Assignment 5

PMAT 611

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Problem 1 (Aluffi problem VI.4.7).

Let R be a commutative Noetherian ring, and let M be a finitely generated module over R. Prove that M admits a finite series

$$\langle 0\rangle = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{m-1} \subsetneq M_m = M$$

in which all quotients M_{i+1}/M_i are of the form $R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ of R.

Solution. We first prove a few lemmas (see Exercises VI.4.4 - VI.4.6 in Aluffi).

Lemma 1 (Exercise VI.4.4)**.** *Let R be a commutative ring. Then $\operatorname{Ann}_R(M)$ is an ideal of R and $\operatorname{Ann}_R(m)$ is an ideal of R for each $m \in M$.*

Proof. Let $a \in \operatorname{Ann}_R(M)$, then $am = 0$ for all $m \in M$. For all $b \in R$ we have $(ab)m = (ba)m = b(am) = 0$ for all m, so $ba \in \operatorname{Ann}_R(M)$. Similarly, if $a, a' \in \operatorname{Ann}_R(M)$ then $(a + a')m = am + a'm = 0$ for all $m \in M$. Hence $\operatorname{Ann}_R(M) \subset R$ is an ideal. Now let $m \in M$ and consider $\operatorname{Ann}_R(m)$. Then $(ab)m = (ba)m = b(am) = 0$ for all $b \in R$ and $(a + a')m = am + a'm = 0$ for all $a, a' \in \operatorname{Ann}_R(m)$. So $\operatorname{Ann}_R(m) \subset R$ is an ideal. □

Lemma 2 (Exercise VI.4.5)**.** *Let R be a commutative ring and M an R-module. Then for an ideal $I \subset R$ we have $I = \operatorname{Ann}_R(m)$ for some $m \in M$ if and only if there is a submodule $N \subset M$ such that $N \cong R/I$.*

Proof. Suppose that $I = \operatorname{Ann}_R(m)$ for some $m \in M$ and define $N = \langle m\rangle$. Consider the R-module homomorphism

$$\begin{aligned} \varphi\colon \langle m\rangle &\longrightarrow R/I \\ am &\longmapsto a + I, \end{aligned}$$

which is well-defined since if $a + I = a' + I$ then $(a - a') \in I$ so $(a - a')m = 0$ and thus $am = a'm$. This is injective, since $a + I = I$ if and only if $a \in I$ and thus $am = 0$. It is also surjective, since for all $a + I \in R/I$ we have $am \longmapsto a + I$. So φ is an isomorphism and thus $N = \langle m\rangle \cong R/I$.

Now suppose that M has a submodule $N \subset M$ such that $N \cong R/I$ for some I. Since R/I is generated by $1 + I$, we have that $N = \langle m\rangle$ for some $m \in N$ such that $m \longmapsto 1 + I$ under the isomorphism $N \cong R/I$. Then $I = \operatorname{Ann}_R(m)$. Indeed, we have $am = 0$ if and only if $0 = am \longmapsto a + I = I$ and thus $a \in I$. □

Lemma 3 (Exercise VI.4.6)**.** *Let R be a commutative ring and let M be an R-module. Consider the family of ideals*

$$\mathcal{A}_R(M) := \{\operatorname{Ann}_R(m) \mid m \in M,\, m \neq 0\}.$$

Then the maximal elements of $\mathcal{A}_R(M)$ are prime ideals of R.

Proof. Let $\mathfrak{m}$ be maximal in the family $\mathcal{A}_R(M)$. Then $\mathfrak{m} = \mathrm{Ann}_R(m)$ for some $m \in M$ and if there is an ideal $I \in \mathcal{A}_R(M)$ such that $\mathfrak{m} \subseteq I$ then $\mathfrak{m} = I$. Suppose that $ab \in \mathfrak{m}$ for some elements $a, b \in R$ such that $a \notin \mathfrak{m}$. Then we have $abm = 0$ and assume without loss of generality that $am \neq 0$. Then $b(am) = 0$ implies that $b \in \mathrm{Ann}_R(am)$. However, for all $r \in \mathrm{Ann}_R(m) = \mathfrak{m}$ we have $rm = 0$ and thus $r(am) = a(rm) = 0$ so $r \in \mathrm{Ann}_R(am)$. This tells us that $\mathfrak{m} \subseteq \mathrm{Ann}_R(am)$, but $\mathfrak{m}$ is maximal and thus $\mathfrak{m} = \mathrm{Ann}_R(am)$. Hence $b \in \mathfrak{m}$.

