Assignment 4
PMAT 611

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Problem 1 (Aluffi problem VIII.3.6). Let $f\colon R \longrightarrow S$ be a ring homomorphism, and let $\varphi\colon N_1 \longrightarrow N_2$ be a homomorphism of S-modules. Prove that φ is an isomorphism if and only if $f_*(\varphi)$ is an isomorphism. (Functors with this property are said to be *conservative*.) In fact, prove that f_* is *faithfully* exact: a sequence of S-modules

$$0 \longrightarrow L \xrightarrow{\ \alpha\ } M \xrightarrow{\ \beta\ } N \longrightarrow 0$$

is exact if and only if the sequence of R-modules

$$0 \longrightarrow f_*(L) \xrightarrow{f_*(\alpha)} f_*(M) \xrightarrow{f_*(\beta)} f_*(N) \longrightarrow 0$$

is exact. In particular, a sequence of R-modules is exact if and only if it is exact as a sequence of abelian groups. (This is completely trivial, but useful nonetheless.)

Solution. As sets, we note that $\mathrm{Forget}(N) = \mathrm{Forget}(f_*(N))$ for all S-modules N, and so we denote $n \in f_*(N)$ (as an R-module) for all $n \in N$ (as an S-module).

- We first show that $\varphi : N_1 \longrightarrow N_2$, as a homomorphism of S-modules, is

  (i) injective if and only if $f_*(\varphi)$ is injective and

  (ii) surjective if and only if $f_*(\varphi)$ is surjective.

  To show (i), first suppose that φ is injective and let $n \in \ker f_*(\varphi)$. Then

  $$f_*(\varphi)(1_R \cdot n) = \underbrace{f(1_R)}_{1_S}\varphi(n) = 1_s\varphi(n) = 0$$

  and thus $n = 0$. Analagously, suppose that $f_*(\varphi)$ is injective and let $n \in \ker\varphi$. Then

  $$\varphi(1_s n) = 1_R \cdot f_*(\varphi)(n) = 0$$

  and thus $n = 0$. So we have $\ker\varphi = 0$ if and only if $\ker f_*(\varphi) = 0$. To show (ii), suppose φ is surjective and let $n \in f_*(N_2)$ (as an R-module). Then $n \in N_2$ as an S-module and thus there is a $m \in N_1$ such that $\varphi(m) = n$. Hence $f_*(\varphi)(m) = n$. Similarly, suppose $f_*(\varphi)$ is surjective and let $n \in N_2$ as an element of an S-module. Then $n \in f_*(N_2)$ as an R-module and there is an $m \in f_*(N_1)$ such that $f_*(\varphi)(m) = n$, and thus $\varphi(m) = n$. Hence φ is an isomorphism if and only if $f_*(\varphi)$ is an isomorphism.

  in particular, we have shown that

  - $n \in \ker\varphi$ (as an element of the S-module N_1) if and only if $n \in \ker f_*(\varphi)$ (as an element of the R-module $f_*(N_1)$),

- $n \in \operatorname{im} \varphi$ (as an element of the S-module N_2) if and only if $n \in \operatorname{im} f_*(\varphi)$ (as an element of the R-module $f_*(N_2)$).

- Let L, M and N be S-modules and consider the complex

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0.$$

  of S-modules as well as the corresponding complex of R-modules

$$0 \longrightarrow f_*(L) \xrightarrow{f_*(\alpha)} f_*(M) \xrightarrow{f_*(\beta)} f_*(N) \longrightarrow 0$$

  from applying f_*.

  From the above analysis, we have that α is injective if and only if $f_*(\alpha)$ is injective, and that β is surjective if and only of $f_*(\beta)$ is surjective. Furthermore, we have

$$\operatorname{Forget}(\ker \beta) = \operatorname{Forget} \ker f_*(\beta) \qquad \text{and} \qquad \operatorname{Forget}(\operatorname{im} \alpha) = \operatorname{Forget} \operatorname{im} f_*(\alpha).$$

  Thus, $\ker \beta = \operatorname{im} \alpha$ if and only if $\ker f_*(\beta) = \operatorname{im} f_*(\alpha)$. Hence, the first sequence is exact if and only if the second one is.

