Assignment 3

PMAT 611

Mark Girard

10 March 2014

Problem 1 (Aluffi problem VII.2.6)**.** Let S be a multiplicative subset of a ring R and let M be an R-module.

i) Let N be an $S^{-1}R$-module. Prove that $(S^{-1}M) \otimes_{S^{-1}R} N \cong M \otimes_R N$.

ii) Let A be an R-module. Prove that $(S^{-1}A) \otimes_R M \cong S^{-1}(A \otimes_R M)$.

Solution. .

i) *Proof.* First note that $M \otimes_R N$ has an $S^{-1}R$-module structure given by $\frac{r}{s}(m \otimes n) = m \otimes (\frac{r}{s} \cdot n)$, where $\frac{r}{s} \cdot n$ represents the given $S^{-1}R$-action on N. Define an $S^{-1}R$-module homomorphism

$$\begin{aligned}(S^{-1}M) \otimes_{S^{-1}R} N &\xrightarrow{\varphi} M \otimes_R N \\ \frac{m}{s} \otimes n &\longmapsto m \otimes \frac{1}{s} \cdot n\end{aligned}$$

by extending this definition $S^{-1}R$-linearly. The inverse of this mapping may be given by

$$\begin{aligned}(S^{-1}M) \otimes_{S^{-1}R} N &\xleftarrow{\psi} M \otimes_R N \\ \frac{m}{1} \otimes n &\longleftarrow m \otimes n\end{aligned}$$

by similarly extending this $S^{-1}R$-linearly. Then $\varphi(\psi(m \otimes n)) = \varphi(\frac{m}{1} \otimes n) = m \otimes \frac{1}{1}n = m \otimes n$ and conversely $\psi(\varphi(\frac{m}{s} \otimes n)) = \psi(m \otimes \frac{1}{s}n) = \frac{1}{s}\psi(m \otimes n) = \frac{1}{s}(\frac{m}{1} \otimes n) = \frac{m}{s} \otimes n$. Hence ψ and φ are mutual inverses and $(S^{-1}M) \otimes_{S^{-1}R} N \cong M \otimes_R N$ as desired. □

ii) Note that both $(S^{-1}A) \otimes_R M$ and $S^{-1}(A \otimes_R M)$ are R-modules. Define an R-module homomorphism

$$\begin{aligned}(S^{-1}A) \otimes_R M &\longrightarrow S^{-1}(A \otimes_R M) \\ \frac{a}{s} \otimes m &\longmapsto \frac{a \otimes m}{s}\end{aligned}$$

and extending R-linearly, and deifne its inverse by

$$\begin{aligned}(S^{-1}A) \otimes_R M &\longleftarrow S^{-1}(A \otimes_R M) \\ \frac{a}{s} \otimes m &\longleftarrow \frac{a \otimes m}{s}.\end{aligned}$$

Hence we obtain the desired isomorphism $(S^{-1}A) \otimes_R M \cong S^{-1}(A \otimes_R M)$.

Problem 2 (Aluffi problem VII.2.7)**.** Show that changing the base ring in a tensor may or may not make a difference. Show that

(i) $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ and

(ii) $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \ncong \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$.

Solution. .

(i) As $\mathbb{Q}$-modules we have that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q} \cong \mathbb{Q}$.

Clearly, we have that $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q} \cong \mathbb{Q}$. So it remains to show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$. As a $\mathbb{Q}$-module, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ has $\mathbb{Q}$-action given by $a \cdot (b \otimes c) = (ab) \otimes c$. We can define an isomophism $\mathbb{Q} \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ by

$$b \longmapsto b \otimes 1.$$

Indeed, this is injective, since $b \otimes 1 = 0 \otimes 0$ if and only if $b = 0$. It is also surjective since

$$\frac{p}{q} \otimes \frac{r}{s} = r\left(\frac{ps}{qs} \otimes \frac{1}{s}\right) = s\left(\frac{pr}{qs} \otimes \frac{1}{s}\right) = \frac{pr}{qs} \otimes \frac{s}{s} = \frac{pr}{qs} \otimes 1$$

for all $\frac{p}{q}, \frac{r}{s} \in \mathbb{Q}$ and thus every element in $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ can be written as

$$\sum_i m_i \frac{p_i}{q_i} \otimes \frac{r_i}{s_i} = \left(\sum_i \frac{m_i p_i r_i}{q_i s_i}\right) \otimes 1.$$

It is also $\mathbb{Q}$-linear, since $ab \longmapsto (ab) \otimes 1 = a \cdot (b \otimes 1)$. Hence $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q} \cong \mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$.