So we have that $ab \in \mathfrak{m}$ if and only if $a \in \mathfrak{m}$ or $b \in \mathfrak{m}$ and thus $\mathfrak{m}$ is a prime ideal in R. □

Definition (Associated prime)**.** Let R be a commutative ring and M be an R-module. An ideal $I \subset R$ is an *associated prime* of M if $I = \mathrm{Ann}_R(m)$ for some nonzero $m \in M$ and I is a prime ideal of R. The set of associated primes of M is denoted $\mathrm{Ass}_R(M)$.

Corollary 4 (Exercise VI.4.6)**.** *If R is a commutative Noetherian ring, then* $\mathrm{Ass}_R(M) \neq \emptyset$.

Proof. Suppose that there is no maximal element of $\mathcal{A}_R(M) := \{\mathrm{Ann}_R(m) \mid m \in M,\ m \neq 0\}$. Then for every element in $\mathcal{A}$ we can find a larger element that contains it and there is an ascending chain of ideals of R

$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$$

such that for each i there is an element $m_i \in M$ with $I_i = \mathrm{Ann}_R(m_i)$, a contradiction to the fact that R is Noetherian. Hence there is at least one maximal element $\mathfrak{m} = \mathrm{Ann}_R(m)$ of $\mathcal{A}_R(M)$. By the previous lemma, $\mathfrak{m}$ is a prime ideal of R and thus $\mathfrak{m} \in \mathrm{Ass}_R(M)$. □

We also note the definition of a Noetherian module and make a few useful notes about Noetherian modules.

Definition (Noetherian module)**.** Let R be a ring. An R-module M is *Noetherian* if every submodule is finitely generated.

Proposition 5. *Let R be a ring and M be a Noetherian R-module. Then M satisfies the ascending chain condition. That is, if there is a sequence of submodules $M_i \subseteq M$ such that*

$$M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$$

then there is an $n \in \mathbb{N}$ such that $M_i = M_n$ for all $i \geq n$.

Proof. Suppose otherwise. Then without loss of generality we may assume that there is a sequence of submodules of M

$$M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots .$$

Then $N = \bigcup_{i=0}^{\infty} M_i$ is a submodule of M. Since M is Noetherian, it is finitely generated so there is a finite set $a_1, \ldots, a_m$ of generators of M. But each a_i must be contained in some M_j, so there is an n such that $a_1, \ldots, a_m \in M_n$. But then $M_i = M$ for all $i \geq n$. □

Lemma 6. *Let M be an R-module, and let N be a submodule of M. If N and M/N are both finitely generated, then M is finitely generated.*

Proof. Since M/N is finitely generated there is a finite set of generators $a_1+N, \ldots, a_n+N$ of M/N. Similarly there is a finite set of generators $b_1, \ldots, b_l$ of N. Let $m \in M$. Then

$$m + N = \sum_{i=1}^{n} r_i(a_i + N) = \Big(\sum_{i=1}^{n} r_i a_i\Big) + N.$$

This implies that $m - \sum_{i=1}^n r_i a_i \in N$ and thus

$$m - \sum_{i=1}^{n} r_i a_i = s_1 b_1 + \cdots + s_l b_l$$

for some $s_j \in R$. Hence $m = r_1 a_1 + \cdots + r_n a_n + s_1 b_1 + \cdots + s_l b_l$ so M is generated by $a_1, \ldots, a_n, b_1, \ldots, b_l$ and thus is fintiely generated. □

Proposition 7. *Let R be a ring, M an R-module and $N \subset M$ a submodule. Then M is Noetherian if both N and M/N are Noetherian.*

Proof. Let P be a submodule of M, then we have to prove that P is finitely generated. Since $P \cap N$ is a submodule of M and N is Noetherian, we have that $P \cap N \subset N$ is finitely generated. By the Second Isomorphism Theorem for modules, we have that

$$\frac{P}{P \cap N} \cong \frac{P+N}{N}$$

and hence $\frac{P}{P \cap N}$ is isomorphic to a submodule of M/N. Since M/N is Noetherian, we have that $\frac{P}{P \cap N}$ is finitely generated. Hence P is finitely generated since, by Lemma 6, $P \cap N$ and $\frac{P}{P \cap N}$ are both finitely generated. □

Proposition 8. *Let R be a Noetherian ring. Then an R-module M is Noetherian if and only if it is finitely generated.*

Proof. If M is Noetherian then it is finitely generated since it is a submodule of itself and every submodule of M is finitely generated. So suppose that M is finitely generated, say by elements $a_1, \ldots, a_n \in M$. Then there is a surjection $R^{\oplus n} \twoheadrightarrow M$. By the previous proposition, it suffices to show that $R^{\oplus n}$ is Noetherian as an R-module.

