r) = (\iota \circ \beta)(r)$ and thus $\frac{\alpha(r)}{1} = \frac{\beta(r)}{1}$, which implies $\alpha(r) = \beta(r)$. So $\alpha = \beta$. So ι is a monomorphism.

Consider another ring R' and two maps $\mathbb{Q} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} R'$ such that $\alpha \circ \iota = \beta \circ \iota$.

$$\mathbb{Z} \overset{\iota}{\hookrightarrow} \mathbb{Q} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} R'$$

For all $n \in \mathbb{Z}$, we have $(\alpha \circ \iota)(n) = \alpha\left(\frac{n}{1}\right) = \frac{\alpha(n)}{\alpha(1)} = \frac{\alpha(n)}{1} = \alpha(n)$. Similarly, we have $(\beta \circ \iota)(n) = \beta(n)$, and thus $\alpha = \beta$. So ι is an epimorohism.

Proposition 1.6. *In* Ring, *a homomorphism* $\varphi \colon R \longrightarrow R'$ *is:*

$$\begin{array}{ccc} \textit{monomorphism} & \Longleftrightarrow & \textit{injective ring homomorphism} \\ \textit{epiomorphism} & \Longleftarrow & \textit{surjective ring homomorphism.} \end{array}$$

Proof. Exercise. □

Question: In Ring, is a quotient an epimorphism?

1.3 Localization

Definition 1.7. Let $S \subset R$ a subset of a commutitive ring R. A map $\lambda \colon R \longrightarrow R'$ is a ***localization*** of R at S if

i) $\lambda(S) \subset (R')^*$ (i.e. $\lambda(s)$ is a unit in R' for all $s \in S$)[1].

ii) for all $\varphi \colon R \longrightarrow R''$ with $\varphi(R) \subset (R'')^*$ there exists a unique map $\theta \colon R' \longrightarrow R''$ such that the diagram

[1]For a ring R, the subset $R^* \subset R$ denotes the set of all units (invertible elements).

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & R'' \\ & {\scriptstyle\lambda}\searrow \quad \nearrow{\scriptstyle \exists!\theta} & \\ & R' & \end{array}$$

commutes.

Exercise 1.8. Given a ring R', show that $(R')^*$ is a group.

Proposition 1.9. *In* CRing*, localizations exist.*

Proof. Let $S \subseteq R$ and consider M the monoidal closure of $S \cup R^*$ (i.e. M is closed under multiplication and contains the identnty). Define an equivalence relation of $M \times R$ by

$$(m, a) \sim (n, b) \quad \text{if there exists an } s \in M \text{ such that} \quad s(mb - na) = 0.$$

We need check that this defines an equivalence relation.

reflexivity: $(m, a) \sim (m, a)$? Yes, since $1 \cdot (ma - ma) = 0$.

symmetry: Does $(m, a) \sim (n, b)$ imply $(m, a) \sim (n, b)$. Note, $-1 \in R^* \subset M$, so $-s \in M$ for each $s \in M$ since M is a monoid. Then

$$s(mb - na) = 0 \quad \text{implies}(-s)(mb - na) = 0.$$

transitivity: (left as an exercise)

Define $S^{-1}R = M/\sim$. Then $S^{-1}R$ is a commutative ring with the operations

$$[m, a] + [n, b] = [mn, an + bm]$$
$$[m, a] \cdot [n, b] = [mn, ab].$$

To show that $S^{-1}R$ is a localization, we need the map

$$\lambda_S \colon R \longrightarrow S^{-1}R$$
$$r \longmapsto [1, r].$$

It remains to show that $R \xrightarrow{\lambda_S} S^{-1}R$ enjoys the proposed universal property (will complete in the following lecture).

Note that $S^{-1}R$ has an identity even if R does not! Indeed, if R does not have 1, then instead we may define λ_S as

$$\lambda_S \colon R \longrightarrow S^{-1}R$$
$$r \longmapsto [a, ar]$$

for some nonzero $a \in R$, and $[a, a]$ is the identity in $S^{-1}R$ for any $a \in M$. □

Example 1.10. Consider $R = \mathbb{Z}$, $S = \mathbb{Z}^\times$ and $M = \mathbb{Z}^\times$. Then $S^{-1}R = \mathbb{Q}$.

2 Lecture 2

(14 January 2014)

Proof (*of Proposition 1.9 from last time*). We need to show that $\lambda_S \colon R \longrightarrow S^{-1}R$ enjoys the universal property. Suppose $\varphi\colon R \longrightarrow R'$ is a ring homomorphism such that $\varphi(s) \in (R'')^*$ for all $s \in S$. We need to find a unique $\theta\colon S^{-1}R \longrightarrow R''$ such that

$$\begin{array}{ccc} R & \xrightarrow{\quad\varphi\quad} & R'' \\ & {\scriptstyle\lambda_S}\searrow \quad \nearrow{\scriptstyle\theta} & \\ & S^{-1}R & \end{array}$$

commutes. Note that $\varphi(r) = \theta(\lambda_S(r)) = \theta([1,r])$ and $[m,r] = [m,1][1,r]$ for all $m \in M$ and $r \in R$. So define

$$\theta([m,r]) = \theta([m,1][1,r]) = \varphi(m)^{-1}\varphi(r).$$

□

Proposition 2.1. *In* Ring*, every localization is an epimorphism.*

Proof. Let R be a ring and fix $S \subset R$. Suppose $\lambda\colon R \longrightarrow R'$ is a localization of R at S. Consider maps α, β

$$R \xrightarrow{\lambda} R' \underset{\beta}{\overset{\alpha}{\longrightarrow}} R''$$

such that $\alpha \circ \lambda = \beta \circ \lambda$, and set $\phi = \alpha \circ \lambda = \beta \circ \lambda$. Then there exists a unique $\theta\colon R \longrightarrow R''$ such that $\phi = \theta \circ \lambda$. Thus $\theta = \alpha = \beta$, so λ is an epimorphism. □

Example 2.2. Let R be a CRing and let R and let $I \lhd R$ be a prime ideal[1]. Set $S \equiv R \smallsetminus I$. Note that $R^* \subset S$, since $r \in S$ implies $r \notin I$. Indeed, if I contained any units, then it would not be proper. Now $S = S \cup R^*$, and S is a monoid since for any $a, b \in S$ we have $ab \in S$ (since $a, b \notin S$ implies $ab \notin I$ by primality of I). So

$$S^{-1}R = \{[s,r] \mid s \in S,\, r \in R\} = \{[s,r] \mid s \notin I,\, r \in R\}.$$

Denote $R_I \equiv S^{-1}R$ where $S = R \smallsetminus I$.

Example 2.3. Let $R = \mathbb{Z}$ and $I = \langle p \rangle$ for some prime number p. Then

$$S^{-1}R = \left\{ \frac{r}{s} \,\middle|\, \gcd(s,p) = 1,\, r \in \mathbb{Z} \right\} = \mathbb{Z}_{(p)}.$$

2.1 R-modules

Let M be an abelian group. The category of abelian groups will be denoted Ab[2] Then $\mathrm{End}_{\mathsf{Ab}}(M)$ is a ring.

Exercise 2.4. Prove that $\mathrm{End}_{\mathsf{Ab}}(M)$ is a ring with the properties

$$\begin{aligned} \alpha + \beta\colon M &\longrightarrow M \\ m &\longmapsto \alpha(m) + \beta(m) \end{aligned}$$

[1] Note that a prime ideal must be proper (we should define our prime ideals this way).
[2] Although, according to Clifton, perhaps the notation AGrp would be better to denote the fact that it is a subcategory of Grp.

and

$$\begin{aligned}\beta \circ \alpha \colon M &\longrightarrow M \\ m &\longmapsto \beta(\alpha(m))).\end{aligned}$$

Note 2.5. For a ring R and M an abelian group, consider a ring homomorphism

$$\sigma \colon R \longrightarrow \mathrm{End}_{\mathsf{Ab}}(M)$$

and define

$$\begin{aligned}\rho \colon R \times M &\longrightarrow M \\ (r \times m) &\longmapsto \sigma(r)(m) \ \ (= r \cdot m).\end{aligned}$$

This has the following properties

$$\begin{aligned}r(m+n) &= rm + rn \\ (rs)m &= r(sm) \\ (r+s)m &= rm + sm \\ 1_R m = m.\end{aligned}$$

Then M is an R-***module***. Homomorphisms of R-modules are given by "R-linear maps:"

$$\varphi \colon M \longrightarrow N \qquad \text{such that } \varphi(rm) = r\varphi(m).$$

Example 2.6. Any ring homomorphism $\varphi \colon R \to R'$ can define an interesting R-module. Define $\sigma \colon R \longrightarrow \mathrm{End}_{\mathsf{Ab}}(R')$ by

$$r \longmapsto \sigma(r) \qquad \text{where} \qquad \sigma(r) \colon r' \longmapsto \varphi(r)r'.$$

Then R' is an R-module.

Example 2.7. Every abelian group G is a $\mathbb{Z}$-module (in exactly one way):

$$\begin{aligned}\mathbb{Z} &\longrightarrow \mathrm{End}_{\mathsf{Ab}}(G) \\ n &\longmapsto [n]\end{aligned}$$

where $[n] \colon G \longrightarrow G$ is the map $[n] \colon g \mapsto ng = g + \cdots + g$.

Question: What are monomorphism like in R-Mod? Namely, are "injective" and "monic" the same in R-Mod?

Clearly, injectivity of a homomomorphism implies that it is monic. Suppose that $\varphi \colon M \longrightarrow N$ is a monomorphism and let $I = \{m \in M \mid \varphi(m) = 0_N\}$[3]. Consider the maps

$$I \underset{0}{\overset{\iota}{\rightrightarrows}} M \xrightarrow{\ \varphi\ } N,$$

where ι is the inclusion and 0 is the zero map. Clearly $\varphi \circ \iota = 0$ and $\varphi \circ 0 = 0$. Since φ is monic, we have have $\iota = 0$. Thus $I = \{0\}$.

Definition 2.8. Let M, N be R-modules. If there is a map

$$M \hookrightarrow N$$

then M is a submodule of N. (Categorically speaking, the injective homomorphism itself is the submodule.)

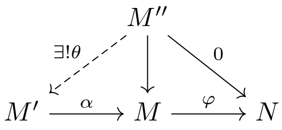
[3]This is the 'kernel' of φ, but we don't want to use this notation yet. Later on, we will see that the kernel of an R-module homomorphism is actually a submodule, and we'll use the notation $\ker \varphi$ for I.

2.2 Kernels and Cokernels

Definition 2.9. Consider a map $\varphi\colon M \longrightarrow N$. Then think of $\ker\varphi$ as a map. A map

$$\alpha\colon M' \longrightarrow M$$

is a *kernel* of $\varphi\colon M \longrightarrow N$ if $\varphi\circ\alpha = 0$ and we have a universal property: for all maps $M'' \xrightarrow{\beta} M$ there exists a unique map $\theta\colon M'' \longrightarrow M'$ such that the diagram



commutes.

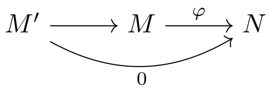
3 Lecture 3

(16 January 2014)

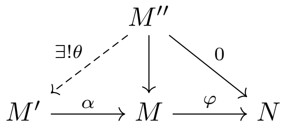
Theorem 3.1. *In R-*Mod*, for a map* $\varphi : M \longrightarrow N$

1. *The following are equivalent:*
   - φ *is a monomorphism*
   - φ *is injective*
   - φ *is a kernel*
2. *The following are equivalent:*
   - φ *is a epimorphism*
   - φ *is suejective*
   - φ *is a cokernel.*

Definition 3.2. (Correct definition) In any category with a zero element and zero maps, a map $M' \longrightarrow M$ is a ***kernel*** for $M \xrightarrow{\varphi} N$ if

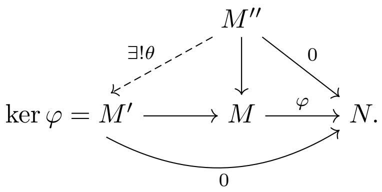


(i.e. the composition of the two maps is the zero map) and we have the universal property: for all maps $M'' \longrightarrow M$ there exists a unique map $\theta : M'' \longrightarrow M'$ such that the diagram



commutes.

Example 3.3. i) In Ab, $\ker\varphi$ is an abelian group, and we have

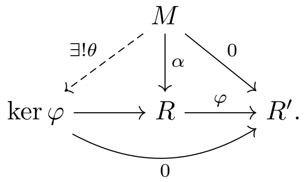


ii) This picture does **not** happen in Ring, since identity in M'' must be mapped to the identity in M. So "$\ker\varphi$" may not be in the category of rings.

Note 3.4. If R is commutative, consider R as an R-module. For any ring homomorphism $R \xrightarrow{\varphi} R'$, where R' has an R-module structure, $\ker\varphi = \{r \in R \mid \varphi(r) = 0\}$. Then $\ker\varphi$ is in R-Mod!

Question: Is $\iota : \ker\varphi \longrightarrow R$ R-linear? Yes, since $\iota(rr') = r\iota(r')$.

Question: Is $\ker\varphi \hookrightarrow M$ a kernel in the categorical sense? Yes. Consider the diagram



Since $\varphi(\alpha(m)) = 0$ and thus $\alpha(m) \in \ker\varphi$ for each $m \in M$. So define $\theta = \alpha$. Then R-linearity of θ comes from the R-linearity of α.

Note 3.5. If $N' \hookrightarrow N$ is a submodule, then N/N' is an R-module with

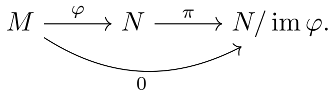
$$N' \hookrightarrow N \longrightarrow N/N',$$

where the morphism is given by

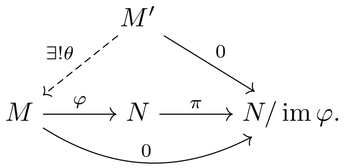
$$\begin{aligned} n &\longmapsto n + N' \\ rn &\longmapsto rn + N' = r(n + N') \end{aligned}$$

where $rN' \subset N'$ since N' is a submodule of N, so it is closed under the R-action.

Proof of Theorem 3.1 part 1. Let φ be a monomorphism

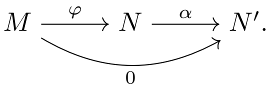


As abelian groups [1] (if we forget the R-action), we have the diagram from the universal property for abelian groups



Since θ exists for the abelian groups, it s unique. But it also preserves R-linearity, so it is the unique R-module homomorphism we are looking for.

Now suppose φ is a categorical kernel of $\alpha : N \longrightarrow N'$, i.e.



Since φ is a kernel in R-Mod, it is also a kernel in Ab. So $M \simeq \{n \in N \mid \alpha(n) = 0\}$. ☐

Exercise 3.6. Check that this isomorphism of abelian groups in the proof above

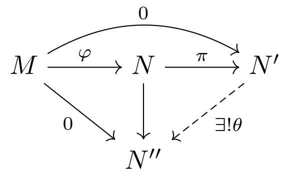
$$M \simeq \{n \in N \mid \alpha(n) = 0\}$$

is R-linear.

Exercise 3.7. In Ab, show that a map is an epimorphism if and only if it is a surjection.

[1] Clifton, on the similarities between Ab and R-Mod: "This has all happened before, this will all happened again."

Definition 3.8. In any category, a map $N \xrightarrow{\pi} N'$ is a ***cokernel*** of a map $M \xrightarrow{\varphi} N$ if $\pi \circ \varphi = 0$ and given a map $\alpha : N \longrightarrow N''$ such that $\alpha \circ \varphi = 0$ there exists a unique map $\theta : N' \longrightarrow N''$ such that $\alpha = \theta \circ \pi$. Namely, we have the universal property such that the following diagram commutes:



Example 3.9. In Ab, $\operatorname{coker} \varphi = N / \operatorname{im} \phi$,

$$M \xrightarrow{\varphi} N \longrightarrow N / \operatorname{im} \varphi.$$

Example 3.10. In Grp, $N / \operatorname{im} \phi$ is not necessarily a group! So take $\overline{\operatorname{im} \varphi}$ as the "normal closure" in N, i.e. the smallest normal subgroup of N that contains $\operatorname{im} \varphi$. Then $\operatorname{coker} \varphi = N / \overline{\operatorname{im} \phi}$. (In Ab, every subgroup is normal.)

Question: What are cokernels in R-Mod? They are the same as in Ab! We just need to confirm that the maps are R-linear.

Exercise 3.11. What are kernels and cokernels in Ring?

Next week we will discuss *exact sequences*.

4 Lecture 4

(20 January 2014)

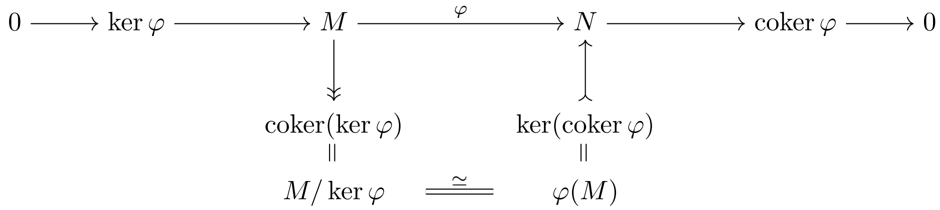
If we start with a morphism of R-modules, we can define a kernel and a cokernel.

$$0 \longrightarrow \ker\varphi \longrightarrow M \xrightarrow{\varphi} N \longrightarrow \operatorname{coker}\varphi \longrightarrow 0,$$

where $\operatorname{coker}\varphi$ is fancy language for $N/\varphi(M)$. This sequence is *exact*. That is, the composition of any two maps in the sequence is the zero map.

One of our big results from last time is that $\ker\varphi$ is a submodule of M. (Though we must be careful with language, since we also consider $\ker\varphi$ and $\operatorname{coker}\varphi$ as maps.)

What is the cokernel of the kernel of φ? From the definition of the cokernel, $\operatorname{coker}(\ker\varphi) = M/\ker\varphi$. Similarly, what is the kernel of the cokernel? It is the set that is 'annihilated' by φ, and thus $\ker(\operatorname{coker}\varphi) = \varphi(M)$.



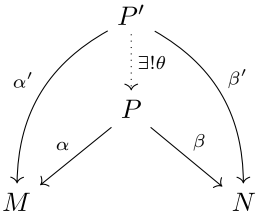
4.1 Products and Coproducts

We have looked at the categorical kernels and cokernels in the category of R-modules. There are a few other basic categorical things to examine before moving on (things we should familiarize ourselves with to get to know R-Mod).

Definition 4.1. In any category, a ***product*** of two objects M and N in that category is another object P if

i) it is equipped with maps $\alpha\colon P \to M$ and $\beta\colon P \to N$

ii) for any other object P' and maps $\alpha'\colon P' \to M$ and $\beta'\colon P' \to N$ there exists a unique map $\theta\colon P' \to P$ such that the following diagram commutes



.

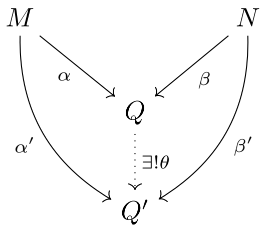
Notation. If such a thing exists for two objects M and N, it is commonly denoted $P = M \prod N$ with maps $\alpha = \Pi_M$ and $\beta = \Pi_N$.

Question: does R-Mod admit products? Let's try the cartesian product $P = M \times N$ with the standard projection maps, $\Pi_M(m,n) = m$ and $\Pi_N(m,n) = n$. As a product of abelian groups this is again an abelian group, and this has the structure of an R-module by $r(m,n) = (rm, rn)$. We need to see if $M \times N$ equipped with the projection maps satisfies the universal property. Given any R-module P' and maps $\alpha'\colon P' \to M$ and $\beta'\colon P' \to N$, define the map $\theta\colon P' \to P$ by

$$\theta(p') = (\alpha'(p'), \beta'(p')).$$

Definition 4.2. In any category, a ***coproduct*** of two objects M and N in that category is another object Q if

1. it is equipped with maps $\alpha\colon M \to Q$ and $\beta\colon N \to Q$
2. for any other object Q' and maps $\alpha'\colon M \to Q'$ and $\beta'\colon N \to Q'$ there exists a unique map $\theta\colon Q \to Q'$ such that the following diagram commutes



.

Notation. If such a thing exists for two objects M and N, it is commonly denoted $Q = M \coprod N$ with maps $\alpha = \amalg_M$ and $\beta = \amalg_N$.

Question: does R-Mod admit coproducts? Let's try the same $Q = M \times N$ with the standard inclusion maps

$$\iota_M\colon M \to M \times N \qquad \iota_N\colon N \to M \times N$$

given by $\iota_M(m) = (m, 0)$ and $\iota_N(m) = (0, N)$. We need to see if this satisfies the universal property. Given any R-module Q' and maps $\alpha'\colon M \to Q'$ and $\beta'\colon N \to Q'$, any map $\theta\colon Q \to Q'$ must satisfy

$$\theta(m, n) = \theta((m, 0) + (0, n)) = \theta(m, 0) + \theta(0, n) = \theta \circ \iota_M(m) + \theta \circ \iota_N(n) = \alpha'(m) + \beta'(n).$$

So this map indeed exists and is unique.