- In particular, if $R = \mathbb{Z}$ then $f_*(M)$ is just M as an abelian group where we forget the R-module structure. Hence we get that a sequence of S-modules is an exact sequence if and if it is exact as a sequence as abelian groups.

Problem 2 (Aluffi problem VIII.3.14)**.** Let $f\colon R \longrightarrow S$ be an onto ring homomorphism; thus, $S \cong R/I$ for some ideal I of R.

i) Prove that, for all R-modules M, $f^!(M) \cong \{m \in M \,|\, \forall a \in I,\, am = 0\}$, while $f^*(M) \cong M/IM$. (Exercise III.7.7 may help.)

ii) Prove that, for all S-modules N, $f^! f_*(N) \cong N$ and $f^* f_*(N) \cong N$.

iii) Prove that f_* is fully faithful (Definition VII.1.6).

iv) Deduce that if there is an onto homomorphism $R \longrightarrow S$, then S-Mod is equivalent to a full subcategory of R-Mod.

Solution. (**Note**: Normally, I take great care in understanding and working on all of the problems, but I was incredibly busy the past week and didn't have the time to work on the homework as much as I usually do. When I'm not so busy, I'd like to come back to this problem and spend some time on it..., but the solution here is not complete.)

The following lemma will be useful (See Exercise III.7.7 in Aluffi – I'd give a proof if I had more time.).

Lemma 1. *Given a short exact sequence of R-modules*

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0.$$

and an R-module L, there is an exact sequence of R-modules

$$0 \longrightarrow \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(P, L) \longrightarrow \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(P, L) \longrightarrow \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(M, L).$$

Note that we have a short exact sequence of R-modules

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \cong S \longrightarrow 0. \tag{1}$$

i) Note that $f^!(M) = \mathrm{Hom}_{\text{R-}\mathsf{Mod}}(R/I, M)$ by definition. By the above lemma, we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\text{R-}\mathsf{Mod}}(R/I, M) \longrightarrow \mathrm{Hom}_{\text{R-}\mathsf{Mod}}(R, M) \longrightarrow \mathrm{Hom}_{\text{R-}\mathsf{Mod}}(I, M).$$

I'm not quite sure what to do next, but I think we want to construct the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_{\text{R-}\mathsf{Mod}}(R/I, M) & \longrightarrow & \mathrm{Hom}_{\text{R-}\mathsf{Mod}}(R, M) & \longrightarrow & \mathrm{Hom}_{\text{R-}\mathsf{Mod}}(I, M) \\ & & \downarrow & & \downarrow{\cong} & & \downarrow \\ 0 & \longrightarrow & \{m \in M \,|\, am = 0, \forall a \in I\} & \longrightarrow & M & \longrightarrow & ? \end{array}$$

and use the five lemma to show that $f^!(M) = \mathrm{Hom}_{\text{R-}\mathsf{Mod}}(R/I, M) \cong \{m \in M \,|\, am = 0, \forall a \in I\}$.

Note that $f^*(M) = M \otimes_R R/I$ by definition. Since the functor $M \otimes_R -$ is right-exact for any R-module M, applying $M \otimes -$ to the exact sequence in (1) yields the exact sequence

$$M \otimes_R I \longrightarrow M \otimes_R R \longrightarrow M \otimes_R R/I \longrightarrow 0.$$

Again, not quite sure what to do, but I think we want to show a commutative diagram with exact rows

$$\begin{array}{ccccccc} M \otimes_R I & \longrightarrow & M \otimes_R R & \longrightarrow & M \otimes_R R/I & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow & & \\ IM & \longrightarrow & M & \longrightarrow & M/IM & \longrightarrow & 0, \end{array}$$

and use the five lemma to show that $f^*(M) = M \otimes_R R/I \cong M/IM$.

