# Assignment 1
PMAT 611

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**Problem 1** (Aluffi V.4 Problem 11)**.** Let $R$ be a commutative ring, and let $\mathfrak{p}$ be a prime ideal of $R$. Prove that the set $S = R \smallsetminus \mathfrak{p}$ is multiplicatively closed. Prove that there is an inclusion-preserving bijection between the prime ideals of $R_{\mathfrak{p}}$ and the prime ideals of $R$ contained in $\mathfrak{p}$. Deduce that $R_{\mathfrak{p}}$ is a local ring.

**Solution.** We first show that $S$ is multiplicatively closed. Indeed, if $s, s' \in S$, then $s, s' \notin \mathfrak{p}$. Since $\mathfrak{p}$ is prime, $ss' \notin \mathfrak{p}$ and thus $ss' \in S$.

For convenience, I use the notation $\frac{r}{s}$ to denote the elements $[s, r]$ of $S^{-1}R$. Let $\lambda : R \longrightarrow S^{-1}R$ be the localization homomorphism $r \overset{\lambda}{\longmapsto} \frac{r}{1}$. To prove the existence of the necessary bijection, we make use of the following lemmas.

**Lemma 1.** *If $I$ is an ideal of $R$ such that $I \cap S = \emptyset$, then $S^{-1}I$ is a proper ideal of $S^{-1}R$.*

**Lemma 2.** *If $J$ is a proper ideal of $S^{-1}R$, then $\lambda^{-1}(J)$ is an ideal of $R$ such that $\lambda^{-1}(J) \cap S = \emptyset$.*

**Lemma 3.** *If $J$ is an ideal of $S^{-1}R$, then $S^{-1}\left(\lambda^{-1}(J)\right) = J$.*

**Lemma 4.** *If $I$ is an ideal of $R$ such that $I \cap S = \emptyset$, then $\lambda^{-1}\left(S^{-1}I\right) = \{a \in R \,|\, \exists s \in S \text{ such that } sa \in I\}$.*

First note that $I \cap S = \emptyset$ if and only if $I \subset \mathfrak{p}$.

*Proof of Lemma 1*: Let $\frac{a}{s}, \frac{a'}{s'} \in S^{-1}I$ such that $a, a' \in I$. Then $sa'$, $s'a$ and $sa' + s'a$ are in $I$. So

$$\tfrac{a}{s} + \tfrac{a'}{s'} = \tfrac{sa'+s'a}{ss'} \in S^{-1}I.$$

For $\frac{r}{s} \in S^{-1}R$, we have $ra \in I$ so

$$\tfrac{r}{s'} \cdot \tfrac{a}{s'} = \tfrac{ra}{ss'} \in S^{-1}I.$$

Thus $S^{-1}I$ is an ideal of $S^{-1}R$.

Suppose $S^{-1}I$ is not a proper ideal of $S^{-1}R$. Then $S^{-1}I = S^{-1}R$ contains $\frac{1}{1}$, and thus $\frac{1}{1} = \frac{a}{s}$ for some $s \in S$ and $a \in I$. Thus $ts = ta$ for some $t \in S$. Since $a \in I \subset \mathfrak{p}$, we have that $ts = ta \in \mathfrak{p}$, but neither $s$ nor $t$ are in $\mathfrak{p}$. This is a contradiction to the primality of $\mathfrak{p}$. □

*Proof of Lemma 2*: Let $J \subset S^{-1}R$ be a proper ideal. Then

$$\lambda^{-1}(J) = \left\{a \in R \,\middle|\, \tfrac{a}{1} \in J\right\}.$$

Let $a, a' \in \lambda^{-1}(J)$ then

$$\tfrac{a}{1} + \tfrac{a'}{1} = \tfrac{a+a'}{1} \in J$$

so $a + a' \in \lambda^{-1}(J)$. Let $r \in R$, then

$$\tfrac{r}{1} \cdot \tfrac{a}{1} = \tfrac{ra}{1} \in J$$

so $ra \in \lambda^{-1}(J)$. Thus $\lambda^{-1}(J)$ is an ideal of $R$.