Definition 4.3. The ***direct sum*** of two R-modules M and N is the cartesian product $M \times N$ equipped with the maps of $M \times N$ as a product and coproduct. This is denoted by $M \oplus N$.

$$M \underset{\iota_M}{\overset{\pi_M}{\leftrightarrows}} M \oplus N \underset{\iota_N}{\overset{\pi_N}{\rightleftarrows}} N \quad .$$

Some people say that the products and coproducts 'coincide' in the category of R-modules. This is not exactly correct, since the product and coproduct also require the definition of the maps.

4.2 Free Modules

Definition 4.4. Fix a (not necessarily commutative) ring R. The free R-module generated by a set S is

$$\mathrm{Free}_{R\text{-}\mathsf{Mod}}(S) \colon = \left\{ \alpha \in \mathrm{Hom}_{\mathsf{Set}}(S, R) \,\middle|\, \alpha(s) = 0 \ p.p. \right\}$$

where *p.p.* is french (presque partout) for a.e. (almost everywhere), meaning "for all but finitely many elements in S".

There is really a hidden "forgetful functor" hidden in the above definition. We should write $\mathrm{Hom}_{\mathsf{Set}}(S, \mathrm{Forget}(R))$.

To see that $\mathrm{Free}_{R\text{-}\mathsf{Mod}}(S)$ really is an R-module, first note that $\mathrm{Hom}_{\mathsf{Set}}(S, \mathrm{Forget}(R))$ is an R-module. Consider $\alpha, \beta \in \mathrm{Hom}_{\mathsf{Set}}(S, \mathrm{Forget}(R))$. Then $(\alpha + \beta)(s) = \alpha(s) + \beta(s)$ and $(r\alpha)(s) = r\alpha(s)$, with multiplication and addition in R.

Proposition 4.5. *For any R-module M, there is a canonical bijection of sets*

$$\mathrm{Hom}_{\mathsf{Set}}(S, \mathrm{Forget}(M)) \simeq \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(\mathrm{Free}_{R\text{-}\mathsf{Mod}}(S), M).$$

The forgetful functor Forget from R-Mod to Set is dual to the functor $\text{Free}_{R\text{-Mod}}$. For any category for which you can forget structure and produce sets, if there is a left-adjoint then it is this Free functor that we discuss here. (If someone asks you at a bar "Give me an interesting example of adjoint functors," then Forget and Free will do the trick.) One of the main points of this course will be the hom-tensor duality, which can be examined in terms of an adjuction of this form.

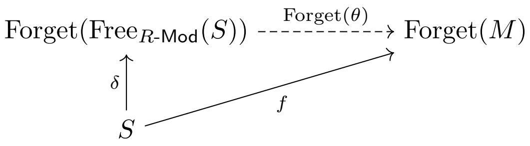
Proposition 4.6. $\text{Free}_{R\text{-Mod}}(S)$ *comes with a map*

$$\begin{aligned} \delta\colon S &\to \text{Forget}(\text{Free}_{R\text{-Mod}}(S)) \\ s &\mapsto \delta_s \end{aligned}$$

where $\delta_s\colon S \to R$ *is the 'delta' function*

$$\delta_s(x) = \begin{cases} 1, & x = s \\ 0, & x \neq s, \end{cases}$$

such that for all functions $f\colon S \to \text{Forget}(M)$ *there exists a unique map* θ *from* $\text{Free}_{R\text{-Mod}}(S)$ *to* M *such that the diagram*



commutes.

Proof. Take any $\alpha \in \text{Free}_{R\text{-Mod}}(S)$, then the support of α is finite by definition. Then we can write α as a finite sum

$$\alpha = \sum_{s\in S} \alpha(s)\delta_s = \sum_{s\in \text{supp}(\alpha)\subset S} \alpha(s)\delta_s.$$

The the map θ must preserve sums and R-action

$$\theta(\alpha) = \theta\left(\sum_{s\in S} \alpha(s)\delta_s\right) = \sum_{s\in S} \theta\left(\alpha(s)\delta_s\right) = \sum_{s\in S} \alpha(s)\theta(\delta_s).$$

□

Proof of Proposition 4.5. Given $\theta\colon \text{Free}_{R\text{-Mod}}(S) \to M$, define $f_\theta\colon S \to \text{Forget}(M)$ by $s \mapsto \theta(\delta_s)$. Given $f\colon S \to \text{Forget}(M)$, define θ_f..... This defines a bijection.

The "canonical" part requires a discussion of functors, which we will delay until later. □

4.3 Submodules generated by a set

Definition 4.7. Given an R-module M and some set $S \subset M$, the R-module generated by S is

$$\langle S \rangle = \left\{ \sum_{s\in S} r_s \cdot s \,\middle|\, \text{finitely many } r_s \text{ are nonzero} \right\}.$$

Contrast this with the definition of $\text{Free}_{R\text{-Mod}}(S)$. The definitions are certainly similar, but can we think of a set S such that $\langle S \rangle$ is not the same as $\text{Free}_{R\text{-Mod}}(S)$? Think about this for next lecture (e.g. $R = \mathbb{Z}$, $M = \mathbb{Z}/2\mathbb{Z}$ and $S = \{1\}$).

5 Lecture 5

(22 January 2014)

There is a relationship between 'forgetting' the structure of an R-module and thinking of of as a set and constructing a 'free' module from that set.

$$R\text{-}\mathsf{Mod} \underset{\mathrm{Free}_{R\text{-}\mathsf{Mod}}}{\overset{\mathrm{Forget}}{\rightleftarrows}} \mathsf{Set}$$

Proposition 5.1 (from last time)**.** *There is a canonical bijection of sets*

$$\mathrm{Hom}_{\mathsf{Set}}(S, \mathrm{Forget}(M)) \simeq \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(\mathrm{Free}_{R\text{-}\mathsf{Mod}}(S), \mathrm{Forget}(M)).$$

Here 'canonical' really means 'functorial'.

(There was a long aside here from Clifton about the definition of categories and functors, including a definition of the forgetful functor.[1])

Note: the forgetful functor is covariant while $\mathrm{Free}_{R\text{-}\mathsf{Mod}}$ is contravariant. We have the correspondence[2]

$$\begin{aligned} \mathrm{Free}_{R\text{-}\mathsf{Mod}}: &\quad \mathsf{Set} &&\longrightarrow R\text{-}\mathsf{Mod} \\ \text{objects:} &\quad S &&\longmapsto \{\alpha \in \mathrm{Hom}_{\mathsf{Set}}(S,R) \mid |\mathrm{supp}(\alpha)| < \infty\} \\ \text{maps:} &\quad S \mapsto S' &&\mapsto (\mathrm{Hom}_{\mathsf{Set}}(S,R) \leftarrow \mathrm{Hom}_{\mathsf{Set}}(S',R)) \end{aligned}$$

where a map $f : S \to S'$ becomes a map $\mathrm{Free}_{R\text{-}\mathsf{Mod}}(f) : \mathrm{Hom}_{\mathsf{Set}}(S', R) \to \mathrm{Hom}_{\mathsf{Set}}(S, R)$ by composition, i.e. for a map $\varphi \in \mathrm{Hom}_{\mathsf{Set}}(S', R)$ we have $\mathrm{Free}_{R\text{-}\mathsf{Mod}}(f)(\varphi) = \varphi \circ f$

$$\mathrm{Free}_{R\text{-}\mathsf{Mod}}(f)(\varphi) : S \xrightarrow{f} S' \xrightarrow{\varphi} R.$$

5.1 R-algebras

Before moving on, we should consider some things about the closely related category of R-algebras. Recall that the endomorphisms of an abelian group form a ring $\mathrm{End}_{\mathsf{Ab}}(M)$.

To have an R-algebra, we have a ring A equipped with an action of R that satisfies

$$(r_1 \cdot a_1)(r_2 \cdot a_2) = (r_1 r_2) \cdot (a_1 a_2), \tag{5.1}$$

where $\cdot$ is the action of an element in R acting on elements in A.

Definition 5.2. (Let R be a commutative ring.) A ring A together with an action of R on a A is an R-***algebra*** if A as an abelian group is an R-module and for all $r_1, r_2 \in R$ and $a_1, a_2 \in A$ eq. (5.1) is satisfied.

Let R be commutative and consider an R-algebra A with a map given by

$$\begin{aligned} R &\longrightarrow A \\ r &\longmapsto r1_A. \end{aligned}$$

This is a ring homomorphism, since

$$\begin{aligned} r_1 + r_2 &\longmapsto (r_1 + r_2)1_A = r_1 1_A + r_2 1_A \\ r_1 r_2 &\longmapsto (r_1 r_2)1_A = r_1 1_A \, r_2 1_A. \end{aligned}$$

[1] "The wisdom of the ages is in my unhallowed hands and I dare not touch it" - Clifton (regarding the use of 1 vs id as the name of the identity arrow in categories), while trying to quote Charles Dickens. The actual quote he was going for is "But the wisdom of our ancestors is in the simile; and my unhallowed hands shall not disturb it, or the Country's done for." - Charles Dickens, *A Christam Carol.*

[2] "When I eat I don't like having my food touching other food on my plate and I feel the same way about mathematics" - Clifton, on using common analysis notation in algebra. But he does it here anyway...?

Claim 1. The commutativity of R implies that the image of R under this map lies in the centre of A.

Exercise 5.3. Show that every ring homomorphism from a commutative ring R to a (not necessarily commutative) ring A, $\varphi : R \to A$

$$\varphi(R) \subset Z(A) \text{ centre of } A$$

and this determines an R-algebra structure on A.

Question: Does the forgetful functor in the category R-Alg have an adjoint of the form

$$\begin{aligned} \mathrm{Free}_{R\text{-}\mathsf{Alg}} : \mathsf{Set} &\longrightarrow R\text{-}\mathsf{Alg} \\ S &\longmapsto \quad ? \end{aligned}$$

Suppose S is finite, $S = \{s_1, \ldots, s_n\}$. Then $\mathrm{Free}_{\mathsf{Com}\text{-}R\text{-}\mathsf{Alg}}(S) = R[x_1, \ldots, x_n]$, where Com-R-Alg is the category of commutative R-algebras (this is a definition, and thus only defined for a finite set).

Claim 2. The functor $\mathrm{Free}_{\mathsf{Com}\text{-}R\text{-}\mathsf{Alg}}$ is a left adjoint to Forget : Com-R-$\mathsf{Alg} \to S$

Definition 5.4. An R-module M is ***finitely generated*** if there is a surjective[3] R-module map

$$\mathrm{Free}_{R\text{-}\mathsf{Mod}}(S) \longrightarrow M$$

for some finite set S.

Definition 5.5. A commutative R-algebra A is ***finitely generated*** if there is a surjective R-algebra map

$$\mathrm{Free}_{\mathsf{Com}\text{-}R\text{-}\mathsf{Alg}}(S) \longrightarrow A$$

for some finite set S.

Question: If A is finitely generated as an R-algebra, is it finitely generated as an R-module? No! Consider $A = R[x]$, which is finitely generated as an R-algebra. The necessary surjection is the natural map $R \longrightarrow R[x]$. But this is *not* finitely generated as an R-module. In fact, its basis is $\{1, x, x^2, \ldots\}$.

Definition 5.6. A commutative R-algebra is said to be ***finite*** if it is finitely generated as an R-module. It has ***finite type*** if it is finitely generated as an R-algebra

Note 5.7. A ring R (with identity) as an R-module. The generating set consists of the identity. Furthermore, $R^{\oplus n}$ is finitely generated from a set with n elements. To determine if an R-algebra is *finite* vs of *of finite type*, the necessary surjections one needs to determine are

$$R^{\oplus n} \twoheadrightarrow A \qquad\qquad R[x_1, \cdots, x_n] \twoheadrightarrow A$$

Note 5.8. For next time, make sure to read about Noetherian R-modules.

[3]Clifton originally wrote 'epimorphism' here, but crossed it out and put 'surjective'

6 Lecture 6

(28 January 2014)

(Zach started the lecture with a review of the first two weeks of the course.)

6.1 Homological algebra

Definition 6.1. A ***sequence*** R-modules $M_\bullet$ is a countable collection of R-modules

$$\cdots \xrightarrow{d_{-2}} M_{-2} \xrightarrow{d_{-1}} M_{-1} \xrightarrow{d_0} M_0 \xrightarrow{d_1} M_1 \xrightarrow{d_2} M_2 \longrightarrow \cdots,$$

such that the composition of any consecutive two maps is zero.

Clearly, for each map $\operatorname{im} d_i \subseteq \ker d_{i+1}$ for each i.

Example 6.2. For any homomorphism of R-modules φ, there is the sequence

$$0 \longrightarrow \ker\varphi \longrightarrow M \xrightarrow{\varphi} N \longrightarrow \operatorname{coker}\varphi \longrightarrow 0.$$

Definition 6.3. For each sequence $M_\bullet$ of R-modules we can attach a sequence with R-modules $H^i(M_\bullet) = \ker d_i / \operatorname{im} d_{i+1}$ with arrows in the opposite direction:

$$\cdots \longrightarrow M_0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow \cdots$$

$$\begin{array}{ccccccccc}
\cdots \longleftarrow & \ker d_0/\operatorname{im} d_1 & \longleftarrow & \ker d_1/\operatorname{im} d_2 & \longleftarrow & \ker d_1/\operatorname{im} d_2 & \longleftarrow \cdots \\
 & \| & & \| & & \| & \\
\cdots & H^0(M_\bullet) & & H^1(M_\bullet) & & H^2(M_\bullet) & \cdots
\end{array}$$

The sequence $H^*(M_\bullet)$ is call the ***cohomology*** of $M_\bullet$.

A sequence of R-modules is ***exact at*** M_i if $\operatorname{im} d_i = \ker d_{i+1}$. That is, the kernel of one map is exactly the image of the previous map, and so $H^i(M_\bullet) \simeq 0$. If $M_\bullet$ is exact everywhere, then the sequence is ***exact***. Thus, a sequence is exact if

$$\ker d_{i+1}/\operatorname{im} d_i \simeq 0 \quad \text{for all } i.$$

Example 6.4. Some properties of sequences of R-modules

1. Given an exact sequence[1]

$$0 \longrightarrow M \xrightarrow{\varphi} N \longrightarrow 0,$$

   the map φ is an isomorphism, since $\ker\varphi = 0$ and $\operatorname{im}\varphi = N$.

2. Given an exact sequence

$$0 \longrightarrow M_1 \xrightarrow{d_1} M_2 \xrightarrow{d_2} M_3 \longrightarrow 0,$$

   d_1 is injective and d_2 is surjective. So M_1 is a submodule of M_2 and M_3 is a quotient. So this sequence is isomorphic to

$$0 \longrightarrow \ker\varphi \hookrightarrow M \xrightarrow{\varphi} M/\ker\varphi \longrightarrow 0.$$

   This is a *short exact sequence.*

[1] In an abelian category, any arrow that is epic and monic is an isomorphism.

7 Lecture 7

(30 January 2014)

Correction to notation from last time

Last lecture we were getting confused with our notation and confusing cohomologies with homologies, mixing notations from different chapters in Aluffi's book. Now we go back to the correct notation for this section of the book.

Definition 7.1. A ***complex***[1] of R-modules is a collection of R-modules denoted $M_\bullet$ with maps

$$\cdots \longrightarrow M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M_{-1} \xrightarrow{d_{-1}} M_{-2} \longrightarrow \cdots .$$

The ***homology*** $H_*(M_\bullet)$ of the complex $M_\bullet$ is the collection of R-modules

$$H_i(M_\bullet) \equiv \ker d_i / \operatorname{im} d_{i+1}.$$

7.1 Category of complexes

Definition 7.2. The *category of complexes of R-moduels* has objects as complexes and a map of complexes $M_\bullet \xrightarrow{\alpha} N_\bullet$ is a collection of maps $\alpha_i : M_i \to N_i$ such that each square of the diagram

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
M_1 & \xrightarrow{\alpha_1} & N_1 \\
\downarrow{\scriptstyle d_1^M} & & \downarrow{\scriptstyle d_1^N} \\
M_0 & \xrightarrow{\alpha_0} & N_0 \\
\downarrow{\scriptstyle d_0^M} & & \downarrow{\scriptstyle d_0^N} \\
M_{-1} & \xrightarrow{\alpha_{-1}} & N_{-1} \\
\downarrow{\scriptstyle d_{-1}^M} & & \downarrow{\scriptstyle d_{-1}^N} \\
\vdots & & \vdots
\end{array}$$

commutes.

Definition 7.3. A ***short exact sequence of complexes of R-modules*** is a sequence of maps of complexes

[1] Usually we will use the word *sequence* to mean pretty much the same thing as a complex.

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_1 & \xrightarrow{\beta_1} & M_1 & \xrightarrow{\alpha_1} & N_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_1^L} & & \downarrow{\scriptstyle d_1^M} & & \downarrow{\scriptstyle d_1^N} & & \\
0 & \longrightarrow & L_0 & \xrightarrow{\beta_0} & M_0 & \xrightarrow{\alpha_0} & N_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_0^L} & & \downarrow{\scriptstyle d_0^M} & & \downarrow{\scriptstyle d_0^N} & & \\
0 & \longrightarrow & L_{-1} & \xrightarrow{\beta_{-1}} & M_{-1} & \xrightarrow{\alpha_{-1}} & N_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_{-1}^L} & & \downarrow{\scriptstyle d_{-1}^M} & & \downarrow{\scriptstyle d_{-1}^N} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

such that each line in the diagram is an exact sequence of R-modules.

Theorem 7.4. *If* $0 \longrightarrow L_\bullet \xrightarrow{\beta} M_\bullet \xrightarrow{\alpha} N_\bullet \longrightarrow 0$ *is a (short) exact sequence of complexes, then this induces a (long) exact sequence of* R*-modules in homology*

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & H_2(L_\bullet) & \longrightarrow & H_2(M_\bullet) & \longrightarrow & H_2(N_\bullet) & \\
 & & & & & \downarrow & \\
 & H_1(N_\bullet) & \longleftarrow & H_1(M_\bullet) & \longleftarrow & H_1(L_\bullet) & \\
 & \downarrow & & & & & \\
 & H_0(L_\bullet) & \longrightarrow & H_0(M_\bullet) & \longrightarrow & H_0(N_\bullet) & \longrightarrow 0.
\end{array}
$$

The maps connecting the homologies $H_i(N_\bullet) \longrightarrow H_{i-1}(L_\bullet)$ *is called the* connecting homomorphism *at* i*.*

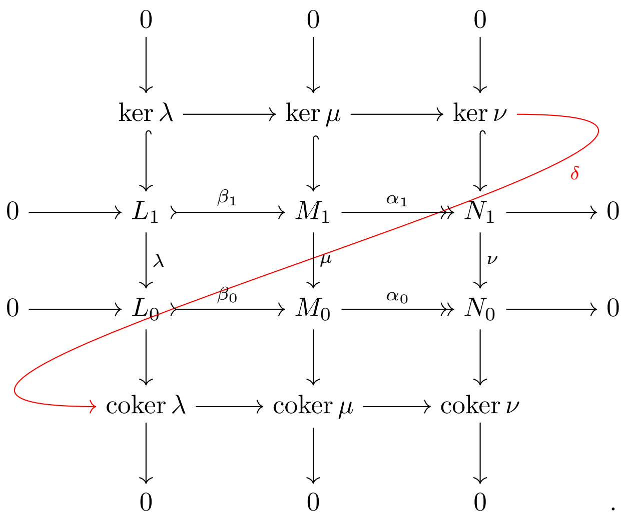
Example 7.5. Special case:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_1 & \xrightarrow{\beta_1} & M_1 & \xrightarrow{\alpha_1} & N_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \nu} & & \\
0 & \longrightarrow & L_0 & \xrightarrow{\beta_0} & M_0 & \xrightarrow{\alpha_0} & N_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

In this case, we have $H_0(N_\bullet) = N_0/\operatorname{im}\nu = \operatorname{coker}\nu$ and $H_1(N_\bullet) = \ker\nu/0 = 0\ker\nu$, and analogously for λ and μ. So the long exact sequence takes the form

$$\begin{array}{ccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ &&&&&& \downarrow \\ && \ker\nu & \xleftarrow{\ !\ } & \ker\mu & \xleftarrow{\ !\ } & \ker\lambda \\ && {\scriptstyle \text{connecting homomorphism}}\downarrow &&&& \\ && \operatorname{coker}\lambda & \xrightarrow{\ !\ } & \operatorname{coker}\mu & \xrightarrow{\ !\ } & \operatorname{coker}\nu \longrightarrow 0. \end{array}$$

This fact goes by the nice name of the *snake lemma*. In particular, we have the diagram



Proof of the Snake Lemma. We want to show that there is a well-defined homomorphism $\ker\nu \longrightarrow \operatorname{coker}\lambda$. Start with an element $n \in \ker\nu$, which is just an element in N_1. Since α_1 is surjective, there is an element in $m \in M_1$ such that $\alpha_1(m) = n$. Consider $\mu(m) \in M_0$. By the commutivity of the diagram,

$$\alpha_0(\mu(m)) = \nu(\alpha_1(m)),$$

and thus $\nu(\alpha_1(m)) = 0$ since $\alpha_1(m) = n \in \ker\nu$. So $\alpha_0(\mu(m)) = 0$ and thus $\mu(m) \in \ker\alpha_0$. By exactness of the rows, $\ker\alpha_0 = \operatorname{im}\beta_0$ and thus $\mu(m) \in \operatorname{im}\beta_0$. Since β_0 is injective, there is a unique $\ell \in L_0$ such that $\beta_0(\ell) = \mu(m)$. Finally, we have a map $L_0 \longrightarrow \operatorname{coker}\lambda$, so we can define a map $\delta : \ker\nu \longrightarrow \operatorname{coker}\lambda$ by $n \mapsto \delta(n) \in \operatorname{coker}\lambda$.