ii)

iii) (Didn't have time for the rest... I'd like to try these in the future and ask you for help. Sorry there's nothing here now!)

iv)

Problem 3 (Aluffi problem VI.2.13)**.** The set of subspaces of given dimension in a fixed vector space is called a *Grassmannian.* In Exercise VI.2.12 you have constructed a bijection between the Grassmannian of r-dimensional subspaces of k^n and the set of reduced row echelon matrices with n columns and r nonzero rows.

For $r = 1$, the Grassmannian is called the *projective space.* For a vector space V, the corresponding projective space $\mathbb{P}V$ is the set of 'lines (1-dimensional subspaces) in V. For $V = k^n$, $\mathbb{P}V$ may be denoted $\mathbb{P}_k^{n-1}$; and the field k may be omitted if it is clear from the context. Show that $\mathbb{P}^{n-1}$ may be written as a union $k^{n-1} \cup k^{n-2} \cup \cdots \cup k^1 \cup k^0$, and describe each of these subsets 'geometrically'. Thus, $\mathbb{P}^{n-1}$ is the union of n 'cells', the largest one having dimension $n-1$ (accounting for the choice of notation). Similarly, all Grassmannians may be written as unions of cells. These are called *Schubert cells.*

Prove that the Grassmannian of $(n-1)$-dimensional subspaces of k^n admits a cell decomposition entirely analogous to that of $\mathbb{P}_k^{n-1}$.

Solution. First consider the set of 1-dimensional subspaces of k^n. Each subspace is a line and each line may be defined by a non-zero vector $v = (v_1, \ldots, v_n)$. Two vectors $v, v' \in k^n$ define the same line if $v = zv'$ for some non-zero $z \in k$, i.e.

$$(v_1, \ldots, v_n) = z(v_1', \ldots, v_n') = (zv_1', \ldots, zv_n').$$

So we may define an equivalence relation on the set of non-zero vectors in k^n by

$$v \sim v' \qquad \text{if and only if } v = zv' \text{ for some } z \in k.$$

Then the projective space $\mathbb{P}^{n-1}$ may be given by

$$\mathbb{P}_k^{n-1} = (k^n \smallsetminus \{0\})/\sim .$$

Each line in k^n is given by an equivalence class of vectors, and for each class we may choose a representative of the form $[z_1, z_2, \ldots, z_n]$.

Each vector $(z_1, z_2, \ldots, z_n)$ whose first entry is non-zero is equivalent to the vector

$$\left(1, \frac{z_2}{z_1}, \ldots, \frac{z_n}{z_1}\right) = (1, z_2', \ldots z_n')$$

which has a one in its first entry. Note that the vectors

$$(1, z_2, z_3, \ldots z_n) \text{ and } (1, z_2', z_3', \ldots z_n')$$

define the same line in k^n if and only if $z_k = z_k'$ for all $k = 2, \ldots, n$. So the space of lines in k^n that may given by vectors whose first entry is non-zero is the set

$$\{[1, z_2, \ldots, z_n] \,|\, z_k \in k\} \cong k^{n-1},$$

and thus isomorphic to k^{n-1}. The set of lines that are defined by vectors whose first entry is zero and second entry is non-zero is the set

$$\{[0, 1, z_3, \ldots, z_n] \,|\, z_k \in k\} \cong k^{n-2}.$$

Similarly, the set of lines in k^n that are defined by vectors whose first two entries are zero and third entry is non-zero is

$$\{[0, 0, 1, z_4, \ldots, z_n] \,|\, z_k \in k\} \cong k^{n-3}.$$

Continuing this process, we see that

$$\mathbb{P}^{n-1} \cong k^{n-1} \cup k^{n-2} \cup \cdots \cup k^1 \cup \{0\}$$

as desired.