(ii) We note that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{R}^4$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{R}^2$ as $\mathbb{R}$-modules. Indeed, every element in both $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ can be written as linear combinations of the elements

$$\{1 \otimes 1,\, 1 \otimes i,\, i \otimes 1,\, i \otimes i\}. \tag{1}$$

with real coefficients, so (1) is a spanning set for both $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and thus they are both $\mathbb{R}$-vector spaces with dimension at most 4. However, the set in (1) is linearly independent and thus a basis for $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, wheras $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ as an $\mathbb{R}$-vector space has a basis given by

$$\{1 \otimes 1,\, i \otimes 1\}.$$

Indeed, in $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ we have that $1 \otimes 1 = ((-i)(i)) \otimes 1 = (-i)(i \otimes 1) = i \otimes (-i) = -(i \otimes i)$ and similarly we have $i \otimes 1 = i(1 \otimes 1) = 1 \otimes i$.

Problem 3 (Aluffi problem VIII.2.24)**.** Let R be a ring. Let P be a flat R-module and Let

$$0 \longrightarrow M \xrightarrow{\mu} N \xrightarrow{\nu} P \longrightarrow 0$$

be an exact sequence R-modules.

(i) Prove that M is flat if and only of N is flat.

(ii) Prove that for all R-modules Q, the induced sequence

$$0 \longrightarrow M \otimes_R Q \longrightarrow N \otimes_R Q \longrightarrow P \otimes_R Q \longrightarrow 0$$

is exact.

Solution. We first prove part (ii) of the problem statement and use this to prove part (i).
(NOTE: I originally proved this using a very lengthy diagram chase, but later realized that the proof can be greatly simplified by use of Tor. I spent so much time figuring it out and writing it up that I didn't want to just delete it, so I first present here my very long diagram chase. My shorter proof is given as an addendum at the end of the solution to this problem.)

(ii) *Proof* (*diagram chase method*)*.* Let Q' be a free R-module that maps surjectively onto Q. For example we could take the module $Q' = \mathrm{Free}_{\textsf{R-Mod}}(\mathrm{Forget}(Q))$, and let $Q'' \xrightarrow{\alpha} Q'$ be the kernel of this mapping such that the sequence

$$0 \longrightarrow Q'' \xrightarrow{\alpha} Q' \xrightarrow{\beta} Q \longrightarrow 0$$

is exact. Then we have the commuting diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & \ker(\mathrm{id}_Q \otimes \mu) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & Q'' \otimes_R M & \xrightarrow{\alpha \otimes \mathrm{id}_M} & Q' \otimes_R M & \xrightarrow{\beta \otimes \mathrm{id}_M} & Q \otimes_R M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}_{Q''} \otimes \mu} & & \downarrow{\scriptstyle \mathrm{id}_{Q'} \otimes \mu} & & \downarrow{\scriptstyle \mathrm{id}_{Q} \otimes \mu} & & \\
 & & Q'' \otimes_R N & \xrightarrow{\alpha \otimes \mathrm{id}_N} & Q' \otimes_R N & \xrightarrow{\beta \otimes \mathrm{id}_N} & Q \otimes_R N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}_{Q''} \otimes \nu} & & \downarrow{\scriptstyle \mathrm{id}_{Q'} \otimes \nu} & & \downarrow{\scriptstyle \mathrm{id}_{Q} \otimes \nu} & & \\
0 & \longrightarrow & Q'' \otimes_R P & \xrightarrow{\alpha \otimes \mathrm{id}_P} & Q' \otimes_R P & \xrightarrow{\beta \otimes \mathrm{id}_P} & Q \otimes_R P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \qquad (*)$$

where the bottom row is exact by flatness of P and the middle column is exact since Q' is free and thus flat. Exactness of the remaining rows and columns comes from the right-exactness of the tensor pruduct.