Note: for the choice of m, we have found a way to produce a unique element in $\operatorname{coker}\lambda$. So we need to show that this element in $\operatorname{coker}\lambda$ does not depend on our choice of $m \in M$.

Claim 3. This rule for $n \mapsto \delta(n)$ is well-defined.

Proof of claim. Consider $m, m' \longmapsto n$. Then $m - m' \in \ker\alpha_1 = \operatorname{im}\beta_1 \simeq L_1$ and $\mu(m - m') = \mu(m) - \mu(m')$. So we have $\ell, \ell' \in L_0$ such that $\beta_0(\ell) = \mu(m)$ and $\beta_0(\ell') = \mu(m')$. But then $\ell, \ell' \in \operatorname{im}\lambda \subset L_0$ and $(\ell - \ell') \mapsto 0 \in \operatorname{coker}\lambda$. Thus ℓ and ℓ' get mapped to the same element in $\operatorname{coker}\lambda$. □

This concludes the proof of the snake lemma. □

Definition 7.6. A short exact sequence of R-modules

$$0 \longrightarrow L \xrightarrow{\ \alpha\ } M \underset{\beta}{\overset{\gamma}{\rightleftarrows}} N \longrightarrow 0$$

is said to be ***split*** if there exists a map $N \xrightarrow{\gamma} M$ such that $\beta \circ \gamma = \mathrm{id}_N$. The map γ is called a ***section*** of β.

(Geoff then gave a long monologue about diagram chasing to state and prove the five lemma by proving the two four lemmas.)

8 Lecture 8

(4 February 2014)

(Geoff continued his long monologue to finish the proof of the four and five lemmas.)

Corollary 8.1.

Theorem 8.2. *Given a short exact sequence of complexes of R-modules,*

$$0 \longrightarrow L_\bullet \xrightarrow{\alpha} M_\bullet \xrightarrow{\beta} N_\bullet \longrightarrow 0$$

there is a (long) exact sequence of homologies

$$\cdots \longrightarrow H_1(L_\bullet) \longrightarrow H_1(M_\bullet) \longrightarrow H_1(N_\bullet) \longrightarrow H_0(L_\bullet) \longrightarrow \cdots .$$

Lets consider how this relates to the short five lemma. We considered the specific case

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_0 & \xrightarrow{\beta_0} & M_0 & \xrightarrow{\alpha_0} & N_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \nu} & & \\
0 & \longrightarrow & L_1 & \xrightarrow{\beta_1} & M_1 & \xrightarrow{\alpha_1} & N_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

and extended this to

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \ker\lambda & \longrightarrow & \ker\mu & \longrightarrow & \ker\nu & \xrightarrow{\delta} & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_0 & \xrightarrow{\beta_0} & M_0 & \xrightarrow{\alpha_0} & N_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \nu} & & \\
0 & \longrightarrow & L_1 & \xrightarrow{\beta_1} & M_1 & \xrightarrow{\alpha_1} & N_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & \xrightarrow{\delta} & \operatorname{coker}\lambda & \longrightarrow & \operatorname{coker}\mu & \longrightarrow & \operatorname{coker}\nu & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

If λ and ν are isomorphisms, then $\ker\lambda = \ker\nu = 0$ and $\operatorname{coker}\lambda = \operatorname{coker}\nu = 0$. So we have the exact sequence

$$\underbrace{\ker\lambda}_{=0} \longrightarrow \ker\mu \longrightarrow \underbrace{\ker\nu}_{=0} \xrightarrow{\delta} \underbrace{\operatorname{coker}\lambda}_{=0} \longrightarrow \operatorname{coker}\mu \longrightarrow \underbrace{\operatorname{coker}\nu}_{=0}.$$

And thus $\operatorname{coker}\mu = \ker\mu = 0$. So μ is an isomorphism.

8.1 Two notions of split

Definition 8.3 (Aluffi's definition)**.** A short exact sequence of R-modules

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

is (***Aluffi***) ***split*** if there exists an isomorphism $M \xrightarrow{\simeq} L \oplus N$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \| & & \downarrow{\simeq} & & \| & & \\ 0 & \longrightarrow & L & \longrightarrow & L \oplus N & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

commutes.

Definition 8.4 (Other definition)**.** A short exact sequence of R-modules

$$0 \longrightarrow L \xrightarrow{\alpha} M \underset{\beta}{\overset{\gamma}{\rightleftarrows}} N \longrightarrow 0$$

is said to be ***split*** if there exists a map $N \xrightarrow{\gamma} M$ such that $\beta \circ \gamma = \mathrm{id}_N$. The map γ is called a ***section*** of β.

Question: Is *split* the same as *Aluffi split*?

Example 8.5. Consider the following sequence:

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

It is exact, but $\mathbb{Z}/4\mathbb{Z}$ is not isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. So it is not Aluffi split. But is it split in the other sense?

8.2 Linear algebra with R-modules

Definition 8.6. Let R be a ring, M an R-module and $S \subseteq M$ a subset. Then S is a ***linearly independent*** subset of M if

$$\sum_{i \in I} r_i s_i = 0 \qquad \text{implies} \qquad r_i = 0 \text{ for all } i \in I.$$

Here, I is an indexing set with $i \mapsto s_i \in M$, and the sums above must be finite (i.e. the sums must be in $\langle S \rangle$).

Definition 8.7. Let R be a ring, M an R-module and $S \subseteq M$ a subset. The ***span*** of S denoted $\operatorname{span} S$ is $\langle S \rangle$, and S is said to ***generate*** M if $\langle S \rangle = M$.

Definition 8.8. Let R be a ring, M an R-module and $S \subseteq M$ a subset. Then S is a ***basis*** for M if S generates M and is lineary independent in M.

Proposition 8.9. *If M admits a basis B, then M is free and $M \simeq \mathrm{Free}_{R\text{-}\mathsf{Mod}}(B)$.*

Example 8.10. If $M = R$ as an R-module, then $R = \mathrm{Free}_{R\text{-}\mathsf{Mod}}(\{1\})$.

Let $R = \mathbb{Z}/6\mathbb{Z}$, and let M be the ideal $M = \{0, 2, 4\}$. Is this free? It has a generator, since $S = \{2\}$ generates M, but S is not linearly independent! Indeed, $3 \cdot 2 = 0$ in $\mathbb{Z}/6\mathbb{Z}$ so S is not linearly independent (even though it has only one element).

Definition 8.11 (IBN rings - Invariant Basis Number)**.** A ring R is ***IBN*** if for any free R-module M, all bases of M have the same cardinality.

Definition 8.12. If R is an IBN ring and M is a free R-module, then the ***rank*** of M, denoted by $\operatorname{rank}_R(M)$, is the cardinality of any basis for M.

Example 8.13. Let R be an integral domain and let K be the field of fractions of R. Let M be a free R-module with a basis B such that $M = \text{Free}_{R\text{-}\mathsf{Mod}}(B)$. Then S is a linearly independent subset of $\text{Free}_{R\text{-}\mathsf{Mod}}(B)$ if and only if S is a linearly independent subset of $V \equiv \text{Free}_K(B)$.

We have a map of R-modules

$$M = \text{Free}_{R\text{-}\mathsf{Mod}}(B) \hookrightarrow \text{Free}_K(B) = V$$

induced by the localization homomorphism $R \hookrightarrow K$ given by $r \longmapsto \frac{r}{1}$. This is an example of localization of modules.

Proof. Suppose S is linearly independent in M. Then $\sum_{i\in I} r_i s_i = 0$ for finitely many $r_i \neq 0$ implies $r_i = 0$ for all i. Suppose that $\sum_{i\in I} rk_i s_i = 0$ for finitely many $k_i \in K$. Then we can write the k_i's with a common denominator d such that $k_i = \frac{c_i}{d}$. Then

$$\sum_{i\in I} k_i s_i = \sum_{i\in I} \frac{c_i}{d} s_i = \frac{1}{d}\sum_{i\in I} \frac{c_i}{1} s_i = 0.$$

Multiplying both sides by d gives $\sum_{i\in I} \frac{c_i}{1} s_i = 0$. Since the map $\text{Free}_R(B) \longrightarrow \text{Free}_K(B)$ is injective, we have $\sum_{i\in I} c_i s_i \overset{!}{=} 0$. (We still need to show that this map is indeed injective. Why is this true?)

The proof will be finished next time. □

Exercise 8.14. For next time, look at exercise 4.8 in chapter 5 so we can explore these types of rings and localizations of modules.

Question: What do non-IBN rings look like? Is there a good example of one?

9 Lecture 9

(**6 February 2014**)

Lecture today was canceled, but here's what the outline would have been:

0. Show that Aluffi-split is the same as split, following Hungerford's Theorem IV.1.18 in [2] (see http://www.dropbox.com/s/up190pbgns6vcwp/Hungerford%20excerpt.pdf).

1. Explain how to localize an R-module M at a subset $S \subset R$, giving a map of R-modules $M \longrightarrow S^{-1}M$, following Aluffi's Exercise V.4.8 in [1].

2. Show that $M \longrightarrow S^{-1}M$ is injective if M is free and S has no zero-divisors.

3. Use Points 1 and 2 to show that every integral domain is an IBN ring, following Aluffi's Proposition VI.1.9.

4. Point to an example of a non-IBN ring: https://www.dropbox.com/s/rmmk494pyy9na9y/Leavitt-IBN.pdf.

5. Show that every commutative ring is an IBN ring, following Aluffi's Exercises VI.1.10 and VI.1.11.

6. Show that every division ring is an IBN ring.

7. Discuss how to improve Hungerford's Corollary IV.2.12, using module localisation.

10 Lecture 10

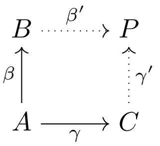
(11 February 2014)[1]

We are finally going to get to discuss the Hom-Tensor duality today. (Ideally, it shouldn't be put off for so long, but Aluffi puts it off until late in the book, so we did too).

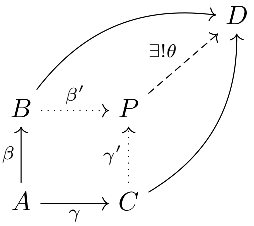
10.1 Tensor products of modules

Question: Do pushouts exist in CRing?

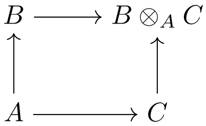
Definition 10.1. In any category, a ***pushout*** of a pair of maps β and γ consists of an object P and maps β' and γ' such that the following diagram commutes:



Furthermore, this must me universal in the sense that for any object D with maps from B and C to D, there exists a unique map $\theta : P \longrightarrow D$ such that the diagram commutes:



Example 10.2. In CRing , do pushouts exist? We can try $B \times C$, but we run into trouble. We need multilinearity.



Definition 10.3. Let R be a ring and M and N R-modules. Consider $\text{Free}_R(M \times N)$, which has elements

$$\sum_{i \in I} r_i(m_i, n_i)$$

which are finite sums (finitely many r_i nonzero). Consider the submodule

$$K = \left\langle \begin{array}{c} (m, r_1 n_1 + r_2 n_2) - r_1(m, n_1) - r_2(m, n_2) \\ (s_1 m_1 + s_2 m_2, n) - s_1(m_1, n) - s_2(m_2, n) \end{array} \;\middle|\; \begin{array}{l} r_1, r_2, s_1, s_2 \in R, \\ m, m_1, m_2 \in M, \\ n, n_1, n_2 \in N \end{array} \right\rangle. \tag{10.1}$$

[1] "Number one in olympics, third world country otherwise" – Clifton, lamenting the fact that one of the chalk boards in the classroom is broken.

This the submodule of $\text{Free}_R(M \times N)$ that is generated by elements of the form in the brackets in (10.1). Then the ***tensor product*** of M and N over R is defined as

$$M \otimes_R N = \text{Free}_R(M \times N)/K.$$

This maps elements of $\text{Free}_R(M \times N)$ into $M \otimes_R N$ by

$$\begin{aligned} (m,n) &\longmapsto (m,n) + K =: m \otimes n \\ \sum_{i\in I} r_i(m_i,n_i) &\longmapsto \sum_{i\in I} r_i(m_i,n_i) + K = \sum_{i\in I}(r_i(m_i,n_i) + K) = \sum_{i\in I} r_i(m_i \otimes n_i) \end{aligned}$$

and we can 'define' $r(m \otimes n) = (rm) \otimes n = m \otimes (rn)$ (although it's not really a definition, it is a consequence). Hence

$$M \otimes_R N = \left\{ \sum_{i\in I} r_i m_i \otimes n_i \,\middle|\, r_i \in R,\ m_i \in M,\ n_i \in N \right\}.$$

Note that this is <u>not</u> the same as $\{m \otimes n \,|\, m \in M,\ n \in N\}$! The elements of this form are the "pure" elements of $M \otimes_R N$.

The tensor product is starting to take the form of a pushout in the category of R-modules:

$$\begin{array}{ccc} M & \longrightarrow & M \otimes_R N \\ \uparrow & & \uparrow \\ R & \longrightarrow & N \end{array}$$

Actually we have a nice universal property that the tensor product fulfills. Suppose we have a map $f : M \times N \longrightarrow P$ (where P is an R-module) such that f is 'R-bilinear':

$$\begin{aligned} f(s_1m_1 + s_2m_2, n) &= s_1 f(m_1, n) + s_2 f(m_2, n) \\ f(m, r_1n_1 + r_2n_2) &= r_1 f(m, n_1) + r_2 f(m, n_2). \end{aligned}$$

Then there is a unique map of R-modules $\varphi : M \otimes_R N \longrightarrow P$ that makes the following diagram commute:

$$\begin{array}{ccc} M \times N & \xrightarrow{f} & P \\ {\scriptstyle (m,n)\mapsto m\otimes n}\downarrow & \nearrow_{\varphi} & \\ M \otimes_R N & & \end{array}$$

Note that $M \times N$ is *not* an R-module here. It exists in some 'shadow' of the category of R-modules. Since φ is a map of R-modules, it must satisfy

$$\varphi\left(\sum_{i\in I} r_i(m_i \otimes n_i)\right) = \sum_{i\in I} r_i \varphi(m_i \otimes n_i) = \sum_{i\in I} f(m_i, n_i)$$

since we must have $\varphi(m \otimes n) = f(m,n)$ from the commutivity of the diagram

Note 10.4. Let R be a ring and fix an R-module N. Consider the functor

$$\begin{aligned} \otimes_R N : M &\longmapsto M \otimes_R N \\ R\text{-}\mathsf{Mod} &\longrightarrow R\text{-}\mathsf{Mod}. \end{aligned}$$

This is left adjoint of a functor that we are already familiar with. Indeed, for a fixed N consider the functor

$$\mathrm{Hom}_{R\text{-}\mathsf{Mod}}(\cdot, N) : M \longmapsto \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(M, N).$$

Note that $\mathrm{Hom}_{R\text{-}\mathsf{Mod}}(M, N)$ is an R-module.

We have already seen an example of a left adjoint functor. Indeed,

$$\mathrm{Hom}_{R\text{-}\mathsf{Mod}}\left(\mathrm{Free}_R(S), N\right) \cong \mathrm{Hom}_{\mathsf{Set}}\left(S, \mathrm{Forget}(N)\right)$$

Theorem 10.5. *For the functors*

$$R\text{-}\mathsf{Mod} \underset{\otimes_R N}{\overset{\mathrm{Hom}_{R\text{-}\mathsf{Mod}}(\cdot,N)}{\rightleftarrows}} R\text{-}\mathsf{Mod}$$

we have the canoncal isomorphism

$$\mathrm{Hom}_{R\text{-}\mathsf{Mod}}\left(M \otimes_R N, P\right) \cong \mathrm{Hom}_{R\text{-}\mathsf{Mod}}\left(M, \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(N, P)\right).$$

Proof. Forgetting the "canonicalness" for a moment, let's just see if we can actually construct a bijection between the two sides. Let $\alpha \in \mathrm{Hom}_{R\text{-}\mathsf{Mod}}\left(M, \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(N, P)\right)$ with

$$\alpha : M \longrightarrow \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(N, P).$$

Define $f_\alpha : M \times N \longrightarrow P$ be defined by $f_\alpha(m, n) = \alpha(m)(n)$. Then f_α is R-bilinear, since

$$\begin{aligned} f_\alpha(s_1 m_1 + s_2 m_2, n) &= \alpha(s_1 m_1 + s_2 m_2)(n) \\ &= s_1 \alpha(m_1)(n) + s_2 \alpha(m_2)(n) \\ &= s_1 f_\alpha(m_1, n) + s_2 f_\alpha(m_2, n) \end{aligned}$$

and

$$\begin{aligned} f_\alpha(m, r_1 n_1 + r_2 n_2) &= \alpha(m)(r_1 n_1 + r_2 n_2) \\ &= r_1 \alpha(m)(n_1) + r_2 \alpha(m)(n_2) \\ &= r_1 f_\alpha(m, n_1) + r_2 f_\alpha(m, n_2). \end{aligned}$$

So we have, by the universal property, a map φ_α such that the diagram

$$\begin{array}{ccc} M \times N & \xrightarrow{f_\alpha} & P \\ \downarrow & \nearrow_{\varphi_\alpha} & \\ M \otimes_R N & & \end{array}$$

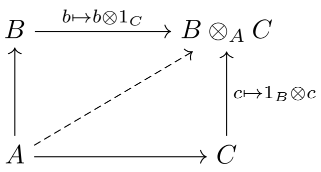
commutes.

Exercise 10.6. Show that the map $\alpha \longmapsto \varphi_\alpha$ is a bijection.

□

The statement of Theorem 10.5 is really a nice unraveling of the ungainly universal property of tensor products.

In particular, tensor products are pushouts and we have the following picture:



where A is a CRing and B and C are A-modules.

11 Lecture 11

(13 February 2014)

Recall the definition of the tensor product $M \otimes_R N$:

$$M \otimes_R N := \mathrm{Free}_{R\text{-}\mathsf{Mod}}\left(\mathrm{Forget}(M \times N)\right)/K$$

where K is the submodule of $\mathrm{Free}_{R\text{-}\mathsf{Mod}}\left(\mathrm{Forget}(M \times N)\right)$ that is generated by elements of the form

$$\begin{aligned}(m_1 + m_2, n) - (m_1, n) - (m_2, n)\\ (m, n_1 + n_2) - (m, n_1) - (m, n_2)\\ (rm, n) - (m, rn).\end{aligned}$$

By construction, we have created a left R-module out of two left R-modules. But we can instead take M to be a right R-module and N a left R-module in order to construct the tensor product as a bi-module. Then the last relation in the generators of K above would instead be given by

$$(mr, n) - (m, rn)$$

The construction of the tensor product exists for any ring R, but for most of our analysis (for today in particular), we will take R to be a commutative ring.

Let R be a commutative ring. Let's look at the following picture in the category of R-modules:

$$\begin{array}{ccc} A & \longrightarrow & A \otimes_0 B \\ {\scriptstyle 0\mapsto 0_A}\uparrow & & \uparrow \\ 0 & \longrightarrow & B \end{array}$$

where A and B are 0-modules. Then $A \otimes_0 B$ is just $A \times B$. (I'm confused here. Isn't the zero module the only module over the zero ring?)

Let's consider a similar picture in the category of cmmutative rings:

$$\begin{array}{ccc} A & & \\ \uparrow & & \\ 0 & \longrightarrow & B \end{array}$$

But $\mathrm{Hom}_{\mathsf{CRing}}(0, A)$ is empty unless A is also the zero ring. So A and B must be the zero ring and this picture becomes:

$$\begin{array}{ccc} A & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & B. \end{array}$$

Still in the category of CRing, consider

$$\begin{array}{ccc} A & \longrightarrow & A \otimes_{\mathbb{Z}} B \\ \uparrow & & \uparrow \\ \mathbb{Z} & \longrightarrow & B. \end{array}$$

As before there is only one map $\mathbb{Z} \longrightarrow A$ (the map that takes 1 ro 1).