Using the observation that the Grassmannian of $n-1$ dimensional spaces $\mathrm{Gr}_k(n-1,n)$ is isomorphic to the set of $n \times n$ matrices over k in reduced row echelon form with $n-1$ non-zero rows, we have that $\mathrm{Gr}_k(n-1,n)$ is isomorphic to the union sets of the following form:

$$\left\{ \begin{bmatrix} 1 & 0 & \dots & \dots & z_1 \\ 0 & 1 & 0 & \dots & z_2 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & z_{n-1} \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix} \middle| \; z_1, \dots, z_{n-1} \in k \right\} \cong k^{n-1}$$

$$\left\{ \begin{bmatrix} 1 & 0 & \dots & 0 & z_1 & 0 \\ 0 & 1 & \dots & 0 & z_2 & 0 \\ \vdots & \vdots & \ddots & & & \vdots \\ 0 & 0 & \dots & 1 & z_{n-2} & \\ 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix} \middle| \; z_1, \dots, z_{n-2} \in k \right\} \cong k^{n-2}$$

$$\vdots$$

$$\vdots$$

$$\left\{ \begin{bmatrix} 1 & z_1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix} \middle| \; z_1 \right\} \cong k^1$$

$$\left\{ \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix} \right\} \cong k^0$$

and thus we have

$$\mathrm{Gr}_k(n-1,n) \cong k^{n-1} \cup k^{n-2} \cup \dots \cup k^1 \cup \{0\}.$$

Problem 4 (Aluffi problem VI.2.14)**.** Ahow that the Grassmannian $\mathrm{Gr}_k(2,4)$ of 2-dimensional subspaces of k^4 is the union of 6 Schubert cells: $k^4 \cup k^3 \cup k^2 \cup k^2 \cup k^1 \cup k^0$.

Solution. We list the forms of all of the possible reduced row echelon forms of 4×4 matrices over k with 2 non-zero rows. First, there are all matrices whose leading entry is a one. This is all matrices of one of the following forms:

$$\left\{ \begin{bmatrix} 1 & 0 & z_1 & z_2 \\ 0 & 1 & z_3 & z_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| z_1, z_2, z_3, z_4 \in k \right\}, \left\{ \begin{bmatrix} 1 & z_1 & 0 & z_2 \\ 0 & 0 & 1 & z_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| z_1, z_2, z_3 \in k \right\}, \left\{ \begin{bmatrix} 1 & z_1 & z_2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| z_1, z_2 \in k \right\},$$

and clearly these sets are isomorphic to k^4, k^3 and k^2 respectively.

Next are all matrices whose leading entry is a zero and second entry is a one. This is all matrices of one of two forms:

$$\left\{ \begin{bmatrix} 0 & 1 & 0 & z_1 \\ 0 & 0 & 1 & z_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| z_1, z_2 \in k \right\} \quad \text{or} \quad \left\{ \begin{bmatrix} 0 & 1 & z_1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| z_1 \in k \right\}$$

which are clearly isomorphic to k^2 and k^1 respectively.

Finally, there is only one matrix in reduced row echelon form whose leading two entry is zero and third entry is a one. This is

$$\left\{ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\}$$

and this set is isomorphic to k^0.

Hence, we have that $\mathrm{Gr}_k(2,4) \cong k^4 \cup k^3 \cup k^2 \cup k^2 \cup k^1 \cup k^0$ as desired.

Problem 5 (Aluffi problem VI.3.8)**.** Let R be a commutative ring, M be a finitely-generated R-module, and let J be an ideal of M contained in the Jacobson radical of R. Prove that $M = 0$ if and only if $JM = M$.

Solution. We first prove the following lemmas (see Exercises VI.3.6 and VI.3.7 in Aluffi).