We prove this by induction. Note that $R^{\oplus 1} = R$ is Noetherian by assumption. So suppose that $R^{\oplus n}$ is Noetherian for some $n \geq 1$. Since $R^{\oplus n}$ may be viewed as a submodule of $R^{\oplus(n+1)}$ such that

$$\frac{R^{\oplus(n+1)}}{R^{\oplus n}} \cong R,$$

which is Noetherian, and $R^{\oplus n}$ is Noetherian by assumption, it follows from Proposition 7 that $R^{\oplus(n+1)}$ is Noetherian as well. □

We can now prove the proposition of the problem.

Proposition 9 (Problem statement)**.** *Let R be a commutative Noetherian ring, and let M be a finitely generated module over R. Then M admits a finite series*

$$\langle 0 \rangle = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{m-1} \subsetneq M_m = M$$

in which all quotients M_i/M_{i-1} are of the form $R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ of R.

Proof. First note that $\mathrm{Ass}_R(M) \neq \emptyset$ by the Corollary 4 above, so there exists a prime ideal $\mathfrak{p}_1 \subset R$ such that $\mathfrak{p}_1 = \mathrm{Ann}_R(m_1)$ for some $m_1 \in M$. By Lemma 2, there is a submodule $M_1 \subset M$ such that $M_1 \cong R/\mathfrak{p}_1$. Furthermore, we have $M_1/M_0 = M_1 \cong R/\mathfrak{p}_1$ where $M_0 = 0$.

We follow by induction. Suppose that for some $n \geq 1$ we have a series of submodules $M_0, M_1, \ldots, M_n$ of M such that

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n \subsetneq M$$

such that for each $i = 1, \ldots, n$ we have $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ where each $\mathfrak{p}_i$ is prime. If $M/M_n \cong \mathfrak{p}$ for some prime ideal $\mathfrak{p}$ then we are done. Otherwise $M/M_n \neq 0$ and we have $\mathrm{Ass}_R(M/M_n) \neq \emptyset$. So there is a submodule $M' \subset M/M_n$ such that $M' \cong R/\mathfrak{p}_{n+1}$ for some prime $\mathfrak{p}_{n+1}$. Set M_{n+1} as the inverse image of M' such that $M_{n+1}/M_n = M' \cong R/\mathfrak{p}_{n+1}$.

Since R is Noetherian and M is finitely generated, M is Noetherian as an R-module. Thus by Proposition 5 we must have that the sequence

$$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots$$

eventually terminates with $M_m = M$ for some $m \in \mathbb{N}$. This proves the claim. □

Problem 2 (Aluffi problem VI.4.13).

Let R be a commutative ring. A tuple $(a_1, \ldots, a_n)$ of elements in R is a *regular sequence* if a_1 is a non-zero-divisor in R and each a_i is a non-zero-divisor modulo $(a_1, \ldots, a_{i-1})$ for $i > 1$.

For $a, b \in R$, consider the following complex of R-modules:

$$0 \longrightarrow R \xrightarrow{d_2} R \oplus R \xrightarrow{d_1} R \xrightarrow{\pi} \frac{R}{(a,b)} \longrightarrow 0 \tag{*}$$

where π is the canonical projection, $d_1(r,s) = ra + sb$ and $d_2(t) = (bt, -at)$. That is, d_1 and d_2 correspond to the matrices

$$\begin{pmatrix} a & b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a \\ -b \end{pmatrix}.$$

i) Prove that this is indeed a complex for every a and b.

ii) Prove that if (a, b) is a regular sequence, then this complex is exact.

The complex (*) is called the *Kozul complex* of (a, b). Thus, when (a, b) is a regular sequence, the Kozul complex provides us with a free resolution of the module $R/(a, b)$.

Solution. *Proof.* .

i) Note that $d_1(r, s) = ar + sb \in (a, b)$ for all $r, s \in R$, hence $\pi(d_1(r, s)) = 0$ and thus $\text{im}(d_1) \subset \ker \pi$. Similarly,

$$d_1(d_2(t)) = d_1(bt, -at) = bta - atb = 0$$

for all t and thus $\text{im}(d_2) \subset \ker(d_1)$. So (*) is indeed a complex.

ii) We examine the complex at each spot to determine exactness. Since the sequence (a, b) is regular, we have that a is a non-zero-divisor and b is a non-zero-divisor modulo (a). That is, $bc \notin (a)$ for all $c \notin (a)$.