Consider this diagram:

$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{\text{epi.}} & \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{F}_p \\ \uparrow & & \uparrow \\ \mathbb{Z} & \xrightarrow{\text{epi.}} & \mathbb{F}_p. \end{array}$$

The two maps labeled 'epi.' are epimorphisms. Replace every part of the diagram above by the sets of prime ideals of that field. In particular, the set of prime ideals in $\mathbb{Z}$ is $\{\langle p \rangle \mid p \text{ prime}\}$ and the set of prime ideals in $\mathbb{Q}$ and $\mathbb{F}_p$ just $\{\langle 0 \rangle\}$ since they are fields. We can consider the diagram

$$\begin{array}{ccc} & & \langle 0 \rangle \\ & & \big\downarrow{\scriptstyle \text{mono.}} \\ \langle 0 \rangle \cup \{\langle p \rangle \mid p \text{ prime}\} & \xleftarrow[\text{mono.}]{} & \langle 0 \rangle \end{array}$$

and we want to find the pullback in the category of sets. What are pullbacks in Set? Fibred products:

$$\begin{array}{ccc} U & \longleftarrow & U \times_S \\ \downarrow & & \downarrow \\ S & \longleftarrow & V. \end{array}$$

In particular, we have that the pullback must be the empty set, so

$$\begin{array}{ccc} \langle 0 \rangle & \longleftarrow & \emptyset \\ \big\downarrow{\scriptstyle \text{mono.}} & & \downarrow \\ \langle 0 \rangle \cup \{\langle p \rangle \mid p \text{ prime}\} & \xleftarrow[\text{mono.}]{} & \langle 0 \rangle. \end{array}$$

So $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{F}_p$ has no prime ideals. But the only commutative ring with identity that has no prime ieals is the zero ring![1] so $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{F}_p = 0$. There is another way to see that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{F}_p$ is zero. Take the identity element $1 \otimes 1$. Then

$$1 \otimes 1 = p\left(\frac{1}{p} \otimes 1\right) = \frac{1}{p} \otimes (p \cdot 1)$$

but $p \cdot 1 = 0$ in $\mathbb{F}_p$, so

$$\frac{1}{p} \otimes 0 = \frac{1}{p} \otimes (0 \cdot 0) = \frac{0}{p} \otimes 0 = 0 \otimes 0,$$

hence $1 \otimes 1 = 0 \otimes 0$.

Going back to the Hom-Tensor duality: Recall that we have an adjunciton forumla between Free and Forget:

$$\mathrm{Hom}_{R\text{-}\mathsf{Mod}}(\mathrm{Free}_{R\text{-}\mathsf{Mod}}(S), M) \cong \mathrm{Hom}_{\mathsf{Set}}(S, \mathrm{Forget}(M)).$$

The Forget functor is covariant but the $\mathrm{Free}_{R\text{-}\mathsf{Mod}}$ functor is contravariant. Indeed, given a map of R-modules $M \to N$ there is a map of sets $\mathrm{Forget}(M) \to \mathrm{Forget}(N)$. But for a map of sets $S \to T$, there is a map $\mathrm{Free}_{R\text{-}\mathsf{Mod}}(S) \leftarrow \mathrm{Free}_{R\text{-}\mathsf{Mod}}(S)$ that is given by precomposition.

The second theorem of this type gave a formula

$$\mathrm{Hom}_{R\text{-}\mathsf{Mod}}(M \otimes_R N, P) \cong \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(M, \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(N, P))$$

with the picture

[1] According to Zach, there is a trivial exercise in [3] that shows that this fact.

$$R\text{-Mod} \underset{\mathrm{Hom}_{R\text{-Mod}}(N,-)}{\overset{-\otimes_R N}{\rightleftarrows}} R\text{-Mod}.$$

Let's explore the functorial properties of $\otimes_R$ and Hom. Starting with Hom, think about it as a functor in the first spot

$$\mathrm{Hom}_{R\text{-Mod}}(-, N) : M \longmapsto \mathrm{Hom}_{R\text{-Mod}}(M, N).$$

Given a map $M_1 \to M_2$ of R-modules, we have the picture

$$\begin{array}{ccc} M_1 & & \mathrm{Hom}_{R\text{-Mod}}(M_1, N) \\ \downarrow & & \uparrow \\ M_2 & & \mathrm{Hom}_{R\text{-Mod}}(M_2, N) \end{array}$$

where the map on the right is given by precomposition. So we say that "Hom is contravariant in its first slot".

In the second variable, we have the functor

$$\mathrm{Hom}_{R\text{-Mod}}(M, -) : N \longmapsto \mathrm{Hom}_{R\text{-Mod}}(M, N).$$

Given a map $N_1 \to N_2$ of R-modules, we have the picture

$$\begin{array}{ccc} N_1 & & \mathrm{Hom}_{R\text{-Mod}}(M, N_1) \\ \downarrow & & \downarrow \\ N_2 & & \mathrm{Hom}_{R\text{-Mod}}(M, N_2) \end{array}$$

where the map on the right is given by postcomposition. So we say that "Hom is covariant in its second slot".[2]

Now let's consider the tensor product as a functor in both of its spots. In the first spot, we have

$$- \otimes_R N : M \longrightarrow M \otimes_R N.$$

Given a homomorphism of R-moduels $\varphi : M_1 \to M_2$, we have the picture

$$\begin{array}{ccc} M_1 & & M_1 \otimes_R N \\ \downarrow & & \downarrow \\ M_2 & & M_2 \otimes_R N \end{array}$$

where the map on the right is given by $m \otimes n \longmapsto \varphi(m) \otimes n$ and can be extended to all elements of $M_1 \otimes_R N$ by R-linearity. Then φ is in fact an R-linear map, since φ is R-linear, and we have

$$\varphi(rm) \otimes n = (r\varphi(m)) \otimes n = \varphi(m) \otimes rn.$$

As a functor in the other variable, we have

$$M \otimes_R - : N \longrightarrow M \otimes_R N,$$

and given a homomorphism of R-moduels $\varphi : N_1 \to N_2$, we have the picture

[2] Note that we don't use the R-module properties at all. In fact, the covariant- and contravariantness of the Hom functor works really in any category.

$$\begin{array}{ccc} N_1 & & M \otimes_R N_1 \\ \downarrow & & \downarrow \\ N_2 & & M \otimes_R N_2. \end{array}$$

(Something's not quite with our picture, since we need one of the functors in the picture

$$R\text{-Mod} \underset{\mathrm{Hom}_{R\text{-Mod}}(N,-)}{\overset{-\otimes_R N}{\rightleftarrows}} R\text{-Mod}.$$

to be covariant and one to be contravariant. We will correct this next time.)

11.1 Functors applied

Consider the functor $- \otimes_R N : M \longmapsto M \otimes_R N$ applied to the short exact sequence of R-modules

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0.$$

Since $0 \otimes_R N = 0$, applying $- \otimes_R N$ we get the sequence

$$0 \longrightarrow A \otimes_R N \xrightarrow{\alpha \otimes \mathrm{id}} B \otimes_R N \xrightarrow{\beta \otimes \mathrm{id}} C \otimes_R N \longrightarrow 0.$$

Is this sequence also exact?

Let's first check to see if the map $\beta \otimes \mathrm{id}$ is surjective. Since β is surjective, for every pure element $c \otimes n$ in $C \otimes_R N$ there is a $b \in B$ such that $\beta(b) = c$. For arbitrary R-linear combinations of pure elements in $C \otimes_R N$, each pure element is achieved from something in $B \otimes_R N$, so the sum is achieved from the sum of those elements in $B \otimes_R N$.

What about the map $\alpha \otimes \mathrm{id}$? It needs to be injective. But it need not be. Next time we will talk about when and why this fails for certain R-modules, as well as introduce Tor. In particular, a functor is said to be *exact* if it takes exact sequences to exact sequences.

Exercise 11.1. For next lecture, think about when the above sequence is or is not exact.

Definition 11.2. An R-module N is said to be ***flat*** if the functor $- \otimes_R N$ is exact.

In order to make the above sequence exact, we need to add something to the left of the sequence. This turns out to be Tor (which we will discuss next time).

12 Lecture 12

(25 February 2014)

(Geoff gave a review of homological algebra from the last few weeks of class.)

Note 12.1. Any small abelian category may be embedded in the category of R-modules for some suitably chosen ring R. Thus, studying R-modules gives us powerful tools that we can apply to any abelian category.

Theorem 12.2. *A sequence of R-modules*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

has $B \cong A \oplus C$ if and only if there exists a map $\gamma : C \longrightarrow B$ such that $\beta \circ \gamma = \mathrm{id}_C$.

Proof. Assume $\gamma \in \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(C, B)$ such that $\beta \circ \gamma = \mathrm{id}_C$. Then define the R-module homomorphism $\varphi : A \oplus C \longrightarrow B$ by $(a, c) \longmapsto \alpha(a) + \beta(c)$. Then we have the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\iota_A} & A \oplus C & \xrightarrow{\pi_C} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}_A} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{id}_C} & & \\ 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0. \end{array} \tag{*}$$

Since this diagram commutes, by the short five lemma we have that φ is an isomorphism.

Now suppose $B \cong A \oplus C$. Then we have an isomorphism of R-modules given by $\varphi : A \oplus C \longrightarrow B$ with the same commuting diagram (*) above. Define an R-module homomorphism $\gamma : C \longrightarrow B$ by $\gamma = \varphi \circ \iota_C$. Note that $\pi_C \circ \iota_C = \mathrm{id}_C$, since

$$\pi_C(\iota_C(c)) = \pi_C(0, c) = c \qquad \text{for all } c \in C.$$

Also, from the commutativity of the daigram we have $\beta \circ \varphi = \pi_C$. Hence

$$\beta \circ \gamma = \underbrace{\beta \circ \varphi}_{=\pi_C} \circ \iota_C = \pi_C \circ \iota_C = \mathrm{id}_C\,.$$

So the general notion of split is the same as the definition given by Aluffi for R-modules. ☐

12.1 Back to adjoints

Recall this theorem that we had earlier.

Theorem 12.3. *Let M, N, P be R-modules over a fixed ring R. Then there is a canonical isomorphism*

$$\mathrm{Hom}_{R\text{-}\mathsf{Mod}}(M \otimes_R N, P) \cong \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(M, \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(N, P)).$$

In particular, $\mathrm{Hom}_{R\text{-}\mathsf{Mod}}(N, P)$ itself has the structure of an R-module. The isomorphism in the theorem comes as an isomorphism between two functors. These functors are both from R-Mod to R-Mod and are

$$\mathrm{Hom}_{R\text{-}\mathsf{Mod}}(N, -) : P \longmapsto \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(N, P)$$
$$- \otimes_R N : M \longmapsto M \otimes_R N.$$

These are both *covariant* functors. That is, given two R-modules P_1, P_2 and an R-module homomorphism $\varphi : P_1 \longrightarrow P_2$, we have a natural R-module homomorphism $\mathrm{Hom}_{R\text{-}\mathsf{Mod}}(N, P_1) \longrightarrow \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(N, P_2)$.

$$\begin{array}{ccc} P_1 & & \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(N, P_1) \\ \big\downarrow \varphi & & \big\downarrow \psi \longmapsto \varphi \circ \psi \\ P_2 & & \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(N, P_2). \end{array}$$

Similarly, given R-modules M_1, M_2 and an R-module homomorphism $\varphi : M_1 \longrightarrow M_2$, there is a natural R-module homomorphism $\varphi \otimes \mathrm{id}_N : M_1 \otimes_R N \longrightarrow M_2 \otimes_R N$.

$$\begin{array}{ccc} M_1 & & M_1 \otimes_R N \\ \big\downarrow \varphi & & \big\downarrow \varphi \otimes \mathrm{id}_N : m_1 \otimes n \longmapsto \varphi(m_1) \otimes n \\ M_2 & & M_1 \otimes_R N. \end{array}$$

Recall that we also had another theorem regarding adjunctions:

Theorem 12.4. *For the free functor* $\mathrm{Free}_R : R\text{-}\mathsf{Mod} \longrightarrow \mathsf{Set}$ *and the forgetful functor* $\mathrm{Forget} : \mathsf{Set} \longrightarrow \mathsf{R\text{-}Mod}$, *a set S and an R-module M we have the canonical isomorphism*

$$\mathrm{Hom}_{R\text{-}\mathsf{Mod}}(\mathrm{Free}_R(S), M) \cong \mathrm{Hom}_{\mathsf{Set}}(S, \mathrm{Forget}(M)).$$

As above, we have the correspondence for R-modules M_1 and M_2

$$\begin{array}{ccc} M_1 & & \mathrm{Forget}(M_1) \\ \big\downarrow \varphi & & \big\downarrow \mathrm{Forget}(\varphi) \\ M_2 & & \mathrm{Forget}(M_2), \end{array}$$

where $\mathrm{Forget}(\varphi) : \mathrm{Forget}(M_1) \longrightarrow \mathrm{Forget}(M_2)$ is just the same mapping φ but between M_1 and M_2 as sets. So Forget is a covarinat functor.

Given sets S_1 and S_2 with a set-function $f : S_1 \longrightarrow S_2$, we have the correspondence

$$\begin{array}{ccccc} S_1 & \xrightarrow{\delta^1} & \mathrm{Free}(S_1) & = & \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(S_1, R) \\ \big\downarrow f & \searrow & \big\downarrow \mathrm{Free}(f) & & \big\downarrow \delta^1_s \mapsto \delta^2_{f(s)} \\ S_2 & \xrightarrow{\delta^2} & \mathrm{Free}(S_2) & = & \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(S_2, R), \end{array}$$

where $\delta^1 : S_1 \longrightarrow \mathrm{Free}(S_1)$ is the mapping $s \longmapsto 1_R \cdot s$.

Proposition 12.5. *Given R-modules M_1, M_2, N, we have the canonical isomorphism*

$$(M_1 \oplus M_2) \otimes_R N \cong (M_1 \otimes N) \oplus (M_2 \otimes N)\,.$$

Proof. We can define the necessary isomorphism by

$$(m_1, m_2) \otimes n \longmapsto (m_1 \otimes n, m_2 \otimes n)$$

with its inverse given by

$$(m_1 \otimes n_1, m_2 \otimes n_2) \longmapsto (m_1, 0) \otimes n_1 + (0, m_2) \otimes n_2.$$

□

Example 12.6. The standard examples of tensor pruducts of $\mathbb{Z}$-modules are

1. $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} \cong 0$
2. $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/\gcd(n,m)\mathbb{Z}$

12.2 Flat and Projective

Definition 12.7. An R-module P is ***flat*** if the functor $- \otimes_R P$ is exact. That is, given a short exact sequence

$$0 \longrightarrow A \hookrightarrow B \twoheadrightarrow C \longrightarrow 0$$

of R-modules the sequence

$$0 \longrightarrow A \otimes_R P \longrightarrow B \otimes_R P \longrightarrow C \otimes_R P \longrightarrow 0$$

is exact. Note that the right side of the sequence is always exact, but the left side might not be.

Proposition 12.8. *Free modules are flat.*

Example 12.9. Is $\mathbb{Z}/2\mathbb{Z}$ a flat $\mathbb{Z}$-module? Consider the map $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$, there the map is just multiplication by 2. This is injective, but tensoring with $\mathbb{Z}/2\mathbb{Z}$ we get

$$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z},$$

and thus $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2 \otimes \mathrm{id}_{\mathbb{Z}/2\mathbb{Z}}} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$. However, this is the zero map, which is not injective. So $\mathbb{Z}/2\mathbb{Z}$ is not flat.

Definition 12.10. An R-module P is ***projective*** if the functor $\mathrm{Hom}_{R\text{-}\mathsf{Mod}}(-, P)$ is exact. That is, given a short exact sequence

$$0 \longrightarrow A \hookrightarrow B \twoheadrightarrow C \longrightarrow 0$$

of R-modules the sequence

$$0 \longrightarrow \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(A, P) \longrightarrow \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(B, P) \longrightarrow \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(C, P) \longrightarrow 0$$

is exact.

Proposition 12.11. *Projective modules are flat.*

Example 12.12. (Clifton started talking about some example regarding a map $\mathbb{C} \longrightarrow \mathbb{C}$ given by $z \longmapsto z^2$)

13 Lecture 13

(**27 February 2014**)

From the last few lectures, we've had an unresolved proposition that has been sitting around unproved.

Proposition 13.1 (unresolved as of yet)**.** *Given a short exact sequence of R-modules*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

applying the functor $-\otimes_R N$ yields the exact sequence

$$A \otimes_R N \longrightarrow B \otimes_R N \longrightarrow C \otimes_R N \longrightarrow 0$$

for every R-module N.

That is, the functor $-\otimes_R N$ is *right*-exact, but not necessarily left-exact. Recall the definition of flatness.

Definition 13.2. An R-module P[1] is ***flat*** if

$$0 \longrightarrow A \otimes_R P \longrightarrow B \otimes_R P \longrightarrow C \otimes_R P \longrightarrow 0$$

is exact for every short exact sequence of R-modules $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$.

Proof of Proposition 13.1. Explicitly, we have the short exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

and the corresponding (not necessarily exact) sequence

$$A \otimes_R N \xrightarrow{\alpha\otimes \mathrm{id}_N} B \otimes_R N \xrightarrow{\beta\otimes \mathrm{id}_N} C \otimes_R N \longrightarrow 0.$$

We have already shown that this sequence is exact at $C \otimes_R N$, so it remains to show that it is exact at $B \otimes_R N$.

What is the cokernel of α? As a map, the cokernel is a map $B \xrightarrow{\mathrm{coker}(\alpha)} B/\ker\alpha$ such that we have the universal property

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C \\ & & {\scriptstyle \mathrm{coker}(\alpha)}\downarrow & \nearrow {\scriptstyle \exists !} & \\ & & K \equiv B/\ker\alpha & & \end{array}.$$

Hence $C \cong B/\ker\alpha$. Applying the functor $F = -\otimes_R N$, we have the equivalent diagram

$$\begin{array}{ccccc} F(A) & \xrightarrow{F(\alpha)} & F(B) & \xrightarrow{F(\beta)} & F(C) \\ & & {\scriptstyle \mathrm{coker}\, F(\alpha)}\downarrow & \nearrow {\scriptstyle \exists ! \theta} & \\ & & F(B)/F(\alpha)(F(A)) & & \end{array}.$$

[1] where P comes from the French word for flat

Then the sequence $F(A) \xrightarrow{F(\alpha)} F(B) \xrightarrow{F(\beta)} C \longrightarrow 0$ is exact at $F(B)$ if and only if $F(\alpha)(F(A)) = \operatorname{im} F(\alpha) = \ker F(\beta)$, hence $F(B)/F(\alpha)(F(A)) = F(B)/\ker F(\beta)$, in which case θ here is an isomorphism.

We can then apply the functor $G = \operatorname{Hom}_{R\text{-}\mathsf{Mod}}(N, -)$ and use the isomorphism theorem.

The rest of this proof will be completed next time. □

Example 13.3. Consider the mapping $\mathbb{C} \xrightarrow{z \mapsto z^2} \mathbb{C}$. This has a *smooth locus* and a *singular locus*. That is, for each $w \neq 0$, the map $\pm w^{1/2} \mapsto w$ is 2:1, but is only 1:1 for $w = 0$.

For the map $\mathbb{C}^\times \xrightarrow{z \mapsto z^2} \mathbb{C}^\times$, we have the induced map on the polynomial rings

$$\begin{array}{ccc} \mathbb{C}^\times & & \mathbb{C}[y] \\ \big\downarrow{\scriptstyle z \mapsto z^2} & & \big\uparrow{\scriptstyle \varphi : x \longrightarrow y^2} \\ \mathbb{C}^\times & & \mathbb{C}[x]. \end{array}$$

What is the set of prime ideals in $\mathbb{C}[x]$? Namely,

$$\operatorname{Spec}(\mathbb{C}[x]) = \{(x - a) \mid a \in \mathbb{C}\} \cup \{0\}.$$

What do prime ideals in $\operatorname{Spec}(\mathbb{C}[y])$ get mapped to in $\operatorname{Spec}(\mathbb{C}[x])$ under the map induced by φ?

$$\begin{array}{ccc} \operatorname{Spec}(\mathbb{C}[y]) & & \mathfrak{p} \\ \big\downarrow & & \big\downarrow \\ \operatorname{Spec}(\mathbb{C}[x]) & & \varphi^{-1}(\mathfrak{p}) \end{array}$$

where $\varphi^{-1}(y - b) = (x - b^2)$ if $\mathfrak{p} = (y - b)$ and $\varphi^{-1}(\mathfrak{p}) = 0$ if $\mathfrak{p} = (0)$. This is the 'same' map as the one $\mathbb{C} \xrightarrow{z \mapsto z^2} \mathbb{C}$ except with the extra information that it takes the zero ideal to the zero ideal.