Lemma 1. *Let R be a commutative ring and $M = \langle m_1, \ldots, m_r\rangle$ a finitely-generated R-module. Let $A \in \mathcal{M}_r(R)$ be a matrix such that*

$$A \cdot \begin{bmatrix} m_1 \\ \vdots \\ m_r \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{1}$$

Then $\det(A)m = 0$ *for all* $m \in M$*.*

Proof (*of Lemma 1*)*.* Denote by $\text{adj}(A)$ the matrix that is the adjoint of A,

$$\text{adj}(A) = \begin{bmatrix} A^{(11)} & \ldots & A^{(r1)} \\ \vdots & \ddots & \vdots \\ A^{(1r)} & \ldots & A^{(rr)} \end{bmatrix}.$$

From Corollary VI.3.5 in Aluffi, we have that $\text{adj}(A) \cdot A = A \cdot \text{adj}(A) = \det(A)I_r$. Multiplying both sides of (1) above by $\text{adj}(A)^t$ yields

$$(\text{adj}(A)^t \cdot A) \begin{bmatrix} m_1 \\ \vdots \\ m_r \end{bmatrix} = \det(A) \begin{bmatrix} m_1 \\ \vdots \\ m_r \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Hence $\det(A)m_i = 0$ for each generator m_i of M, and thus $\det(A)M = 0$. □

Lemma 2. *Let R be a commutative ring, M a finitely-generated R-module, and let I be an ideal of R. Assuume that $IM = M$. Then there exists an element $b \in I$ such that $(1 + b)M = 0$.*

Proof (*of Lemma 2*)*.* Since $IM = M$, for each generator $m_i \in M$ there is a $b_i \in J$ such that

$$c_i(a_{i1}m_1 + \cdots + a_{ir}m_r) = m_i$$

for some $a_{ij} \in R$. Since J is an ideal and $c_i \in J$, we have that $b_{ij} := c_i a_{ij} \in J$ for all i, j. So we have a $r \times r$-matrix B with elements in J

$$B = \begin{bmatrix} b_{11} & \ldots & b_{1r} \\ \vdots & \ddots & \vdots \\ b_{r1} & \ldots & b_{rr} \end{bmatrix} \qquad \text{such that } B \cdot \begin{bmatrix} m_1 \\ \vdots \\ m_r \end{bmatrix} = \begin{bmatrix} m_1 \\ \vdots \\ m_r \end{bmatrix}.$$

Subtracting both sides of the above equation by $I_r \cdot \begin{bmatrix} m_1 \\ \vdots \\ m_r \end{bmatrix}$ yields

$$(I_r - B) \cdot \begin{bmatrix} m_1 \\ \vdots \\ m_r \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix},$$

and by the previous lemma we have $\det(I_r - B)m = 0$ for all $m \in M$. Since the elements of B are in J, expanding the determinant of $\det(I_r - B)$ we find that this is a polynomial in the entries of B

$$\det(I_r - B) = \sum_{\sigma \in S_r} (-1)^\sigma \prod_{i=1}^{r} (\delta_{i\sigma(i)} - b_{i\sigma(i)}).$$

Hence $\det(I_r - B) = (1 + b)$ for some $b \in J$ since each entry of B is in J. Thus there is an element $b \in J$ such that $(1 + b)m_i = 0$ for all generators $m_i \in M$, and thus $bm = 0$ for all $m \in M$, as desired. □

Recall that, for all elements b in the Jacobson radical of R, we have that $1+b$ is a unit of R. Finally, we prove the desired proposition.

Proposition 3. *Let R be a commutative ring and M a finitely generated R-module. Let J be an ideal of R that is contained in the Jacobson radical. Then $JM = M$ if and only if $M = 0$.*

Proof. Clearly if $M = 0$ then $JM = 0 = M$. So suppose that $JM = M$. From the previous lemma, we have that there exists an element $b \in J$ such that $(1+b)m = 0$ for all $m \in M$. But $1+b$ is a unit in R and thus

$$m = (1+b)^{-1}0 = 0$$

for all $m \in M$ and thus $M = 0$. □