Finally, suppose that $\lambda^{-1}(J) \cap S \neq 0$. Then there exists an $a \in \lambda^{-1}(J)$ such that $a \in S$ and thus $\frac{1}{a} \cdot \frac{a}{1} = \frac{a}{a} = \frac{1}{1} \in J$. Since $J$ includes the unit element, $J$ comprises all of $S^{-1}R$, a contradiction to the properness of $J$. □

*Proof of Lemma 3*: Let $J$ be an ideal of $S^{-1}R$. Then by definition we have

$$S^{-1}\left(\lambda^{-1}(J)\right) = \left\{ \tfrac{a}{s} \,\middle|\, s \notin \mathfrak{p},\, a \in R \text{ such that } \tfrac{a}{1} \in J \right\}.$$

Let $\frac{a}{s} \in J$, then $\frac{a}{1} = \frac{s}{1} \cdot \frac{a}{s} \in J$ since $J$ is an ideal. Therefore $\frac{a}{s} \in S^{-1}\left(\lambda^{-1}(J)\right)$, and thus $J \subset S^{-1}\left(\lambda^{-1}(J)\right)$.

Similarly, let $\frac{a}{s} \notin J$. Then $\frac{a}{s} = \frac{1}{s} \cdot \frac{a}{1} \notin J$ and thus $\frac{a}{1} \notin J$. Hence $\frac{a}{s} \notin S^{-1}\left(\lambda^{-1}(J)\right)$. Thus $J = S^{-1}\left(\lambda^{-1}(J)\right)$. □

*Proof of Lemma 4*: Let $I \subset \mathfrak{p}$ be an ideal of $R$. We first claim that $\frac{a}{1} \in S^{-1}I$ if and only if there exists an $s \in S$ such that $sa \in I$. Indeed, if $sa \in I$ for some $s \in S$, then $\frac{a}{1} = \frac{sa}{s} \in S^{-1}I$. Conversely, if $\frac{a}{1} \in S^{-1}I$ then $\frac{a}{1} = \frac{b}{t}$ for some $b \in I$ and $t \in S$, and thus $s'ta = s'b$ for some $s' \in S$. Since $b \in I$, we have $s'ta = s'b \in I$, so take $s = s't$.

By the claim, we have

$$\begin{aligned} \lambda^{-1}\left(S^{-1}I\right) &= \left\{ a \in R \,\middle|\, \tfrac{a}{1} \in S^{-1}I \right\} \\ &= \{ a \in R \,|\, \exists s \in S \text{ such that } sa \in I \} \end{aligned}$$

as desired. □

**Proposition 1.** *If $I \subset \mathfrak{p}$ is a prime ideal of $R$, then $S^{-1}I$ is a prime ideal of $S^{-1}R$ and $\lambda^{-1}\left(S^{-1}I\right) = I$.*

*Proof.* From Lemma 1, $S^{-1}I$ is a proper ideal of $S^{-1}R$, so we need to show that it is prime. Suppose $\frac{r}{t}, \frac{r'}{t'} \in S^{-1}R$ such that $\frac{r}{t} \cdot \frac{r'}{t'} = \frac{rr'}{tt'} \in S^{-1}I$. Then

$$\tfrac{rr'}{tt'} = \tfrac{a}{s} \qquad \text{for some } s \in S \text{ and } a \in I,$$

and there exists an $s' \in S$ such that $s'srr' = s'tt'a \in I$. Since $I$ is prime and $s, s' \in S$ and thus $s, s' \notin I$, either $r$ or $r'$ must be in $I$. Hence either $\frac{r}{t}$ or $\frac{r'}{t'}$ is in $S^{-1}I$. So $S^{-1}I$ is prime.

Finally, since $I$ is prime, if $sa \in I$ for some $a \in R$ and $s \notin \mathfrak{p}$ then $a \in I$. From Lemma 4, we have

$$\begin{aligned} \lambda^{-1}\left(S^{-1}I\right) &= \{ a \in R \,|\, \exists s \in S \text{ such that } sa \in I \} \\ &= \{ a \in R \,|\, a \in I \} \\ &= I. \end{aligned}$$

□

**Proposition 2.** *If $J$ is a prime ideal of $S^{-1}R$, then $\lambda^{-1}(J)$ is a prime ideal of $R$.*

*Proof.* From Lemma 2, $\lambda^{-1}(J)$ is an ideal of $R$ and $\lambda^{-1}(J) \subset \mathfrak{p}$, so it remains to show that $\lambda^{-1}(J)$ is prime. Let $r, r' \in R$ such that $rr' \in \lambda^{-1}(J)$. Then $\frac{r}{1} \cdot \frac{r'}{1} = \frac{rr'}{1} \in J$. Since $J$ is prime, either $\frac{r}{1}$ or $\frac{r'}{1} \in J$, hence either $r$ or $r'$ is in $\lambda^{-1}(J)$. □