What about for the map $\mathbb{C}^\times \xrightarrow{z \mapsto z^2} \mathbb{C}^\times$? Can we do the same process? We get

$$\mathbb{C}[x]_x \cong \frac{\mathbb{C}[x, x']}{(xx' - 1)} \cong \mathbb{C}[x, x^{-1}].$$

What is its spectrum? It turns out that

$$\operatorname{Spec}(\mathbb{C}[x]_x) = \{(x - a) \mid a \neq 0\} \cup \{(0)\}.$$

So we have a pullback

$$\begin{array}{ccc} \mathbb{C} & \longleftarrow & \mathbb{C}^\times \\ {\scriptstyle z \mapsto z^2}\big\downarrow & & \big\downarrow \\ \mathbb{C} & \hookleftarrow & \mathbb{C}^\times \end{array}$$

with corresponding pushout

$$\begin{array}{ccccc} \mathbb{C}[y] & \longrightarrow & \mathbb{C}[y] \otimes_{\mathbb{C}[x]} \mathbb{C}[x, x^{-1}] & \overset{\cong}{=\!=\!=} & \mathbb{C}[y]_y \\ {\scriptstyle x \mapsto y^2}\big\uparrow & & \big\uparrow & & \\ \mathbb{C}[x] & \twoheadrightarrow & \dfrac{\mathbb{C}[x, x']}{(xx' - 1)} & \overset{\cong}{=\!=\!=} & \mathbb{C}[x, x^{-1}] \end{array}$$

where we note that pushouts in Ring are tensor products.

Example 13.4. $\dfrac{\mathbb{C}[x,y]}{(xy)}$ is not flat as a $\mathbb{C}[x]$-module. Consider the mapping of $\mathbb{C}[x]$-modules

$$\mathbb{C}[x] \xrightarrow{f \longmapsto xf} \mathbb{C}[x].$$

This is injective. Is the map

$$\frac{\mathbb{C}[x,y]}{(xy)} \cong \mathbb{C}[x] \otimes_{\mathbb{C}[x]} \mathbb{C}[x,y] \longrightarrow \mathbb{C}[x] \otimes_{\mathbb{C}[x]} \mathbb{C}[x,y] \cong \frac{\mathbb{C}[x,y]}{(xy)}$$

injective? It takes $1 \otimes 1 \longmapsto x \otimes 1 = 1 \otimes x$. No, since $1 \otimes y \longmapsto x \otimes y = 1 \otimes xy = 0$, since $xy = 0 \in \dfrac{\mathbb{C}[x,y]}{(xy)}$.

Question: Is $\mathbb{C}[y]$ flat as a $\mathbb{C}[x]$-module with module structure given by $\mathbb{C}[x] \xrightarrow{x \longmapsto y^2} \mathbb{C}[y]$? No, since we have already seen that the corresponding geometric picture is not smooth. As an exercise, show this algebraically.

13.1 Free resolutions

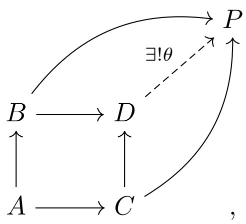
This (and Tor) will be our next topic.

14 Lecture 14

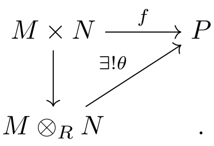
(4 March 2014)

(Review of the last two weeks by Will.)

- Free modules and bases of R-modules.
  - Given an R-module M,a subset $S \subset M$ is *linearly independent* in M if
    $$\sum_{s \in S} r_s s = 0 \quad \Longrightarrow \quad r_s = 0 \text{ for all } s \in S.$$
  - A *spanning set* of M is a subset $S \subset M$ such that $\langle S \rangle = M$. That is, all $m \in M$ may be represented by $m = \sum r_s s$.
  - A *basis* of M is a subset $S \subset$ that is linearly independent and a spanning set.
  - Note that arbitrary R-modules vary greatly from our standard notion of vector spaces here, since even a one element set in M can fail to be linearly independent, and R-modules may have no bases.
  - **Example**: the $\mathbb{Z}$-module $\mathbb{Z}/p\mathbb{Z}$ (where p is prime) has no basis.
  - **Theorem**: If a module has a basis, then it is a free module.
  - A ring R has the *IBN* property if any two bases of an R-module have the same cardinality.
  - **Theorem**: For a free module $\text{Free}_{R\text{-}\mathsf{Mod}}(B)$ over an integral doman R, a subset $S \subset \text{Free}_{R\text{-}\mathsf{Mod}}(B)$ is linearly independent over $\text{Free}_R(B)$ if and only if it is linearly independent over $\text{Free}_{k\text{-}\mathsf{Mod}}(B)$ where k is the field of fractions of R.
- Tensor products: Consider pushouts

  
  ,

  where we define $M \otimes_R N := \text{Free}_{R\text{-}\mathsf{Mod}} / K$ and K is the submodule of $\text{Free}_{R\text{-}\mathsf{Mod}}$ generated by elements of the form
  $$(m_1 + m_2, n) - (m_1, n) - (m_2, n), \quad (m, n_1 + n_2) - (m, n_1) - (m, n_2), \quad (mr, n) - (m, rn).$$
  Here, M is a right-R-module and N is a left-R-module to form $M \otimes_R N$ as a bi-module, with R-action given by
  $$r(m, n) = (m, rn) = (mr, n) = (m, n)r.$$
  - We use tensor products to show that pushouts exists in Ring, but tensor products themselves are not necessarily pushouts in R-Mod.
  - Tensor products fulfill an important universal property: for $f : M \times N \longrightarrow P$ a bilinear map (of sets),

    
    .

- **Theorem**: we have the canonical isomorphism

$$\mathrm{Hom}_{R\text{-}\mathsf{Mod}}(M \otimes_R N, P) \cong \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(M, \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(N, P))$$

  for any R-modules M,N, and P.

- Flat and projective modules
  - An R-module N is *flat* if $- \otimes_R N$ is exact as a functor.
  - An R-module P is *projective* if $\mathrm{Hom}_{R\text{-}\mathsf{Mod}}(P, -)$ is exact as a functor.
  - (An R-module P is *injective* if $\mathrm{Hom}_{R\text{-}\mathsf{Mod}}(-, P)$ is exact as a functor.)
  - Projective modules are always flat.
  - **Examples:**
    * R as an R-module is always projective (and thus flat).
    * $\mathbb{Z}/n\mathbb{Z}$ is not flat.
-

Problem. Given a short exact sequence of R-modules

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

the following sequence is exact for any R-module N,

$$A \otimes_R N \xrightarrow{\bar{\alpha}=\alpha\otimes\mathrm{id}_n} B \otimes_R N \xrightarrow{\bar{\beta}=\beta\otimes\mathrm{id}_n} C \otimes_R N \longrightarrow 0.$$

Solution. Exactness at $C \otimes_R N$ just requires that $\bar{\beta}$ be surjective. This is because, given $c\otimes n$, there is a $b \in B$ such that $c = \beta(b)$ since β is surjective, then $\bar{\beta}(b\otimes n) = \beta(b)\otimes n = c\otimes n$.

To be exact at $B \otimes_R N$, we require that $\bar{\alpha}(A \otimes_R N) = \ker \bar{\beta}$. By the universal property of the cokernel, there is a unique map θ such that yields the commuting diagram

$$\begin{array}{ccccc} A \otimes_R N & \xrightarrow{\bar{\alpha}} & B \otimes_R N & \xrightarrow{\bar{\beta}} & C \otimes_R N \\ & & \downarrow{\scriptstyle \mathrm{coker}\,\bar{\alpha}} & \nearrow{\scriptstyle \exists!\theta} & \\ & & B \otimes_R N/\bar{\alpha}(A \otimes_R N) & & \end{array},$$

where θ is the map $\theta(\sum r_i b_i \otimes n_i + \bar{\alpha}(A \otimes_R N)) = \sum r_i \beta(b_i) \otimes n_i$. We can define the inverse of θ as follows. Since β is surjective, for each $c \in C$ there is a $b \in B$ such that $\beta(b) = c$. Hence we have $c \otimes n = \bar{\beta}(b \otimes n)$. Then define

$$\theta^{-1}(c \otimes n) = b \otimes n + \bar{\alpha}(A \otimes_R N).$$

This is well-defined, since for $b, b' \in B$ such that $\beta(b) = \beta(b') = c$, we have $\beta(b - b') = 0$ and thus $b - b \in \ker \beta = \alpha(A)$ by exactness, so there is an $a \in A$ such that $\alpha(a) = b - b'$. Hence $(b - b') \otimes n \in \bar{\alpha}(A \otimes_R N)$ and thus $(b - b') \otimes n + \bar{\alpha}(A \otimes_R N) = 0$.

So we have that $C \otimes_R N \cong \mathrm{coker}\,\bar{\alpha}$, and thus $\ker \beta = \mathrm{im}\,\bar{\alpha}$. (?)

14.1 Base change

We want to develop a categorical proof of the problem that we just solved above.

We've been studying the category of R-modules since the beginning of this course. Our studies suggest that we can learn something about a ring by studying its R-modules. Given another ring S, we can learn something about the maps (in Ring) between R and S by studying functors from R-Mod to S-Mod. That is, given a ring homomorphism

$$R \xrightarrow{f} S$$

we study the functors[1] f^*, f_* and $f^!$:

$$R\text{-}\mathsf{Mod} \begin{array}{c} \xrightarrow{f^*} \\ \xleftarrow{f_*} \\ \xrightarrow[f^!]{} \end{array} S\text{-}\mathsf{Mod}.$$

- First we study $f_* : S\text{-}\mathsf{Mod} \longrightarrow R\text{-}\mathsf{Mod}$. This takes an S-module A and turns it into an R-module by
$$ra \longmapsto f(r)a.$$
- Now study $f^* : R\text{-}\mathsf{Mod} \longrightarrow S\text{-}\mathsf{Mod}$ (and suppose for a moment that we are restricted to commutative rings CRing). Here $f^*(A) = A \otimes_{R \nearrow f} S$, where S is viewed as an S-module and
$$(ar) \otimes s = a \otimes f(r)s.$$
So $f^* = - \otimes_{R \nearrow f} S$.
- Finally $f^! : R\text{-}\mathsf{Mod} \longrightarrow S\text{-}\mathsf{Mod}$[2] is $f^! = \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(S, -)$, so $f^!(A) = \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(S, A)$.

Theorem 14.1. *These three functors $(f^*, f_*, f^!)$ form a triple of adjoint functors. That is each is a left-adjoint of the next one.*

Corollary 14.2. *.*

- *f^* is right-exact;*
- *$f^!$ is left-exact;*
- *f_* is exact.*

Theorem 14.3. *If (F, G) is an adjiont pair of functors, then F is right-exact and G is left-exact.*

Consider categories $\mathbb{X}$ and $\mathbb{Y}$ with a pair of adjoint functors

$$\mathbb{X} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow[G]{} \end{array} \mathbb{Y}.$$

Then we have the natural isomprphisms

$$\mathrm{Hom}_{\mathbb{Y}}(F(X), Y) \cong \mathrm{Hom}_{\mathbb{X}}(X, G(Y)).$$

From this isomorphism, we have the special case

$$\begin{array}{rcl} \mathrm{Hom}_{\mathbb{Y}}(FG(Y), Y) & \cong & \mathrm{Hom}_{\mathbb{X}}(G(Y), G(Y)) \\ \varepsilon(Y) & \longleftarrow & \mathrm{id}_{G(Y)}, \end{array}$$

[1]This is Aluffi's notation, and Clifton thinks that it is hideous.
[2]pronounced "f-shriek"

where ε is a natural transformation $\varepsilon : FG \longrightarrow \mathrm{id}_{\mathbb{Y}}$. Similarly, we have

$$\begin{array}{rcl} \mathrm{Hom}_{\mathbb{X}}(X, G(F(X))) & \cong & \mathrm{Hom}_{\mathbb{Y}}(F(X), F(X)) \\ \eta(Y) & \longleftarrow & \mathrm{id}_{F(X)}, \end{array}$$

where $\eta(X) : X \longrightarrow G(F(X))$ and η is the natural transformation $\eta : \mathrm{id}_{\mathbb{X}} \longrightarrow GF$. Then we have isomprphisms

$$F \xrightarrow{\sim} FGF$$
$$G \xrightarrow{\sim} GFG.$$

Consider categories where we can have exact sequences. If we have an exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0,$$

consider the seqeunce produced by applying the functor[3]

$$F(A) \xrightarrow{F(\alpha)} F(B) \xrightarrow{F(\beta)} F(C) \longrightarrow 0$$

Claim: We have $\operatorname{coker} F(\alpha) = F(\beta)$.

Suppose $F(A) \xrightarrow{F(\alpha)} F(B) \xrightarrow{\gamma} D$ is a complex. Then we have

$$\begin{array}{ccccc} F(A) & \xrightarrow{F(\alpha)} & F(B) & \xrightarrow{\gamma} & D \\ & & {\scriptstyle F(\beta)}\downarrow & \nearrow{\scriptstyle \exists! \theta} & \\ & & F(C). & & \end{array}$$

Claim: There exists a unique $\theta : F(C) \longrightarrow D$ such that $\gamma = \theta \circ F(\beta)$.

Lemma 14.4. *Given a pair of adjoint functors F and G, we have*

$$\begin{array}{rcl} \mathrm{Hom}(F(C), D) & \cong & \mathrm{Hom}(C, G(D)) \\ \theta & \longmapsto & \theta', \end{array}$$

where $\theta = \varepsilon(D) \circ F(\theta)$ and $\theta' = G(\theta) \circ \eta(C)$.

Since $\varepsilon : FG \longrightarrow \mathrm{id}$ is a natural transformation and $F(\theta') : F(C) \longrightarrow FG(D)$, we have

$$\begin{array}{ccc} F(G(D)) & \xrightarrow{\varepsilon(D)} & D \\ {\scriptstyle F(\theta')}\uparrow & \nearrow{\scriptstyle \gamma} & \\ F(C) & & \end{array}$$

(????????)

[3]e.g. in R-modules, the functor might be $F = - \otimes_R N$.

15 Lecture 15

(6 March 2014)

A reminder of what we are doing:
We have two categories $\mathbb{X}$ and $\mathbb{Y}$ and a pair of adjoint functors

$$\mathbb{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{Y}.$$

What does it mean to say that (F, G) is an adjoint pair of functors?

1. What we have said previously, is that F and G are adjoint if we have a functorial isomorphism

$$\mathrm{Hom}_{\mathbb{Y}}(F(X), Y) \cong \mathrm{Hom}_{\mathbb{X}}(X, G(Y)) \tag{15.1}$$

   for all objects X and Y. In particular, we can form a new category $\mathbb{X} \times \mathbb{Y}$, called the *product category*[1] Then we have the functor on the left-hand side of (15.1):

$$\begin{aligned} \mathbb{X} \times \mathbb{Y} &\xrightarrow{\text{LHS}} \mathsf{Set} \\ (X, Y) &\longmapsto \mathrm{Hom}_{\mathbb{Y}}(F(X), Y). \end{aligned}$$

   Similarly, we have the functor of the right-hand side of (15.1):

$$\begin{aligned} \mathbb{X} \times \mathbb{Y} &\xrightarrow{\text{RHS}} \mathsf{Set} \\ (X, Y) &\longmapsto \mathrm{Hom}_{\mathbb{X}}(X, G(Y)). \end{aligned}$$

   What does it mean for these to be functorially isomorphic? It means that there is an invertible natural transformation

$$\text{LHS} \xrightarrow{\text{natural transformation}} \text{RHS}.$$

   A *natural transformation* from the functor LHS to RHS is a rule T that assigns maps $T(A) : \text{LHS}(A) \longrightarrow \text{RHS}(A)$ for each object $A \in \mathbb{X} \times \mathbb{Y}$ such that for any morphism $A \longmapsto B$ in $\mathbb{X} \times \mathbb{Y}$ the following diagram commutes:

$$\begin{array}{ccc} \text{LHS}(A) & \xrightarrow{T(A)} & \text{RHS}(A) \\ \downarrow & & \downarrow \\ \text{LHS}(B) & \xrightarrow[T(B)]{} & \text{RHS}(B). \end{array}$$

   Exercise 15.1. Finish the proofs of the main adjunction theorems we had earlier in the course by showing that the isomorphisms

$$\mathrm{Hom}(F(X), Y) \cong \mathrm{Hom}(X, G(Y))$$

   we exhibited do indeed satisfy this property that we have here.

2. The other way of thinking about this adjunction is to define the natural transformations

$$\begin{aligned} FG &\xrightarrow{\varepsilon} \mathrm{id} \\ \mathrm{id} &\xrightarrow{\eta} GF \end{aligned}$$

[1] In particular it is a product in the category of categories.

that determine natural transformations

$$GFG \xrightarrow{\text{iso}} G$$
$$F \xrightarrow{\text{iso}} FGF.$$

We still need a bit more requirements that ε and η need to fulfill to ensure that this is a true adjunction, but this will be left to Adam Gerlach's summary of this material that he will present next week.

Now we suppose that we are working in categories that have exact sequences

Proposition 15.2. *Given an adjoint pair (F, G) of functors, F is right-exact and G is left-exact.*

Proof. Given a short exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$. To show that

$$F(A) \xrightarrow{F(\alpha)} F(B) \xrightarrow{F(\beta)} F(C) \longrightarrow 0$$

is exact, we first need to show that $F(\beta) = \operatorname{coker} F(\alpha)$.

Suppose that $F(A) \xrightarrow{F(\alpha)} F(B) \xrightarrow{\gamma} D$ is a complex, that is $\gamma \circ F(\alpha) = 0$. Then we have (we need to show)

$$\begin{array}{ccccc} F(A) & \xrightarrow{F(\alpha)} & F(B) & \xrightarrow{\gamma} & D \\ & & \downarrow{\scriptstyle F(\beta)} & \nearrow{\scriptstyle \exists!\delta} & \\ & & F(C), & & \end{array}$$

and thus $F(\beta) = \operatorname{coker} F(\alpha)$. Indeed, applying the functor G to this gives us

$$\begin{array}{ccccccc} GF(A) & \xrightarrow{GF(\alpha)} & GF(B) & \xrightarrow{G(\gamma)} & G(D) & \longrightarrow & 0 \\ \uparrow{\scriptstyle \eta(A)} & & \uparrow{\scriptstyle \eta(B)} & & & & \\ A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & & \end{array} .$$

By universal property of the cokernel, we know that $\beta = \operatorname{coker} \alpha$, so there is a unique map θ such that $\theta \circ \beta = G(\gamma) \circ \eta(B) = \gamma'$

$$\begin{array}{ccccccc} GF(A) & \xrightarrow{GF(\alpha)} & GF(B) & \xrightarrow{G(\gamma)} & G(D) & \longrightarrow & 0 \\ \uparrow{\scriptstyle \eta(A)} & & \uparrow{\scriptstyle \eta(B)} & \nearrow{\scriptstyle \gamma'} & \uparrow{\scriptstyle \exists!\theta} & & \\ A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & & \end{array} .$$

In particular, we have from the isomorphism

$$\operatorname{Hom}(F(B), D) \xrightarrow{\sim} \operatorname{Hom}(B, G(D))$$
$$\gamma \longmapsto \gamma'.$$

Applying the functor F to the above diagram gives a new diagram

$$\begin{array}{ccccccc} FGF(A) & \xrightarrow{FGF(\alpha)} & FGF(B) & \xrightarrow{FG(\gamma)} & FG(D) & \xrightarrow{\varepsilon(D)} & D \\ \uparrow{\scriptstyle \cong} & & \uparrow & & \uparrow{\scriptstyle F(\theta)} & \nearrow{\scriptstyle \delta = \varepsilon(D) \circ F(\theta)} & \\ F(A) & \xrightarrow{F(\alpha)} & F(B) & \longrightarrow & F(C) & & \end{array}$$

and we claim that $\delta \circ F(\beta) = \gamma$. That is,

$$\varepsilon(D) \circ \underbrace{F(\theta) \circ F(\beta)}_{=F(\theta\circ\beta)=F(\gamma')} = \gamma.$$

Exercise 15.3. Show uniqueness.

It still remains to prove the Lemma (which will also be done by Adam) that $\gamma = \varepsilon(D) \circ F(\gamma')$. □

Recall the theorem and corollary from last time (in particular, note that functors do not necessarily have unique adjoints, since both f_* and $f^!$ are right-adjoints to f^*):

Theorem 15.4. *The three functors $(f^*, f_*, f^!)$ is an adjoint tuple.*

Corollary 15.5. *f^* is right-exact, f_* is exact, and $f^!$ is left-exact.*

Example 15.6. One might have seen an example of this in Galois theory. Consider field extensions of $\mathbb{Q}$

$$\mathbb{Q} \text{ ——— } F \text{ ——— } \overline{\mathbb{Q}}.$$

and the automorphism functor $\mathrm{Aut}_{\mathbb{Q}} : \mathsf{FldExt}_{\mathbb{Q}-\overline{\mathbb{Q}}} \longrightarrow \mathsf{Grp}$

$$\mathrm{Gal}(F/\mathbb{Q}) = \mathrm{Aut}_{\mathbb{Q}}(F).$$

This takes field extensions to groups.