Let $z \in \ker(\mathrm{id}_Q \otimes \mu)$. By exactness of the top row, we have that $\beta \otimes \mathrm{id}_M$ is surjective and thus there is a $y \in Q' \otimes_R M$ such that $(\beta \otimes \mathrm{id}_M)(y) = z$. Then let $x = (\mathrm{id}_{Q'} \otimes \mu)(y) \in Q' \otimes_R N$. However, by commutativity of the top right square, we have that

$$(\beta \otimes \mathrm{id}_N)(x) = (\beta \otimes \mathrm{id}_N)\left((\mathrm{id}_{Q'} \otimes \mu)(y)\right)$$

$$
\begin{aligned}
&= (\mathrm{id}_Q \otimes \mu)\,(\beta \otimes \mathrm{id}_M(y)) \\
&= (\mathrm{id}_Q \otimes \mu)(z) = 0
\end{aligned}
$$

since $z \in \ker(\mathrm{id}_Q \otimes \mu)$, and thus $x \in \ker(\beta \otimes \mathrm{id}_N)$. This implies, by exactness of the middle row, that x is in the image of $\alpha \otimes \mathrm{id}_N$ and thus there is a $w \in Q'' \otimes_R N$ such that $(\alpha \otimes \mathrm{id}_N)(w) = x$. Let $v = (\mathrm{id}_{Q''} \otimes \nu)(w)$, then $v = 0$. Indeed, by commutativity of the lower left square, we have that

$$
\begin{aligned}
(\alpha \otimes \mathrm{id}_P)(v) &= (\alpha \otimes \mathrm{id}_P)\,((\mathrm{id}_{Q''} \otimes \nu)(w)) \\
&= (\mathrm{id}_{Q'} \otimes \nu)\,(\alpha \otimes \mathrm{id}_N(w)) \\
&= (\mathrm{id}_{Q'} \otimes \nu)(x) \\
&= \underbrace{((\mathrm{id}_{Q'} \otimes \nu) \circ (\mathrm{id}_{Q'} \otimes \mu))}_{=0}(y) \\
&= 0,
\end{aligned}
$$

and thus $v = 0$ since $\alpha \otimes \mathrm{id}_P$ is injective. Then $w \in \ker(\mathrm{id}_{Q''} \otimes \nu)$ since $(\mathrm{id}_{Q''} \otimes \nu)(w) = 0$, and thus w is in the image of $\mathrm{id}_{Q''} \otimes \mu$ by exactness of the first column. So there is a $u \in Q'' \otimes_R M$ such that $(\mathrm{id}_{Q''} \otimes \mu)(u) = w$, and let $y' = (\alpha \otimes \mathrm{id}_M)(u) \in Q' \otimes_R M$. We claim that $(\mathrm{id}_{Q'} \otimes \mu)(y - y') = 0$. Indeed, by commutativity of the upper left square, we have that

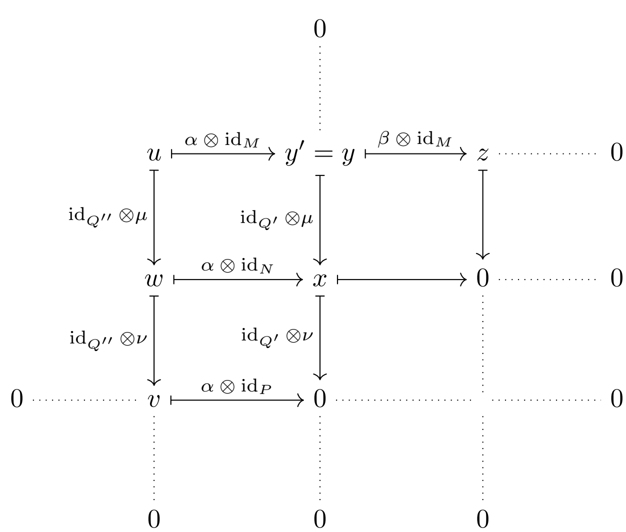
$$
\begin{aligned}
(\mathrm{id}_{Q'} \otimes \mu)(y') &= (\mathrm{id}_{Q'} \otimes \mu)((\alpha \otimes \mathrm{id}_M)(u)) \\
&= (\alpha \otimes \mathrm{id}_N)(\mathrm{id}_{Q''} \otimes \mu)(u) \\
&= (\alpha \otimes \mathrm{id}_N)(w) \\
&= x \\
&= (\mathrm{id}_{Q'} \otimes \mu)(y),
\end{aligned}
$$