**Proposition 3.** *There is an inclusion-preserving bijection between the prime ideals of $R_{\mathfrak{p}}$ and the prime ideals of $R$ contained in $\mathfrak{p}$.*

*Proof.* From Lemma 3 and Propositions 1 and 2, we clearly see that there is a bijection between the prime ideals of $R$ contained in $\mathfrak{p}$ and the prime ideals of $R_{\mathfrak{p}} = S^{-1}R$. This is induced by $S^{-1}(\cdot)$ and $\lambda^{-1}(\cdot)$. It remains to show that this bijection is inclusion preserving. Suppose $I \subset I' \subset \mathfrak{p}$ are prime ideals in $R$. Then

$$S^{-1}I = \left\{ \tfrac{a}{s} \,\middle|\, s \in S,\, a \in I \right\} \subset \left\{ \tfrac{a}{s} \,\middle|\, s \in S,\, a \in I' \right\} = S^{-1}I$$

so $S^{-1}I \subset S^{-1}I'$. Now suppose $J \subset J'$ are prime ideals in $S^{-1}R$. Then

$$\lambda^{-1}(J) = \left\{ a \in R \,\middle|\, \tfrac{a}{1} \in J \right\} \subset \left\{ a \in R \,\middle|\, \tfrac{a}{1} \in J' \right\} = \lambda^{-1}(J')$$

so $\lambda^{-1}(J) \subset \lambda^{-1}(J')$. □

**Proposition 4.** *$S^{-1}R$ is a local ring. That is, it has a unique maximal ideal.*

*Proof.* For each ideal $I \subseteq \mathfrak{p}$ in $R$, $S^{-1}I$ is proper ideal of $S^{-1}R$ by Lemma 1. Since $I \subset \mathfrak{p}$, note that

$$S^{-1}I = \left\{ \tfrac{a}{s} \,\middle|\, s \in S,\, a \in I \right\} \subset \left\{ \tfrac{a}{s} \,\middle|\, s \in S,\, a \in \mathfrak{p} \right\} = S^{-1}\mathfrak{p},$$

so $S^{-1}I \subset S^{-1}\mathfrak{p}$. By Lemma 2, every proper ideal $J$ of $S^{-1}R$ is of the form $J = S^{-1}I$ for some ideal $I \subseteq \mathfrak{p}$ in $R$. So every proper ideal $J$ of $S^{-1}R$ is contained in $S^{-1}\mathfrak{p}$. Since $S^{-1}\mathfrak{p}$ is a proper ideal, it is the unique maximal ideal of $S^{-1}R$. □

**Problem 2** (Aluffi III.5 Problem 4)**.** Let $R$ be a ring. A nonzero $R$-module $M$ is simple (or irreducible) if its only submodules are $\{0\}$ and $M$. Let $M$, $N$ be simple modules, and let $\varphi : M \to N$ be a homomorphism of $R$-modules. Prove that either $\varphi = 0$, or $\varphi$ is an isomorphism.

**Solution.** Suppose that $\varphi \neq 0$. Then $\operatorname{im}\varphi \neq \{0\}$ and $\ker\varphi \neq M$, otherwise $\varphi$ is the zero map. By the discussion in class, $\operatorname{im}\varphi$ is a submodule of $N$ and $\ker\varphi$ is a submodule of $M$. Since $N$ is simple and $\operatorname{im}\varphi \neq \{0\}$, we have $\operatorname{im}\varphi = N$ and thus $\varphi$ is a surjection. Similarly, $\ker\varphi = \{0\}$ since $M$ is simple and $\ker\varphi \neq M$, so $\varphi$ is an injection. Thus $\varphi$ is a bijection and hence an isomorphism, since bijections and isomorphisms are the same in R-$\mathsf{Mod}$.

**Problem 3** (Aluffi III.5 Problem 16)**.** Let $R$ be a commutative ring, $M$ an $R$-module, and let $a \in R$ be a nilpotent element determining a submodule $aM$ of $M$. Prove that $M = 0 \iff aM = M$.

**Solution.** Since $a$ is nilpotent, there exists an integer $n > 1$ such that $a^n = 0$. If $M = \{0\}$ then clearly $aM = M = \{0\}$, so suppose $aM = M$. Let $m \in M$. Then there is an element $m_1 \in M$ such that $m = am_1$. Similarly, since $am_1 \in M$, there is an $m_2 \in M$ such that $m_1 = am_2$, and thus $m = a^2 m_2$. Continuing this process, we eventually find $m = a^n m_n = 0$ for some $m_n \in M$, so $m = 0$ for all $m \in M$. Thus $M = \{0\}$.