How can we pass from a group back to an extension? Consider subgroups $H \subset \mathrm{Aut}_{\mathbb{Q}}(\overline{\mathbb{Q}}) = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then we have a functor

$$\overline{\mathbb{Q}}^H \xleftarrow{\mathrm{Fix}_{\overline{\mathbb{Q}}/\mathbb{Q}}} H$$

where $\mathrm{Fix}_{\overline{\mathbb{Q}}/\mathbb{Q}}(H) = \overline{\mathbb{Q}}^H$ is the subfield of $\overline{\mathbb{Q}}$ of elements that is fixed pointwise by each element in H,

$$\mathbb{Q} \text{ ——— } \overline{\mathbb{Q}}^H \text{ ——— } \overline{\mathbb{Q}}.$$

(Addendum: Kaplansky extensions is an extension of Galois extension theory.)

Homework: In our readings of categories and functors, learn about what it means for two categories to be equivalent.

Back to whatever it was that we were doing before

Consider R and S in CRing.[2] Consider a ring homomorphism $f : R \longrightarrow S$.

1. We have the functor

$$f_* : S\text{-}\mathsf{Mod} \longrightarrow R\text{-}\mathsf{Mod}$$

   that takes an S-module M and defines an R-module $f_*(M)$ with the R-action $f \cdot m := f(r) \cdot m$, where the action here is the standard S-aciton. So we take right S-modules and create right R-modules.

2. We have the functor

$$f^* : R\text{-}\mathsf{Mod} \longrightarrow S\text{-}\mathsf{Mod}$$

   that takes right R-modules to right S-modules. Given an R-module A, this gives us

$$A \longmapsto A \otimes_{R \nearrow_f} S$$

   where we have $(a \otimes s_0) \cdot s = a \otimes (s_0 s)$.

[2]Aluffi really "drops the ball" when it comes to this section, so Clifton encourages us to read the corresponding sections in Hungerford on this topic, so that we are not restricted to commutative rings.

3. We have the functor

$$\begin{aligned} f^! : R\text{-}\mathsf{Mod} &\longrightarrow S\text{-}\mathsf{Mod} \\ B &\longmapsto \mathrm{Hom}_R(S, B) \end{aligned}$$

where for $\alpha : S \longrightarrow B$ we have the S-action $\alpha(r \cdot s) = r \cdot \alpha(s)$

We can use this to prove Theorem 15.4. [3]

1. If (f^*, f_*) is adjoint, then we have an isomorphism

$$\begin{array}{ccc} \mathrm{Hom}(f^*(A), B) & \cong & \mathrm{Hom}(A, f_*(B)) \\ \| & & \| \\ \mathrm{Hom}(A \otimes_R S, B) & & \mathrm{Hom}_R(A, B) \end{array}$$

but $\mathrm{Hom}(A \otimes_R S, B) \cong \mathrm{Hom}(A, \mathrm{Hom}(S, B))$. How is this related to $\mathrm{Hom}_R(A, B)$?

2. If $(f_*, f^!)$ is adjoint, then we have an isomorphism

$$\mathrm{Hom}(A, f_*(B)) \cong \mathrm{Hom}(f^!(A), B)$$

... (there was some more stuff here, but I ran out of time writing it down).

[3] Clifton is moving very fast here and getting sloppy regarding to noting which things belong to which categories, although in all cases it should be "obvious".

16 Lecture 16

(11 March 2014)

A little category theory

(a review of the past two weeks from Adam):

Recall: Suppose $F, G : \mathbb{X} \longrightarrow \mathbb{Y}$ is a functor. A *natural transformation* is a family of maps (in $\mathbb{Y}$) indexed by objects in $\mathbb{X}$, i.e. $(\alpha_X)_{X \in \mathbb{X}}$

$$\alpha_X : F(X) \longrightarrow G(X),$$

such that for any map $f : X \longrightarrow X'$ in $\mathbb{X}$, the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\alpha_X} & G(X) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(X') & \xrightarrow{\alpha_{X'}} & G(X') \end{array}$$

commutes.

Example 16.1. Let G be a group, and consider G as a category.

$$\text{objects:} \quad G$$

$$\text{maps:for each } \sigma \in G : \begin{array}{ccc} G & \xrightarrow{\sigma} & G \\ g & \longmapsto & \sigma \cdot g. \end{array}$$

We can consider functors $G \longrightarrow G$. In particular, we have the identity functor id_G. Natural transformations of the identity functor correspond to elements in the center of G.

$$\begin{array}{ccc} G & \xrightarrow{\alpha_G} & G \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\ G & \xrightarrow{\alpha_G} & G \end{array}$$

Definition 16.2. An *adjunciton* between two categories consists of two functors

$$F : \mathbb{X} \longrightarrow \mathbb{Y}$$
$$G : \mathbb{Y} \longrightarrow \mathbb{X}$$

and two natural transformations

$$\eta : \mathrm{id}_{\mathbb{X}} \longrightarrow G \circ F$$
$$\varepsilon : F \circ G \longrightarrow \mathrm{id}_{\mathbb{Y}},$$

i.e. $\eta_A : A \longrightarrow G(F(A))$ and $\varepsilon_B : F(G(B)) \longrightarrow B$, that satisfy the so-called "triangle equalities":

$$\begin{array}{ccc} F(A) & \xrightarrow{F(\eta_A)} & F(G(F(A))) \\ & \searrow\!\!\!\!= & \downarrow{\scriptstyle \varepsilon_{F(A)}} \\ & & F(A) \end{array} \qquad \begin{array}{ccc} G(B) & \xrightarrow{\eta_{G(B)}} & G(F(G(B))) \\ & \searrow\!\!\!\!= & \downarrow{\scriptstyle G(\varepsilon_B)} \\ & & G(B). \end{array}$$

Note: often this is written as $(\eta, \varepsilon) : F \dashv G : \mathbb{X} \to \mathbb{Y}$.

We have an "unproved" lemma floating around from previously:

Lemma 16.3. *Given an adjunction* $(\eta, \varepsilon) : F \dashv G : \mathbb{X} \to \mathbb{Y}$, *we have*

$$\mathrm{Hom}(F(X), Y) \cong \mathrm{Hom}(X, G(Y)).$$

Proof. We have an isomorphism given by

$$\gamma \overset{\varphi}{\longmapsto} G(\gamma) \circ \eta_X \qquad\qquad \varepsilon_Y \circ F(\gamma') \overset{\psi}{\longleftarrow\!\shortmid} \gamma'.$$

for $\gamma \in \mathrm{Hom}(F(X), Y)$ and $\gamma' \in \mathrm{Hom}(X, G(Y))$. Indeed, we have

$$\varphi(\psi(\gamma')) = \varphi(\varepsilon_Y \circ F(\gamma')) = G(\varepsilon_Y \circ F(\gamma')) \circ \eta_X = G(\varepsilon_Y) \circ G(F(\gamma')) \circ \eta_X = \gamma'$$

which we see from the commutative following diagram

$$\begin{array}{ccccc} X & \xrightarrow{\eta_X} & G(F(X)) & \xrightarrow{G(F(\gamma'))} & G(F(Y)) \\ & \searrow^{\gamma'} & & & \downarrow{\scriptstyle G(\varepsilon_Y)} \\ & & & & G(Y) \end{array}$$

which holds due to the triangle equality (why? Something here is missing... we need to make use of some universal property). On the otherhand, we have

$$\psi(\varphi(\gamma)) = \psi(G(\gamma) \circ \eta_X) = \varepsilon_Y \circ F(G(\gamma) \circ \eta_X) = \varepsilon_Y \circ F(G(\gamma)) \circ F(\eta_X) = \gamma$$

by a similar use of the triangle equality diagrams (and some universal property that we are still not sure of...). □

16.1 Tensor products over non-commutative rings

Consider R and S to be (not necessarily commutative) rings. Let A be a right R-module and B a left R-module. Then the definition of the tensor product is defined as

$$A \otimes_R B := \mathrm{Free}_{\mathsf{Ab}}(A \times B)/K$$

where $\mathrm{Free}_{\mathsf{Ab}} : \mathsf{Set} \longrightarrow \mathsf{Ab}$ is the functor that takes sets to free abeliean groups, and K is the abelian group generated by

$$\left\{ \begin{array}{c} (a_1 + a_2, b) - (a_1, b) - (a_2, b) \\ (a, b_1 + b_2) - (a, b_1) - (a, b_2) \\ (ar, b) - (a, br) \end{array} \middle| \, a, a_1, a_2 \in A,\ b, b_1, b_2 \in B,\ r \in R \right\}$$

Definition 16.4. An abelian group N is an (R, S)-***bimodule*** if it is a left-R-module and a right-S-module and

$$(rn)s = r(ns)$$

for all $n \in N$, $r \in R$ and $s \in S$.

Notation. Sometimes we use the notation ${}_R N_S$ to denote the (R, S)-bimodule structure.

Now suppose that ${}_R A_R$ and ${}_R B$ (i.e. A is an (R, R)-bimodule and B is a left-R-module), then $A \otimes_R B$ is a left-R-module with action

$$r(a \otimes b) = (ra) \otimes_R b.$$

Likewise, if A_R and ${}_RB_R$, then $A \otimes_R B$ is a right-R-module with action

$$(a \otimes b)r = a \otimes (br).$$

Finally, if ${}_RM_R$ and ${}_RN_S$, then $M \otimes_R N$ is an (R, S)-bimodule. (Refer to [2] to check that these really do have the R-module structures as claimed.

We can now state a more general version of our theorem regarding the adjunction of Hom and $\otimes_R$:

Theorem 16.5. *Let R and S be rings and ${}_RM_R$, ${}_RN_S$, ${}_SP_S$. Then*

$$\mathrm{Hom}_{\mathsf{Mod}\text{-}S}(M \otimes_R N, P) \cong \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(M, \mathrm{Hom}_{\mathsf{Mod}\text{-}S}(N, P)).$$

is a functorial isomorphism of abelian groups.

Furthermore, given an (R, S)-bimodule, the functors

$$- \otimes_R N : R\text{-}\mathsf{Mod}\text{-}R \longrightarrow R\text{-}\mathsf{Mod}\text{-}S$$
$$\mathrm{Hom}_{\mathsf{Mod}\text{-}S}(N, -) : S\text{-}\mathsf{Mod}\text{-}S \longrightarrow R\text{-}\mathsf{Mod}\text{-}S$$

form an adjoint pair $(- \otimes_R N, \mathrm{Hom}_{\mathsf{Mod}\text{-}S}(N, -))$.

(Something is not quite right here.... make sure to read up an this in the relevant section in Hungerford, since Aluffi does not have this theorem for non-commutative rings.)

Here we use the notation where $\mathsf{Mod}\text{-}R$ denotes the category of right-R-modules and $R\text{-}\mathsf{Mod}\text{-}S$ denotes the category of (R, S)-bimodules. Note that $M \otimes_R N$ is an (R, S)-bimodule, so it is a right-S-module, so it makes sense to talk about the abelian group $\mathrm{Hom}_{\mathsf{Mod}\text{-}S}(M \otimes_R N, P)$. Similarly, $\mathrm{Hom}_{\mathsf{Mod}\text{-}S}(N, P)$ has a left-R-module structure given by

$$(r \cdot \alpha)(n) := \alpha(rn)$$

for $\alpha : N \longrightarrow P$ and $r \in R$. Note that it is also a right-R-module with

$$(\alpha \cdot s)(n) := \alpha(n)s.$$

Since we are falling behind, we should look at the first three sections of chapter VI in Aluffi about free R-modules where R is an integral domain.

17 Lecture 17

(13 March 2014)

A few loose ends to tie up regarding Hom-Tensor duality.

Lemma 17.1. *Let R be a (commutative) ring. Let S be an R-algebra*[1] *and P be an S-module. Then*

i) $\mathrm{Hom}_{S\text{-}\mathsf{Mod}}(S,P) \cong P$ *as S-modules;*

ii) $\mathrm{Hom}_{S\text{-}\mathsf{Mod}}(S,P) \cong f_*(P)$ *as R-modules;*

iii) $P \otimes_S S \cong P$ *as S-modules;*

iv) $P \otimes_S S \cong f_*(P)$ *as R-modules,*

where we recal $f_*\colon S\text{-}\mathsf{Mod} \longrightarrow R\text{-}\mathsf{Mod}$ is a functor.

Proof. .

i) Note that, for an S-module homomorphism $\alpha\colon S \longrightarrow P$ $\alpha(s) = \alpha(s \cdot 1_S) = s\alpha(1_S)$. So we can define the bijections

$$\begin{array}{ccc} \mathrm{Hom}_{S\text{-}\mathsf{Mod}}(S,P) & \cong & P \\ \alpha & \longmapsto & \alpha(1_S) \\ \alpha_p & \longleftarrow & p \end{array}$$

and the compositions of these two maps is the identity in both directions. Here $\alpha_p\colon S \longrightarrow P$ is the map $\alpha_p(s) = ps$.

ii) This is left as an exercise.

iii) For pure tensors, we have $p \otimes s = s(p \otimes 1_S)$. We we can difine the bijections

$$\begin{array}{ccc} P \otimes_S S & \cong & P \\ p \otimes 1_S & \longleftarrow & p \\ p \otimes s & \longmapsto & sp. \end{array}$$

These are S-module homomorphisms (by extension via S-linearity), and composition yields the identity in both directions.

iv) Note that $f_*(P)$ is an R-module that consists of P plus an R-action given by $r \cdot p = f(r)p$. We can define an R-action on $P \otimes R$ by

$$r \cdot (p \otimes s) = p \otimes (r \cdot s) = (r \cdot p) \otimes p = (f(r)p) \otimes p = p \otimes f(r)s.$$

□

To fix the statement of the theorem from last time:

Theorem 17.2 (II,bis.)**.** *Let R and S be a rings. Then there is a functorial isomorphism of abelian groups*

$$\mathrm{Hom}_{\mathsf{Mod}\text{-}S}(M \otimes_R, P) \cong_{\mathsf{Ab}} \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(M, \mathrm{Hom}_{\mathsf{Mod}\text{-}S}(N,P))$$

where ${}_RM_R$, ${}_RN_S$ and ${}_SP_S$.

(*There was a mistake in the statement of this theorem, but this will be fixed in Aurora's review of this subject next week.*)

[1] Recall: this is simply given by a ring homomorphism $f\colon R \longrightarrow S$.

One final touch on adjoints: we had a problem with proof last time about adjunctions. There was an important definition that we were missing. This was the idea of a *universal pair*.

Definition 17.3. Let $G\colon \mathbb{X} \longrightarrow \mathbb{Y}$ be a functor. A ***universal pair*** for G at X is $(\mathcal{U}_X, \eta_X)$ such that for amy $Y \in \mathbb{Y}$ and $f\colon X \longrightarrow G(Y)$, there exists a unique $f^\sharp\colon \mathcal{U}_X \longrightarrow Y$ such that

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & G(\mathcal{U}_X) \\ & {\scriptstyle f}\searrow & \vdots{\scriptstyle G(f^\sharp)} \\ & & G(Y) \end{array}$$

commutes.

Theorem 17.4. *Let $G\colon \mathbb{X} \longrightarrow \mathbb{Y}$ be a functor such that for each $X \in \mathbb{X}$ there is a universal pair $(F(X), \eta_X)$. Then we have an adjoint $(\eta, \varepsilon)\colon F \dashv G\colon \mathbb{X} \longrightarrow \mathbb{Y}$. The converse is also true.*

This is the theorem we need to complete our proof about adjunctions from last time. A good reference for this theorem (and its proof) is in Maclane.

Lemma 17.5. *Given an adjoint $(\eta, \varepsilon)\colon F \dashv G\colon \mathbb{X} \longrightarrow \mathbb{Y}$, we have the canonical isomorphism $\mathrm{Hom}(F(X), Y) \cong \mathrm{Hom}(X, G(Y))$.*

Proof. We define the mappings

$$\begin{array}{ccc} \mathrm{Hom}(F(X), Y) & \cong & \mathrm{Hom}(X, G(Y)) \\ \gamma & \xrightarrow{\varphi} & G(\gamma) \circ \eta_X \\ \varepsilon_Y \circ F(\gamma') & \xleftarrow{\psi} & \gamma'. \end{array}$$

Composing these maps, for a $\gamma'\colon X \longrightarrow G(Y)$, we have

$$\varphi(\psi(\gamma')) = G(\varepsilon_Y) \circ G(F(\gamma')) \circ \eta_X.$$

We get that $\varphi(\psi(\gamma')) = \gamma'$ from the universal property and the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & G(F(X)) \\ & \searrow & \downarrow{\scriptstyle G(F(\gamma'))} \\ & {\scriptstyle \gamma'}\searrow & G(F(G(Y))) \\ & & \downarrow{\scriptstyle G(\varepsilon_Y)} \\ & & G(Y) \end{array}$$

since γ' must be unique by the universal property, but we already chose this map to be γ' by assumption. □

17.1 Free modules

Finally, we are turining the page and putting the Hom-Tensor duality behind us. We now study linear algebra with free R-modules. Recall that all free modules are isomorphic to a module $\mathrm{Free}_{R\text{-}\mathsf{Mod}}(S)$ for some set S. Given a finite set S, an R-module isomorphic to $\mathrm{Free}_{R\text{-}\mathsf{Mod}}(S)$ is said to be *finitely generated*.

Proposition 17.6. *Let R be an integral domain. Consider finite sets S. Then*

$$\mathrm{Free}_{R\text{-}\mathsf{Mod}}(S) \longmapsto |S|$$

classifies all finitely generated R-modules up to isomprphism.

Corollary 17.7 (VII.1.8 in Aluffi)**.** *Let R be an integral domain and A and B be sets. Then*

$$\mathrm{Free}_{R\text{-}\mathsf{Mod}}(A) \cong \mathrm{Free}_{R\text{-}\mathsf{Mod}}(B) \qquad \Longleftrightarrow \qquad A \cong B,$$

i.e. A and B have the same cardinality.

According to Clifton, this is all you need to know from Chapter VI section 1 in Aluffi. Moving on to VI.2 in the book:

Proposition 17.8. *Let R be an integral domain, $n, m \in \mathbb{N}$. Then*

$$\mathrm{Hom}_{R\text{-}\mathsf{Mod}}(R^n, R^m) \cong \{R\text{-module of } m \times n \text{ matrices with entries from } R\}.$$

However, this isomorphism is not unique! It depends on a choice of bases for R^n and R^m, where these are

$$R^n : = \mathrm{Free}_{R\text{-}\mathsf{Mod}}(\{1, 2, \ldots, n\}) = \underbrace{R \oplus \cdots \oplus R}_{n}.$$

Question: How does a choice of bases for R^n and R^m determine the isomorphism above? Note that R^n has a natural basis given by

$$R^n = \mathrm{span}_R\{e_1, e_2, \ldots, e_n\}$$

where these basis elements are given by

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \cdots, \qquad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

We can also define the standard *dual basis* $\{f_1 \ldots, f_n\}$ of the dual space.

Question: How do we determine the matrix elements of a given homomorphism? For a homomorphism $\alpha \in \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(R^n, R^m)$, the matrix is given by

$$[\alpha] = \begin{bmatrix} \vdots & & \vdots \\ [\alpha(e_1)]_{(f_1,\ldots,f_m)} & \cdots & [\alpha(e_n)]_{(f_1,\ldots,f_m)} \\ \vdots & & \vdots \end{bmatrix}$$

where the column vectors of this matrix are

$$[\alpha(e_1)]_{(f_1,\ldots,f_m)} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}, \qquad \alpha(e_i) = \sum_{j=1}^{m} f_j(e_i).$$

Clearly, any matrix of this form defines a R-module homomorphism $R^n \longrightarrow R^m$. So we have

$$\mathrm{Hom}_{R\text{-}\mathsf{Mod}}(R^n, R^m) \xrightarrow{\cong} \mathcal{M}_{m\times n}(R)$$

as R-modules (or abelian groups).

Question: Where does this go wrong if R is not an integral domain? We need a basis to define the matrices. But if R is not an integral domain then free R-modules do not necessarily have a basis. For example, $\mathbb{Z}/6\mathbb{Z}$ has no basis as a $\mathbb{Z}/6\mathbb{Z}$-module, since every element is linearly dependent.

Question: Supposing $n = m$, what is the image of the set of invertible maps α in $\mathrm{Hom}_{R\text{-}\mathsf{Mod}}(R^n, R^n)$ under this isomorphism? This is $\mathrm{GL}_n(R)$.

Example 17.9. Let $R = \mathbb{Z}$ and $n = 2$. Consider the homomorphism given by the matrix in $\mathcal{M}_2(\mathbb{Z})$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

This is *not* invertible in $\mathcal{M}_2(\mathbb{Z})$. What is the condition for a matrix of the form

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

to be invertible? It turns out that we need $ab \in \mathbb{Z}^*$, i.e. ab is a unit in $\mathbb{Z}$ (where units are invertible elements). In $\mathbb{Z}$, the units are $\{1, -1\}$.