and thus $(\mathrm{id}_{Q'} \otimes \mu)(y') = (\mathrm{id}_{Q'} \otimes \mu)(y)$. But $(\mathrm{id}_{Q'} \otimes \mu)$ is injective, by exactness of the middle column, so $y - y' = 0$ and thus $y = y'$. Finally, note that

$$
z = (\beta \otimes \mathrm{id}_M)(y) = (\underbrace{(\beta \otimes \mathrm{id}_M) \circ (\alpha \otimes \mathrm{id}_M)}_{=0})(w) = 0,
$$

hence $z = 0$ for an arbirtary $z \in \ker(\mathrm{id}_Q \otimes \mu)$ and thus $\ker(\mathrm{id}_Q \otimes \mu) = 0$.

The diagram chase contained in above arguments may be summarized in the following schematic:



Hence the right-most column in the diagram (*) above, which we may repeat here as

$$0 \longrightarrow Q \otimes_R M \longrightarrow Q \otimes_R N \longrightarrow Q \otimes_R P \longrightarrow 0,$$

is exact. This is naturally isomorphic to the sequence

$$0 \longrightarrow M \otimes_R Q \longrightarrow N \otimes_R Q \longrightarrow P \otimes_R Q \longrightarrow 0,$$

which is exact as desired. ☐

We can now prove the first part of the problem statement.

(i) *Proof.* Let $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ be an exact sequence of R-modules. Then we have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & A \otimes_R M & \longrightarrow & A \otimes_R N & \longrightarrow & A \otimes_R P & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha \otimes \mathrm{id}_M} & & \downarrow{\scriptstyle \alpha \otimes \mathrm{id}_N} & & \downarrow{\scriptstyle \alpha \otimes \mathrm{id}_P} & & \\
0 & \longrightarrow & B \otimes_R M & \longrightarrow & B \otimes_R N & \longrightarrow & B \otimes_R P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C \otimes_R M & \longrightarrow & C \otimes_R N & \longrightarrow & C \otimes_R P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the right-most column is exact by flatness of P and the rows are all exact from part (ii) of the problem. Then the Snake Lemma gives us the exact sequence

$$0 \longrightarrow \ker(\alpha \otimes \mathrm{id}_M) \longrightarrow \ker(\alpha \otimes \mathrm{id}_N) \longrightarrow \ker(\alpha \otimes \mathrm{id}_P) \longrightarrow$$
$$\longrightarrow \operatorname{coker}(\alpha \otimes \mathrm{id}_M) \longrightarrow \operatorname{coker}(\alpha \otimes \mathrm{id}_N) \longrightarrow \operatorname{coker}(\alpha \otimes \mathrm{id}_P) \longrightarrow 0.$$

But $\ker(\alpha \otimes \mathrm{id}_P) = 0$ by exactness of the right-most column. Considering only the top half of the sequence above, we get the exact sequence

$$0 \longrightarrow \ker(\alpha \otimes \mathrm{id}_M) \longrightarrow \ker(\alpha \otimes \mathrm{id}_N) \longrightarrow 0$$

and therefore $\ker(\alpha \otimes \mathrm{id}_M) \cong \ker(\alpha \otimes \mathrm{id}_N)$. Hence the functor $- \otimes_R M$ is exact if and only if $- \otimes_R N$ is, and thus M is flat if and only if N is flat, as desired. ☐

Addendum: Proof of part (ii) of Problem 2 using Tor.