**Problem 4** (Aluffi III.6 Problem 18)**.** Let $M$ be an $R$-module, and let $N$ be a submodule of $M$. Prove that if $N$ and $M/N$ are both finitely generated, then $M$ is finitely generated.

**Solution.** Since $N$ and $M/N$ are finitely generated, there exist numbers $n, k \geq 1$ and surjective $R$-module homomorphisms

$$R^{\oplus n} \overset{\nu}{\twoheadrightarrow} N \qquad \text{and} \qquad R^{\oplus k} \overset{\kappa}{\twoheadrightarrow} M/N.$$

Define the maps $i : R^{\oplus n} \longrightarrow R^{\oplus(k+n)}$ and $p : R^{\oplus(k+n)} \longrightarrow R^{\oplus k}$ by $i : (r_1, \dots, r_n) \mapsto (r_1, \dots, r_n, 0, \dots, 0)$ and $p : (r_1, \dots, r_n, s_1, \dots, s_m) \mapsto (s_1, \dots, s_m)$. Then the sequence

$$R^{\oplus n} \xrightarrow{\ i\ } R^{\oplus(k+n)} \xrightarrow{\ p\ } R^{\oplus k} \longrightarrow 0$$

is exact.

Define a map $\mu : R^{\oplus(k+n)} \longrightarrow M$ in the following manner. For each $j = 1, \dots, k$, consider $\kappa(0, \dots, 1, \dots, 0)$ (with a one in the $j^{\text{th}}$ position and zeros elsewhere) as an element in $M/N$ and pick a representative $m_j \in M$ such that $\kappa(0, \dots, 1, \dots, 0) = m_j + N$. Then define

$$\mu(r_1, \dots, r_n, s_1, \dots, s_m) = \nu(r_1, \dots, r_n) + \sum_{j=1}^{k} s_j m_j,$$

where $\nu(r_1, \dots, r_n) \in N \subset M$. This is a homomorphism of $R$-modules. Indeed,

$$\begin{aligned}
\mu(r_1 + r_1', \dots, r_n + r_n', s_1 + s_1', \dots, s_k + s_k') &= \nu(r_1 + r_1', \dots, r_n + r_n') + \sum_{j=1}^{k} (s_j + s_j') m_j \\
&= \nu(r_1, \dots, r_n) + \sum_{j=1}^{k} s_j m_j + \nu(r_1', \dots, r_n') + \sum_{j=1}^{k} s_j' m_j \\
&= \mu(r_1, \dots, r_n, s_1, \dots, s_k) + \mu(r_1', \dots, r_n', s_1', \dots, s_k')
\end{aligned}$$

and for $t \in R$

$$\begin{aligned}
\mu(tr_1, \dots, tr_n, ts_1, \dots, ts_k) &= t\nu(r_1, \dots, r_n) + t \sum_{j=1}^{k} s_j m_j \\
&= t\mu(r_1, \dots, r_n, s_1, \dots, s_k).
\end{aligned}$$

Consider the following diagram.

$$\begin{array}{ccccccc}
R^{\oplus n} & \xrightarrow{\ i\ } & R^{\oplus(k+n)} & \xrightarrow{\ p\ } & R^{\oplus k} & \longrightarrow & 0 \\
{\scriptstyle\nu}\downarrow\!\!\!\downarrow & & {\scriptstyle\mu}\downarrow & & {\scriptstyle\kappa}\downarrow\!\!\!\downarrow & & \\
N & \xhookrightarrow{\ \iota\ } & M & \xrightarrow{\ \pi\ }\!\!\!\!\rightarrow & M/N & \longrightarrow & 0
\end{array}$$

The rows are clearly exact. Furthermore, the diagram commutes. Indeed, $\mu \circ i(r_1, \dots, r_n) = \nu(r_1, \dots, r_n)$ by definition of $\mu$ so the left square commutes. For the right square,

$$\begin{aligned}
\pi \circ \mu(r_1, \dots, r_n, s_1, \dots, s_k) &= \pi \left( \sum_{j=1}^{k} s_j m_j \right) \\
&= \sum_{j=1}^{k} s_j \pi(m_j)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{k} s_j \kappa(0, \ldots, 1, \ldots 0) \\
&= \kappa(s_1, \ldots, s_k) \\
&= \kappa \circ p(r_1, \ldots, r_n, s_1, \ldots, s_k).
\end{aligned}
$$

By the surjective four lemma, $\mu$ is also a surjection. So $M$ is finitely generated.