18 Lecture 18

(18 March 2014)

Summary of last two weeks by Aurora:

- Review of f^*, f_* and $f^!$; adjoint triple $(f^*, f_*, f^!)$.
- From knowledge of adjoints, f^* is right-exact, f_* is exact, and $f^!$ is left-exact.
- Homomorphisms of free R-modules
- Finally she proved that $\mathrm{Hom}_{R\text{-}\mathsf{Mod}}(R^{\oplus n}, R^{\oplus m}) \cong \mathcal{M}_{m,n}(R)$.

18.1 The Determinant

For the rest of the day, let R be a CRing. We will use the notation $\mathfrak{gl}_n(R) \colon = \mathcal{M}_{n\times n} = \mathcal{M}_n$. The determinant is a function

$$\mathfrak{gl}_n(R) \xrightarrow{\det} R.$$

Since each row of an $n \times n$ matrix A can be viewed as an element of R^n, we can instead view the determinant as a R-multilinear function

$$R^n \times \cdots \times R^n \xrightarrow{\det} R.$$

A few properties of the determinant:

- it is *R-multilinear*, by which we mean that it is linear if we fix all but one arguments;
- it is also *skew-symmetric* (or *alternating*);
- $\det(I) = 1$.

In fact, the determinant is uniquely determined by these properties!

Theorem 18.1. *There can be only one!*

A better statement of this theorem is: There exists a unique function that satisfies the above properties.

Definition 18.2. Given a matrix $A \in \mathfrak{gl}_n(R)$, the ***determinant*** is given by

$$\det(A) := \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) \prod_i a_{i\sigma(i)}$$

where S_n is the group of permutations of $\{1, 2, \ldots, n\}$ and $\mathrm{sgn} : S_n \longrightarrow \{\pm 1\}$ is the sign character function

$$\mathrm{sgn}(\sigma) = \begin{cases} +1, & \text{even number of transpositions in } \sigma \\ -1, & \text{odd number of transpositions in } \sigma. \end{cases}$$

and the elements of the matrix A are given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}.$$

Theorem 18.3. *1. As defined above, the determinant is the unique skew-symmetric multilinear function* $\mathfrak{gl}_n(R) \longrightarrow R$ *satisfying* $I \overset{\det}{\longmapsto} 1_R$.

2. $\det(AB) = \det(A)\det(B)$.

3. $A \in \mathfrak{gl}_n(R)$ *is invertible if and only if* $\det(A) \in R^*$ *(group of units of* R*).*
In particular, the determinant defines a group homomorphism

$$\begin{aligned} \mathrm{GL}_n(R) &\longrightarrow R^* \\ A &\longmapsto \det(A) \end{aligned}$$

where $\mathrm{GL}_n(R) := \{A \in \mathfrak{gl}_n(R) \mid A \text{ invertible}\}$.

The proof follows analogously from what one learns in a course on linear algebra over a field.

Proof. Read VI.3 in Aluffi. □

Next time: Grothendiek groups and K-theory.

(I will do the summary next time! Perhaps I can introduce the wedge product.)

19 Lecture 19

(20 March 2014)

19.1 Euler Characteristic and the Grothendieck Group

Consider $R = k$ a field. Suppose we have finite-dimensional k-vector spaces U, V, W such that we have a short exact sequence

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0.$$

From the Rank plus Nullity Theorem, we have that

$$\dim_k(V) = \dim_k(U) + \dim_k(W).$$

So a short exact sequence of finite-dimensional vector spaces gives us an *Euler characteristic*

$$\dim(U) - \dim(V) + \dim(W) = 0$$

where we drop the index k.

We have the category $k\text{-}\mathsf{Vect}^f$ of finite-dimensional vector spaces over k. Consider the set of *isomorphism classes*[1] of objects in $k\text{-}\mathsf{Vect}^f$, denoted

$$\pi_0(k\text{-}\mathsf{Vect}^f)$$

(where π_0 is the standard terminology for the category of isomorphism classes of a category). That is, elements in $\pi_0(k\text{-}\mathsf{Vect}^f)$ are isomorphism classes $[V]$. Now consider the free abelian group generated by elements in $\pi_0(k\text{-}\mathsf{Vect}^f)$,

$$\mathrm{Free}_{\mathsf{Ab}}(\pi_0(k\text{-}\mathsf{Vect}^f))$$

with elements that look like

$$\sum_i n_i[V_i] \quad \text{with } n_i \in \mathbb{Z}.$$

Consider the subgroup G of elements generated by elements of the form $[U] - [V] + [W]$, where $0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$ is a short exact sequence, that is

$$G = \left\langle [U] - [V] + [W] \;\middle|\; 0 \to U \to V \to W \to 0 \text{ exact} \right\rangle.$$

We now define the *Grothendieck group* for the category of finite-dimensional k-vector spaces as

$$K_0(k\text{-}\mathsf{Vect}^f) = \mathrm{Free}_{\mathsf{Ab}}(\pi_0(k\text{-}\mathsf{Vect}^f))/G.$$

Now consider the group homomorphism $d\colon \mathrm{Free}_{\mathsf{Ab}}(\pi_0(k\text{-}\mathsf{Vect}^f)) \longrightarrow \mathbb{Z}$ given by

$$d(n_1[V_1] + n_2[V_2]) = n_1 \dim V_1 + n_2 \dim V_2.$$

This is clearly a surjective mapping, since for any positive integer $n \in \mathbb{Z}$ we can consider the isomorphism classes

$$[\underbrace{k \oplus \cdots \oplus k}_{n}] \longmapsto n.$$

Similarly, $[0] \longmapsto 0$ and $-[k^{\oplus n}] \longmapsto n$.

[1] Since we are only considering finite-dimensional spaces, the category of isomorphism classes does indeed form a set.

Lemma 19.1. *We have* $\ker d = G$.

Proof. The case $\ker d \supseteq G$ is trival, but $\ker d \subseteq G$ is left for the review. □

Importantly, the map $d\colon \mathrm{Free}_{\mathsf{Ab}}(\pi_0(k\text{-}\mathsf{Vect}^f)) \longrightarrow \mathbb{Z}$ induces a map

$$\begin{aligned} \delta\colon K_0(k\text{-}\mathsf{Vect}^f) &\longrightarrow \mathbb{Z} \\ [V] + G &\longmapsto d(V) = \dim V. \end{aligned}$$

In particular, this is an isomorphism.

Proposition 19.2. *The map* $\delta\colon K_0(k\text{-}\mathsf{Vect}^f) \longrightarrow \mathbb{Z}$ *is an isomrphism.*

In general, given any category C such that $\pi_0(\mathsf{C})$ forms a set, we can define a *Grothendieck group* of C, denoted $K_0(\mathsf{C})$, in a similar manner. In the category of finite-dimensional vector spaces, all short exact sequences split, so the Grothendieck group in this case is just the integers. However, for a category C with group $K_0(\mathsf{C})$, how different the group is from $\mathbb{Z}$ measures to what degree the short exact sequences in this category don't split.

19.1.1 Euler characteristic

Definition 19.3. Given a complex $V_\bullet$ of finite-dimensional vector spaces

$$0 \longrightarrow V_N \longrightarrow \cdots \longrightarrow V_1 \longrightarrow V_0 \longrightarrow 0,$$

the ***Euler charachteristic*** is defined as

$$\chi(V_\bullet) = \sum_i (-1)^i \dim(V_i).$$

Proposition 19.4 (Aluffi VI.3.13)**.** *The Euler characteristic of a complex of finite-dimensional vector spaces is given by the Euler characteristic of the homologies of the complex. That is,*

$$\chi(V_\bullet) = \sum_i (-1)^i \dim(V_i) = \sum_i (-1)^i \dim(H_i(V_\bullet)).$$

Proof. (See the proof in Aluffi, pp. 335-336) We proceed by induction. For $N = 0$, we have the complex

$$V_\bullet : 0 \longrightarrow V_0 \longrightarrow 0,$$

with $H_0(V_\bullet) = V_0$ and $H_i(V_\bullet) = 0$ for $i \neq 0$, so we are done.

If $N = 1$, we have the complex

$$V_\bullet : 0 \longrightarrow V_1 \xrightarrow{\alpha_1} V_0 \xrightarrow{\alpha_0} 0,$$

where the homologies are $H_1(V_\bullet) = \ker \alpha_1$ and

$$H_0(V_\bullet) = \ker \alpha_0 / \operatorname{im} \alpha_1 = V_0 / \operatorname{im} \alpha_1 = \operatorname{coker} \alpha_1.$$

But we have the short exact sequence

$$0 \longrightarrow \ker \alpha_1 \longrightarrow V_1 \longrightarrow \operatorname{coker} \alpha_1 \longrightarrow 0,$$

so $\dim V_1 = \dim(\ker \alpha_1) + \dim(\operatorname{coker} \alpha_1)$. In addition, since $V_0 = V_0/\operatorname{im} \alpha_1 \oplus \operatorname{coker} \alpha_1$, we have that $\dim(H_0(V_\bullet)) = \dim(V_0/\operatorname{im} \alpha_1) = \dim(V_0) - \dim(\operatorname{coker} \alpha_1)$. Hence

$$\begin{aligned} -\dim(H_1(V_\bullet)) + \dim(H_0(V_\bullet)) &= -(\dim(\ker \alpha_1)) + (\dim(V_0) - \dim(\operatorname{coker} \alpha_1)) \\ &= -\dim(V_1) + \dim(V_0) \\ &= \chi(V_\bullet). \end{aligned}$$

For an arbitrary $N \geq 2$, we can truncate the complex $V_\bullet$

$$V_\bullet : 0 \longrightarrow V_N \longrightarrow V_{N-1} \longrightarrow \cdots \longrightarrow V_1 \longrightarrow V_0 \longrightarrow 0,$$

to a shorter complex

$$V'_\bullet : 0 \longrightarrow V_{N-1} \longrightarrow \cdots \longrightarrow V_1 \longrightarrow V_0 \longrightarrow 0,$$

and proceed by induction. Note that the homologies are almost all the same. Namely,

$$H_i(V_\bullet) = H_i(V'_\bullet) \qquad \text{for all } i \leq N-2.$$

It remains to find what $H_N(V_\bullet)$ and $H_{N-1}(V_\bullet)$ are.

See Aluffi for the remainder of the proof by induction. □

20 Lecture 20

(25 March 2014)
(Guest lecturer – Clifton is gone today)

20.1 Presentations and Resolutions of Modules

If we have a free module, then arithmetic is really easy: we just need to know how to manipulate elements of the ring R. But if R is not a field, then not every module is free. So it's nice to have tools to be able to look at and characterize other types of modules.

20.1.1 Torsion

Definition 20.1. Suppose M is an R-module. An element $m \in M$ is ***torsion*** if there exists an $r \in R$ with $r \neq 0$ such that $rm = 0$. (In the language of linear algebra, we can say that $\{m\}$ is linearly independent.) The set of all torsion elements of M is denoted by[1] $\mathrm{Tor}_R(M)$, and we can drop the 'R' if it is obvious which ring we are working with. A ***torsion module*** is a module M such that $M = \mathrm{Tor}(M)$, and M is ***torsion-free*** if $\mathrm{Tor}(M) = 0$.

If R has zero divisors, then examining torsion elements doesn't make much sense. So for now we will assume that R is an integral domain (no zero divisors).

Lemma 20.2. *The direct sum of torsion-free modules is torsion-free. Submodules of torsion-free modules are torsion-free.*

In a direct sum, arithmetic is done component-wise, so it will not have torsion elements.

Example 20.3.
- Let $R = \mathbb{Z}[x]$ and $M = (2, x) \subset R$ be an ideal. This is torsion-free since it is a submodule over a torsion-free, but it is not free.
- All free modules are torsion-free.
- Consider the module $M = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
  - Over $R = \mathbb{Z}/2\mathbb{Z}$, this module is torsion-free.
  - Over $R = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The zero element $(0, 0)$ is always torsion. Note that $(0, 1)$ and $(1, 0)$ are torsion elements, but $(1, 1)$ is not in this case.
  - Over $R = \mathbb{Z}$, every element is torsion, so M is a torsion module over $\mathbb{Z}$.

Definition 20.4. An R-module M is ***cyclic*** if one of the following (equivalent statements) holds:

i) M is generated by one element x;

ii) There is an exact sequence

$$R^1 \longrightarrow M \longrightarrow 0$$

iii) $M \cong R/I$ for some ideal I.

Lemma 20.5. *Suppose every cyclic R-module is torsion-free. Then R is a field.*

[1] Not to be confused with the Tor functor that we had been dealing with earlier.

Proof. (For now, we are still assuming that R is an integral domain. It is left as an exercise to show this for other rings.) Suppose $c \in R$ with $c \neq 0$. Then we get an exact sequence

$$(c) \longrightarrow R \longrightarrow R/(c) \longrightarrow 0$$

where (c) is the ideal generated by c. Since $R/(c)$ is torsion-free, either $R/(c) = 0$, in which case $R = (c)$ and thus $c \in R^\times$, or $(c) = 0$. Hence, every non-zero element is a unit and thus R is a field. □

20.1.2 Finitely generated modules and reslutions

Definition 20.6. An R-module is ***finitely generated*** if there exists an integer m such that one of the following (equivalent statements) holds:

i) M is generated by m elements;

ii) there is an exact sequence

$$R^n \xrightarrow{\pi} M \longrightarrow 0$$

iii) $M \cong R^m/N$ for some submodule N of R^m.

Definition 20.7. An R-module is ***finitely presented*** it is finitely generated and there is an exact sequence

$$R^m \xrightarrow{\varphi} R^n \xrightarrow{\pi} M \longrightarrow 0 \quad ,$$

or, equivalently, if $M \cong R^m/N$ for some fintely generated submodule N of R^m.

Definition 20.8. Let M be an R-module. The ***annihilator*** of M is the ideal

$$\mathrm{Ann}_R(M) := \{r \in R \,|\, rm = 0 \,\forall m \in M\}\,.$$

For an element $x \in M$, we can similarly define the annihilator

$$\mathrm{Ann}_R(x) := \{r \in R \,|\, rx = 0\}\,.$$

Remark 20.9. If M is finitely generated, then M is torsion if and only if $\mathrm{Ann}(M) \neq 0$.

Proof. Suppose M is torsion. Choose generators $x_1, \ldots, x_m$ and $r_1, \ldots, r_m$ such that $r_i x_i = 0$. Then the element $\prod_i r_i$ annihilates everything in M. □

Example 20.10. What are some examples of *non*-finitely generated torsion modules such that $\mathrm{Ann}(M) = 0$?

1. Take $M = \bigoplus_p \mathbb{Z}/p\mathbb{Z}$. Then each element has an element of $\mathbb{Z}$ that kills it, but there is no integer that kills everything.
2. $\mathbb{Q}/\mathbb{Z}$ is another module that commonly comes up that is not finitely generated, is torsion, and as zero annihilator.

Lemma 20.11. *Suppose R is Noetherian*[2]*. Then all finitely generated modules are finitely presented.*

[2] We haven't covered this in this class, so we will skip it

Recall that a ring R is *Noetherian* if every ascending chain of ideals is finite. That is, if we have a chain of ideals
$$\cdots \supseteq I_n \supseteq \cdots \supseteq I_2 \supseteq I_1 \supseteq I$$
then eventually $I_n = I_{n+1}$ for all $n \geq N$ for some N. An example of a non-Noetherian ring is a polynomial ring over infinitely many variables.

If we go back to the definitions of *finitely generated* and *finitely presented*, we see from part (ii) that we are constructing exact sequences by attaching free modules to one side. We have a more general name for this.

Definition 20.12. Let M be an R-module. A ***resolution*** of M is an exact sequence

$$\cdots \longrightarrow M_3 \longrightarrow M_2 \longrightarrow M_1 \longrightarrow M \longrightarrow 0.$$

The ***length*** of a resolution is the largest integer n such that $M_n \neq 0$.

We'll be interested in *free resolutions* where $M_i \cong R^{m_1}$ are all free. Note here, that we define free resolutions to have all terms be fintiely generated. If we are working over a Noetherian ring, then every module is guaranteed to have a free resolution. Otherwise, there might be "free" resolutions , but there might not be one where all of the module terms in the resolution are finitely generated.

Lemma 20.13. *If R is Noetherian, then every module M has a free resolution.*

Note that, in our definitions of finitely generated and presented, we have already been working with free resolutions of length 1 and 2.

Proposition 20.14. *Let R be an integral domain. Then R is a field if and only if every finitely generated R-module is free.*

Proof. If R is a field, then every module is free. So suppose that every finitely generated R-module is free. Then every cyclic module is free, and every free module is torsion-free (since R is an integral domain). By Lemma 20.5 (VI.4.5 in Aluffi), R is a field. □

Proposition 20.15 (VI.4.11 in Aluffi)**.** *Suppose that every R-module M has a length 1 resolution. Then R is a PID.*

Proof. Let $I \subseteq R$ be an ideal. Then R/I is an R-module and we have a length 1 resolution

$$R^1 \longrightarrow R/I \longrightarrow 0$$

and we have the short exact sequence

$$0 \longrightarrow I \longrightarrow R^1 \longrightarrow R/I \longrightarrow 0.$$

Since we have a length 1 free resolution, we have that I is free and thus I has one generator. So R is a PID. □

How are we sure that *this* resolution was the right one? What if we chose the wrong one?

20.1.3 Reading a presentation

Suppose we have a finite presentation

$$R^n \xrightarrow{\varphi} R^m \xrightarrow{\pi} M \longrightarrow 0,$$

then $M \cong \operatorname{coker}(\varphi)$. But, as we have already seen before, any homomorphism of free R-modules

$$R^n \xrightarrow{\varphi} R^m$$

can be viewed as a matrix, say A.

Lemma 20.16. *If M_1 and M_2 are free resolutions with corresponding matrices A_1 and A_2, then the matrix corresponding to the finitely presented module $M_1 \oplus M_2$ is the block matrix*

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2. \end{pmatrix}$$

Note that this matrix A is not unique, since it depends on a choice of basis. Different matrices can represent the same φ, and different φ can represent the same M. If we have an isomorphism $\varphi : R^r \longrightarrow R^r$, then

$$R^r \xrightarrow{\varphi} R^r \longrightarrow 0$$

and the matrix C that represents φ is invertible, and it cna only represent the zero module. Then if A represents some module M and $C \in \mathrm{GL}_r(R)$ is an invertible matrix, then the matrix

$$\begin{pmatrix} C & 0 \\ 0 & A. \end{pmatrix}$$

also represents M (since it really represents $0 \oplus M \cong M$). Furthermore, an $m \times n$ matrix A represents the same map as PAQ when $P \in \mathrm{GL}_m(R)$ and $Q \in \mathrm{GL}_n(R)$ are invertible matrices.

Proposition 20.17 (VI.4.13 in Aluffi)**.** *Let $A \in \mathcal{M}_{m\times n}$ be a matrix and suppose $B \in \mathcal{M}_{m\times n}$ is a matrix obtained from A by*

i) row and column operations;

ii) if a unit is the only nonzero entry in a row, then delete that row and column.

Then B represents the same module as A.

(see the book for details).

21 Lecture 21

(27 March 2014)

(Notes copied from German Luna in my absence)

21.1 Classification of finitely generated modules over a PID

Definition 21.1. Let R be an integral domain. The ***rank*** $\operatorname{rank} M$ of a finitely generated R-module M is the maximum number of linearly independent elements in M.

Recall that the spectrum of a ring $\operatorname{Spec} R$ is the set of all proper prime ideals of R

Theorem 21.2 (Theorem 4 of the course)**.** *Let R be a PID, and M a finitely generated R/module.*

- *There exist $\beta_i \in \operatorname{Spec}(R)$ and $r_{ij} \in \mathbb{Z}^+$ such that*

$$M \cong R^{\operatorname{rank} M} \oplus \left(\bigoplus_{i,j=1}^{n} R/\beta_j^{r_{ij}} \right).$$

- *There exists ideals $I_1 \supset I_2 \supset \cdots \supset I_m$ of R such that*

$$M \cong R^{\operatorname{rank} M} \oplus \left(\bigoplus_{k=1}^{m} R/I_k \right).$$

Example 21.3. Consider the PID $R = \Bbbk[t]$ where $\Bbbk$ is a field, and a vector space $M = V$ over $\Bbbk$. Give V the R-module structure in the following manner. For $f(t) \in \Bbbk[t]$ with $f(t) = \sum_{i=0}^{n} f_i t^i$ and a $v \in V$, the R-action is defined by

$$f(t)v = \sum_{i=0}^{n} (f_i t^i) v_i = \sum_{i=0}^{n} f_i \left(t^i \cdot v \right)$$

where $t\cdot \in \operatorname{End}_{\Bbbk\text{-}\mathsf{Vect}}(V)$.

Thus, given a $\Bbbk$-vector space V, there is a bijection between the set of $\Bbbk[t]$-module structures on V and $\operatorname{End}_{\Bbbk\text{-}\mathsf{Vect}}(V)$.