Proof. For all R-modules Q, the short exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

induces the long exact sequence

$$\cdots \longrightarrow \mathrm{Tor}_1^R(N,Q) \longrightarrow \mathrm{Tor}_1^R(P,Q) \longrightarrow M \otimes_R Q \longrightarrow N \otimes_R Q \longrightarrow P \otimes_R Q \longrightarrow 0.$$

However, $\mathrm{Tor}_1^R(P,Q) \cong 0$ since P is flat. Hence the final part of the long exact sequence above gives the desired short exact sequence

$$0 \longrightarrow M \otimes_R Q \longrightarrow N \otimes_R Q \longrightarrow P \otimes_R Q \longrightarrow 0.$$

$\square$

Problem 4 (Aluffi problem VIII.2.24)**.** Let R ba a commutative Noetherian ring with a (single) maximal ideal $\mathfrak{m}$, and let M be a finitely generated flat R-module.

(i) Choose elements $m_1, \dots, m_r \in M$ whose cosets mod $\mathfrak{m}M$ are a basis of $M/\mathfrak{m}M$ as a vector space over the field $R/\mathfrak{m}$. By Nakayama's lemma, $M = \langle m_1, \dots, m_r \rangle$.

(ii) Obtain an exact sequence

$$0 \longrightarrow N \longrightarrow R^{\oplus r} \longrightarrow M \longrightarrow 0$$

where N is finitely generated.

(iii) Deduce that $N = 0$. (Nakayama.)

(iv) Conclude that M is free.

Thus, a finitely generated R-module over a (Noetherian) local ring is flat if and only if it is free.

Solution. (i) Since M is finitely generated, $M/\mathfrak{m}M$ is a finite dimensional vector space. So we can pick a basis $\{m_1 + \mathfrak{m}M, \dots, m_r + \mathfrak{m}M\}$. This gives us the desired elements $\{m_1, \dots, m_r\}$ that, by Nakayama's lemma (see Aluffi problem VI.3.10 – I'm guessing we'll cover this later in the semester?), generate M.

(ii) We have a surjection $R^{\oplus r} \longrightarrow M$, so let $N \longrightarrow R^{\oplus r}$ be the kernel of this mapping. This gives us an exact sequence

$$0 \longrightarrow N \longrightarrow R^{\oplus r} \longrightarrow M \longrightarrow 0. \tag{1}$$

Since R is Noetherian, M is finitely presented (Lemma VI.4.8 in Aluffi) so N is finitely generated.

(iii) From part (ii) in Problem 3, since M is flat, tensoring the exact sequence in (1) with $R/\mathfrak{m}R$ yields an exact sequence

$$0 \longrightarrow N \otimes_R R/\mathfrak{m}R \longrightarrow R^{\oplus r} \otimes_R R/\mathfrak{m}R \longrightarrow M \otimes_R R/\mathfrak{m}R \longrightarrow 0.$$

However, these tensor products give us the modules $N \otimes_R R/\mathfrak{m}R \cong N/\mathfrak{m}N$, $R^{\oplus r} \otimes_R R/\mathfrak{m}R \cong (R/\mathfrak{m}R)^{\oplus r}$ and $M \otimes_R R/\mathfrak{m}R \cong M/\mathfrak{m}M$, which are all $(R/\mathfrak{m}R)$-vector spaces. So this is the exact sequence of vector spaces

$$0 \longrightarrow N/\mathfrak{m}N \longrightarrow (R/\mathfrak{m}R)^{\oplus r} \longrightarrow M/\mathfrak{m}M \longrightarrow 0.$$

Since these are vector spaces and $\dim(R/\mathfrak{m}R)^{\oplus r} = \dim M/\mathfrak{m}M = r$, we must have that $\dim N/\mathfrak{m}N = 0$. By Nakayama's lemma, we have that $N = 0$. Hence the exact sequence in (1) reduces to

$$0 \longrightarrow R^{\oplus r} \longrightarrow M \longrightarrow 0,$$

and thus $M \cong R^{\oplus r}$ so M is free.