Corollary 21.4. *Given a $\Bbbk[t]$-module V (which has a $\Bbbk$-action and thus a $\Bbbk$-module structure), we have*

$$V \cong_{\Bbbk[t]\text{-}\mathsf{Mod}} \bigoplus_{i,j=1}^{n} \frac{\Bbbk[t]}{p_i(t)^{r_{ij}}}$$

where each $p_i(t) \in \Bbbk[t]$ is prime and $r_{ij} \in \mathbb{Z}^+$, or equivalently

$$V \cong_{\Bbbk[t]\text{-}\mathsf{Mod}} \bigoplus_{k=1}^{m} \frac{\Bbbk[t]}{(f_k(t))}$$

with f_i's monic (wlog) and $f_i | f_{i+1}$ for all $1 \leq i \leq m-1$.

Note that the factors are determined by the endomorphism.

Example 21.5. Consider the ideal $I \subset \mathbb{Q}[t]$ given by $I = (t^2 - 2)$, then we have the isomorphism $a + tb + I \longmapsto a + b\sqrt{2}$

$$\frac{\mathbb{Q}[t]}{(t^2-2)} \cong \mathbb{Q}(\sqrt{2}).$$

Example 21.6. If $0 \neq f(t) \in \Bbbk[t]$ is irreducible, then $(f(t))$ is maximal so $\Bbbk[t]/(f(t))$ is a field. For example,

$$\frac{\mathbb{Q}[t]}{(t^3-1)} \cong \frac{\mathbb{Q}[t]}{(t-1)} \oplus \frac{\mathbb{Q}[t]}{(t^2+t+1)}$$

this is not a field, but it is a $\mathbb{Q}$-algebra and thus a $\mathbb{Q}$-module.

Question: Is a $\Bbbk[t]$-module a $\Bbbk[t]$-algebra? For example, we have

$$\frac{\mathbb{Q}[t]}{t^2} = \mathrm{span}_{\mathbb{Q}}\{1, t\}$$

which has nilpotent elements, i.e. $(t^2) = 0$, and similarly it has torsion as a $\mathbb{Q}[t]$-module.

Question: What are the linear transformations defined by these quotients? In

$$\frac{\mathbb{Q}[t]}{(t^2-2)} \cong \mathrm{span}\{1, t\}$$

note that $t(a + bt) \longmapsto 2b + at$. The matrix representation of this linear map is

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix},$$

but $\mathrm{charpoly}(A) = t^2 - 2$!

Now, what are the elementary divisors and invariant factors of T (where T is the linear transformation defined by A)? In this case, they're all the same thing: the characteristic polynomial.

Example 21.7. .

1. Consider $\dfrac{\mathbb{Q}}{(t^3-1)} \cong \mathrm{span}\{1, t, t^2\}$, with an isomprphism

$$t(a + bt + ct^2) \longmapsto (c + at + bt^2).$$

   The matrix corresponding to this linear transformation is

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

2. Consider $\dfrac{\mathbb{Q}}{(t^3)} \Rightarrow \mathrm{span}\{1, t, t^2\}$,

$$T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

3. Consider $\dfrac{\mathbb{Q}}{((t-1)^3)} \Rightarrow \mathrm{span}\{1, t, t^2\}$,

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & 3 \end{bmatrix}.$$

Note 21.8. In general, for a monic and irreducible $f(t)$,

$$\frac{\Bbbk[t]}{(f(t))} \Rightarrow \operatorname{span}_{\Bbbk}\{1, t, \ldots, t^{(\deg f - 1)}\}$$

with $f(t) = \sum_{i=0}^{n} f_i t^i = f^{\mathsf{T}} \cdot p(t)$ where

$$p(t) = \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^{(\deg f - 1)} \end{bmatrix}.$$

Then the linear transformation will be given by

$$T = \left[\begin{array}{c|c} \vec{0}^T & \\ \hline I_{n-1} & f \end{array}\right]$$

and this is called the *rational canonical form.*

If $f(t)$ is irreducible, then we have a block diagonal matrix with blocks in rational canonical form.

22 Lecture 22

(1 April 2014)

(Reviews given by me and Mineko(?))

Definition 22.1. Let R be a commutative ring and M and R-module. The ***dual*** of M is the R-module

$$M^\vee = \mathrm{Hom}_R(M, R).$$

Proposition 22.2. *Let R be a commutative ring, M and F be R-modules where F is free with finite rank. Then*

$$M^\vee \otimes_R F \cong \mathrm{Hom}_R(M, F).$$

Recall: An R-module M is

i) flat if

ii) projective if

iii) injective if

23 Lecture 23

(3 April 2014)

(Continuation of presentation Mineko)

Proposition 23.1 (Exercise VIII.5.5 in Aluffi)**.** *Let R be a commutative ring, M and F be R-modules where F is free with finite rank. Then*

$$M^\vee \otimes_R F \cong \mathrm{Hom}_R(M, F).$$

Proof. Let $\varepsilon\colon M^\vee \otimes_R F \longrightarrow \mathrm{Hom}_R(M,F)$ be given by

$$\begin{aligned} f \otimes x \longmapsto \varepsilon(f \otimes x)\colon M &\to F \\ m &\mapsto f(m) \cdot x. \end{aligned}$$

Let $\gamma\colon \mathrm{Hom}_R(M,F) \longrightarrow M^\vee \otimes_R F$ be given by

$$(h\colon M \to F) \longmapsto \sum_{i=1}^{n} h_i \otimes e_i$$

where e_i are the standard basis elements of $F \cong R^n$, and $h(m) = \sum_{i=1}^n h_i(m)e_i$. To check that γ and ε are inverses of each other, we have

$$\begin{aligned} \varepsilon \circ \gamma(h) &= \varepsilon\big(\sum_{i=1}^{n} h_i \otimes e_i\big) \\ &= \sum_{i=1}^{n} \varepsilon(h_i \otimes e_i) \end{aligned}$$

and thus $(\varepsilon \circ \gamma(h))\,(m) = \displaystyle\sum_{i=1}^{n} h_i(m) \cdot e_i = h(m)$ by definition. Similarly, we have

$$\big(\gamma \circ \varepsilon(f \otimes x)\big)(h) = \gamma\big(\big)$$

Note that

$$f \otimes x = f \otimes \left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i f \otimes e_i$$

and thus $\varepsilon\,(f \otimes x)\,(m) = \displaystyle\sum_{i=1}^{n} x_i f(m)$

□

23.1 Reading a presentation (reprise)

Let M be a finitely presented R-module with presentation

$$R^n \xrightarrow{\ \varphi\ } R^m \xrightarrow{\ \pi\ } M \longrightarrow 0.$$

For example, lets consider $R = \mathbb{Z}$ and the map $\varphi\colon R^2 \longrightarrow R^3$ given by

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 5 & 9 \end{bmatrix}.$$

Performing standard row reduction, we obtain

$$\begin{bmatrix}1 & 3\\ 2 & 3\\ 5 & 9\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 3\\ 1 & 0\\ 5 & 9\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 0\\ 1 & 3\\ 5 & 9\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 0\\ 0 & 3\\ 0 & 9\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 0\\ 0 & 3\\ 0 & 0\end{bmatrix}.$$

Namely, there is an invertible matrix A in $\mathrm{GL}_3(\mathbb{Z})$ such that

$$A\begin{bmatrix}1 & 3\\ 2 & 3\\ 5 & 9\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 3\\ 0 & 0\end{bmatrix}.$$

The cokernel of this new matrix will be isomorphic to the original one, and this one is a lot easier to study. This gives us the free resolution

$$0 \longrightarrow R^2 \xrightarrow{\varphi} R^3 \longrightarrow R/3R \oplus R \longrightarrow 0.$$

But, given $M = R/3R \oplus R$, it would probably be more natural to find the free resolution as

$$0 \longrightarrow R \xrightarrow{\varphi} R^2 \longrightarrow R/3R \oplus R \longrightarrow 0.$$

where the maps in this sequence are

$$\begin{aligned} R &\longrightarrow R^2 \\ x &\longmapsto \begin{bmatrix}3x\\ 0\end{bmatrix}\end{aligned} \qquad\qquad \begin{aligned} R^2 &\longrightarrow R/3R \oplus R \\ \begin{bmatrix}x\\ y\end{bmatrix} &\longmapsto \begin{bmatrix}x \mod 3\\ y.\end{bmatrix}\end{aligned}$$

Note that the Euler characteristic of both of these complexes are equal to 1.

23.2 Free resolutions and homology

Consider an exact sequence of R-modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

and tensor this with another R-module N which yields the exact sequence

$$- \qquad \longrightarrow A \otimes_R N \longrightarrow B \otimes_R N \longrightarrow C \otimes_R N \longrightarrow 0.$$

Adding a 0 to the left of this does not in general produce an exact sequence! What should we put in this spot in order to make it exact? It needs to be the kernel of the map $A \otimes_R N \longrightarrow B \otimes_R N$, which we will call $\mathrm{Tor}_1^R(A, N)$, so that the sequence

$$- \qquad \longrightarrow \mathrm{Tor}_1^R(C, N) \longrightarrow A \otimes_R N \longrightarrow B \otimes_R N \longrightarrow C \otimes_R N \longrightarrow 0.$$

is exact. Continuing this process, we define a series of R-modules that produces the long exact sequence

$$\begin{aligned} \cdots \longrightarrow \mathrm{Tor}_2^R(A, N) \longrightarrow \mathrm{Tor}_2^R(B, N) \longrightarrow \mathrm{Tor}_2^R(C, N) \longrightarrow \\ \longrightarrow \mathrm{Tor}_1^R(A, N) \longrightarrow \mathrm{Tor}_1^R(B, N) \longrightarrow \mathrm{Tor}_1^R(C, N) \longrightarrow \\ \longrightarrow A \otimes_R N \longrightarrow B \otimes_R N \longrightarrow C \otimes_R N \longrightarrow 0. \end{aligned}$$

How can we compute $\mathrm{Tor}_i^R(M, N)$ in general? We have the following recipe:

(1) Find a free resolution of M (if possible[1]):

$$\cdots \longrightarrow R^{m_2} \longrightarrow R^{m_1} \longrightarrow R^{m_0} \longrightarrow M \longrightarrow 0.$$

Truncating this gives us a complex (that is exact at most spots, but not going to be exact at the last spot!)

$$M_\bullet\colon \quad \cdots \longrightarrow R^{m_2} \longrightarrow R^{m_1} \longrightarrow R^{m_0} \longrightarrow 0.$$

(2) Take this complex and tensor it with N

$$M_\bullet \otimes_R N\colon \quad \cdots \longrightarrow R^{m_2} \otimes_R N \longrightarrow R^{m_1} \otimes_R N \longrightarrow R^{m_0} \otimes_R N \longrightarrow 0.$$

There is no reason for this sequence to be exact. In fact, it most likely isn't.

(3) Compute the homologies $H_i(M_\bullet \otimes_R N) = \mathrm{Tor}_i^R(M, N)$.

This recipe is well-defined (that is, it does not depend on the choice of free resolution of M).

Exercise 23.2. Show that

1. $\mathrm{Tor}_0^R M, N) = M \otimes_R N$;
2. For $N = R, R^2, R/3R$ and $R/4R$ (or whatever other modules you like), calculate

$$\mathrm{Tor}_i^R(R/3R \oplus R, N)$$

for $i = 1, 2, 3, \ldots$.

Note that free modules are flat, and thus $\mathrm{Tor}_i^R(M, N) = 0$ for $i > 0$ if N or M is flat. In particular, we have

$$\left(\bigoplus_{i=1}^m R\right) \otimes_R N \cong \bigoplus_{i=1}^m (R \otimes_R N)$$

[1] It may not be possible to find such a resolution, for example if M is not finitely generated

24 Lecture 24

(10 April 2014)

Today we will discuss the Jordan normal form and finish our proof of Theorem IV (classification theorem of finitely generated R-modules over a PID) of the course. In our last lecture next week we will introduce the Ext functor.

Note that there was some discrepancy in the 'old' and and 'revised' course outlines. If you did not solve the homework questions on the revised outline, then you should still prepare for these.

24.1 Jordan normal form

If $\mathbb{k}$ is a field, note that $\mathbb{k}[t]$ is a PID and we can view

$$\frac{\mathbb{k}[t]}{\big(f(t)\big)}$$

as a $\mathbb{k}[t]$-module. We can consider an action on this given by $[\cdot t]$, that is multiplication by t.

Example 24.1. Consider the polynomial $f(t) = \prod_i (t-\lambda_i)^{r_i}$, where $\lambda_i \in \mathbb{k}$ are all distinct. What can we say about the module

$$\frac{\mathbb{k}[t]}{\big(f(t)\big)} = \frac{\mathbb{k}[t]}{\left(\prod_i (t-\lambda_i)^{r_i}\right)}?$$

By the Chinese remainder theorem (and the classification theorem), we can view this module as

$$\frac{\mathbb{k}[t]}{\left(\prod_i (t-\lambda_i)^{r_i}\right)} \cong \bigoplus_{i=1}^{n} \frac{\mathbb{k}[t]}{\big((t-\lambda_i)^{r_i}\big)}.$$

Note that with the action $[\cdot t]\circlearrowleft\frac{\mathbb{k}[t]}{\left((t-\lambda)^r\right)}$ can represent the action of $[\cdot t]$ in the following basis:

$$\left\{1 = (t-\lambda)^0, t-\lambda, (t-\lambda)^2, \ldots, (t-\lambda)^{r-1}\right\}.$$

Note that we have $(t-\lambda)(t-\lambda)^j = (t-\lambda)^{j+1}$ and

$$t(t-\lambda)^j = \lambda(t-\lambda)^j + 1(t-\lambda)^{j+1}.$$

So in this basis, $[\cdot t]$ is

$$[\cdot t]_{\text{basis}} = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & 1 \\ & & & & \lambda \end{bmatrix}.$$

So $\cdot t$ acting on the whole space $\frac{\mathbb{k}[t]}{\left(\prod_i (t-\lambda_i)^{r_i}\right)}$ is

$$[\cdot t] = \begin{bmatrix} \begin{bmatrix} \lambda_1 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_1 \end{bmatrix} & & \\ & \begin{bmatrix} \lambda_2 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_2 \end{bmatrix} & \\ & & \ddots \end{bmatrix}$$

24.2 "Applications"

Assume that your field is algebraically closed (for the moment)... We want to classify the 'orbits' (conjugacy classes) of nilpotent matrices in $\mathfrak{gl}_n(\mathbb{C})$. Let $x \in \mathfrak{gl}_2(\mathbb{C})$. Then x is nilpotent if and only if it is similar to one of the two matrices

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

These exactly correspond to the the partitions of 2: $[1,1]$ and $[2]$. So nilpotent conjugacy classes in $\mathfrak{gl}_n[\mathbb{C}]$ are classified by partitions of n.

$$\vdots$$

$$\mathfrak{gl}_2(\mathbb{C})_{\text{nilp}} = \left\{ \begin{bmatrix} z & x \\ y & -z \end{bmatrix} \middle| xy + z^2 = 0 \right\}.$$

[1]

24.3 Back to Theorem IV

Theorem 24.2 (Theorem IV)**.** *Let R be a PID and M a finitely generated R-module. Then*

$$M \cong R^r \oplus \mathrm{Tor}_R(M)$$

where $r = \mathrm{rank}(M)$ and $\mathrm{Tor}_R(M)$, and

$$\mathrm{Tor}_R(M) \cong \bigoplus_j \frac{R}{I_j}$$

where $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_m$ and $I_j = (a_j)$ where $a_1 | a_2 | \cdots | a_m$.

Given two free R-modules R^n and R^m and a homomorphism, we can represent this as a matrix and find the cokernel

$$0 \longrightarrow R^m \xrightarrow[A]{} R^n \longrightarrow M \longrightarrow 0$$

where the matrix A is in elementary-reduced form

$$A = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_m \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

and we can see that $M \cong R^r \oplus \bigoplus_j \frac{R}{a_j}$ where $r = n - m$. In particular, we see the following proposition.

Proposition 24.3. *If R is a PID then every finitely generated R-module admits a resolution of length 1.*

[1] Aside: "Exceptional" groups has to do with classification of Lie groups

Proposition 24.4. *Let R be a PID and F a finitely generated free R-module, $F \cong R^n$. Let $M \subset F$ be a submodule.*

M is free and there exists a basis $\{x_1, \dots, x_n\}$ for F and $\{a_1, \dots, a_m\}$ with $m \le n$ and $a_i \in R$ such that $\{a_1x_1, \dots, a_mx_m\}$ is a basis for M.

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \xrightarrow{\text{given}} & F \cong R^n \\ & & \big\downarrow \cong & \nearrow_{A} & \\ & & R^m & & \end{array}$$

where A is a matrix

$$A = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_m \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

We will prove the proposition by induction. The base case is given by the following lemma.

Lemma 24.5. *$M = \langle ax \rangle \oplus M$ and $F = \langle x \rangle \oplus F'$.*

$$\begin{array}{ccccc} & & \langle ax \rangle \cong R & & \langle x \rangle \cong R \\ & & \updownarrow & & \updownarrow [a] \\ 0 & \longrightarrow & M & \longrightarrow & F \cong R^n \\ & & \updownarrow & & \updownarrow \\ 0 & \longrightarrow & M' & \longrightarrow & F' \\ & & \downarrow & & \downarrow \\ & & 0 & & 0 \end{array}$$

Proof. (of Theorem IV) Start with a finitely generated R-module M. So we get a surjection and the exact sequence $R^n \longrightarrow M \longrightarrow 0$. Complete it to the exact sequence

$$0 \longrightarrow R^m \xrightarrow[A]{} R^n \qquad \text{where} \quad A = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_m \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

□

Proof. (of the Lemma) Consider $F^\vee = \mathrm{Hom}_{R\text{-Mod}}(F, R)$ and let $\varphi \in F^\vee$.

$$\begin{array}{ccc} \varphi\colon F & \longrightarrow & R \\ \big\uparrow & \overset{\varphi|_M}{\nearrow} & \\ M & & \end{array}$$

Note that $\operatorname{im}(\varphi|_M) = \varphi(M) \lhd R$ (is an ideal of).

Consider the family of ideals that arise in this way

$$\{\varphi(M) \,|\, \varphi \in F^\vee\}.$$

By Zorn's lemma, there exists a maximal one. So by Max[2] (care of Emmy[3]) there is an $\alpha \in F^\vee$ with $\alpha(M) \lhd R$ that is a maximal ideal.

So $\alpha(M) = (a)$ for some a that we can pick. Furthermore, pick $y \in M$ such that $\alpha(y) = a$.

Claim: For all $\varphi \in F^\vee$, we have that $a|\varphi(y)$.

Proof of claim. Consider the ideal $(a, \varphi(y))$. Since R is a PID, there exists some generator $b_\varphi \in R$ such that $(a, \varphi(y)) = (b_\varphi)$. Since R is a PID, it is also a Euclidean domain(?) so we can view b_φ as the gcd of a and $\varphi(y)$. That is, there are $r, s \in R$ such that

$$b_\varphi = ra + s\varphi(y).$$

Set $\psi_\varphi = r\alpha + s\varphi \in F^\vee$. Then

$$\begin{aligned}\psi_\varphi(y) &= r\alpha(y) + s\varphi(y)\\ &= ra + s\varphi(y) = b_\varphi.\end{aligned}$$

So $b_\varphi \in \psi(M)$ and we have the chain of ideals

$$\alpha(M) = (a) \subseteq (a, \varphi(y)) = (b_\varphi) \subseteq \psi_\varphi(M).$$

But (a) was already maximal, so $(a) = (b_\varphi)$ by Max. Hence $a|\varphi(y)$ and $\varphi(y) = ra$. □

How do we use this?

$$\begin{aligned} y \in M &\longrightarrow F \cong R^n \\ y &\longmapsto (y_1, \ldots, y_n)\end{aligned}$$

where $y_i = \pi_i(y)$. Then $\pi \in F^\vee$ and $a|\pi_i(y) = y_i$ so

$$y_i = r_i a \quad \text{for some} \quad r_i \in R.$$

Define $x = (r_i, \ldots, r_n)$, then $ax = (ar_1, \ldots, ar_n) = (y_1, \ldots, y_n) = y$. □

[2] Zorn
[3] Noether

25 Lecture 25

(15 April 2014)

(I totally spaced that there was a lecture and came to class 40 minutes late...)

Definition 25.1. Let R be a ring and M and N be R-modules. Then the R-module E is an ***extension*** of M by N if there are maps α and β such that the sequence

$$M \xrightarrow{\alpha} E \xrightarrow{\beta} N$$

is exact.

Note 25.2. E is *not* uniquely defined by (M, N).

Note 25.3. The functor Ext_1^R characterizes all of the ways that one can create extensions of M by N.

Definition 25.4. $E_1 \xrightarrow{\varphi} E_1$ is a ***double extension*** of M by N if the sequence

???

is exact.

References

[1] Aluffi, Paolo. *Algebra: Chapter 0*. American Mathematical Society, 2009.

[2] Hungerford, Thomas. *Algebra*. Springer, 1974.

[3] Nicholson, Kieth. *Introduction to Abstract Algebra*. Wiley, (4th edition) 2012.