- The complex is clearly exact at $R \xrightarrow{\pi} \frac{R}{(a,b)} \longrightarrow 0$ by surjectivity of π.
- Let $t \in \ker \pi = (a, b)$, then $t = ra + sb$ for some $r, s \in R$. Hence $t = d_1(r, s)$ and thus $\text{im}(d_1) = \ker \pi$ so the complex is exact in the second spot.
- Let $(r, s) \in \ker(d_1)$. Then $d_1(r, s) = ra + sb = 0$ such that $sb = -ra$ and thus $sb \in (a)$. But b is a non-zero-divisor modulo (a). Hence $sb \in (a)$ implies $s \in (a)$ and thus $s = t'a$ for some $t' \in R$. So we have

  $$ra + (t'a)b = 0 \quad \Longrightarrow \quad a(r + t'b) = 0.$$

  But a is a non-zero-divisor and thus $(r + t'b) = 0$, so $r = -t'b$. Setting $t = -t' \in R$ yields $(r, s) = (bt, -at)$ and thus $(r, s) \in \text{im}(d_2)$. Hence $\text{im}(d_2) = \ker(d_2)$ so we have that the complex is exact at $R \xrightarrow{d_2} R \oplus R \xrightarrow{d_1} R$.
- Finally, the complex is exact in the last spot since d_2 is injective. Indeed, for some $t \in R$ suppose that $d_2(t) = (bt, -at) = (0, 0)$. Then $at = 0$, but a is a non-zero-divisor and thus $t = 0$.

□

Problem 3 (Aluffi problem VI.5.5)**.**

Recall that a commutative ring is *local* if it has a single maximal ideal $\mathfrak{m}$. Let R be a local ring and let M be the *direct summand* of a finitely generated free R-module. That is, there exists an R-module N such that $M \oplus N$ is a free R-module.

i) Choose elements $m_1, \ldots, m_r \in M$ whose cosets mod $\mathfrak{m}M$ are a basis of $M/\mathfrak{m}M$ as a vector space over the field $R/\mathfrak{m}$. By Nakayama's lemma, $M = \langle m_1, \ldots, m_r \rangle$.

ii) Obtain a surjective homomorphism $\pi \colon R^{\oplus r} \longrightarrow M$.

iii) Show that π splits, giving an isomorphism $R^{\oplus r} \cong M \oplus \ker \pi$.

iv) Show that $\ker \pi / \mathfrak{m} \ker \pi = 0$. Use Nakayama's lemma to deduce that $\ker \pi = 0$.

v) Conclude that $M \cong R^{\oplus r}$ and this M is in fact free.

Summarizing, over a local ring, every direct summand of a finitely generated free R-module is free. (Contrast this fact with Proposition VI.5.1, which shows that every submodule of a finitely generated free module is free.)

Solution. Recall the two statements of Nakayama's Lemma that we will use:

Lemma 10 (Nakayama)**.** *Let R be a commutative ring and M an R-module. Suppose $I \subset J$ is an ideal of R that is contained in the Jacobson radical J of R.*

(*1*) *If $m_1, \ldots, m_r$ have images in M/IM that generate it as an R/I-module, then $m_1, \ldots, m_r$ generate M as an R-module.*

(*2*) *If $M/IM = 0$ then $M = 0$.*

Since the Jacobson radical is the intersection of all maximal ideals of R, in this case we have that $J = \mathfrak{m}$. We now prove the problem statement.

i) First note that M is finitely generated. Indeed, since there is an R-module N such that $R^{\oplus n} \cong M \oplus N$, there is a surjective map $R^{\oplus n} \longrightarrow M$ given by the projection map

$$R^{\oplus n} \cong M \oplus N \xrightarrow{\pi_M} M.$$

Since $\mathfrak{m} \subset R$ is maximal, $R/\mathfrak{m}$ is a field. Note that $M/\mathfrak{m}M$ has an $R/\mathfrak{m}$-module structure given by

$$(r + \mathfrak{m})(m + \mathfrak{m}M) = rm + \mathfrak{m}M$$

and thus $M/\mathfrak{m}M$ is an $R/\mathfrak{m}$-vector space. Also note that $M/\mathfrak{m}M$ is finitely generated since M is. So $M/\mathfrak{m}M \cong (R/\mathfrak{m})^{\oplus r}$ for some integer r and thus has a basis given by $m_1 + \mathfrak{m}M, \ldots, m_r + \mathfrak{m}M$. By Nakayama's lemma, $m_1, \ldots, m_r$ also generates M.

ii) Since $m_1, \ldots, m_r$ generate M, we have a surjective homomorphism

$$\begin{aligned} \pi \colon R^{\oplus r} &\longrightarrow M \\ (a_1, \ldots, a_r) &\longmapsto a_1 m_1 + \cdots + a_r m_r. \end{aligned}$$

iii) Since π is surjective and $M \oplus N = R^{\oplus n}$ is free (and thus projective), there is an R-module homomorphism $R^{\oplus n} \xrightarrow{\beta} R^{\oplus r}$ such that the diagram

$$\begin{array}{ccc} R^{\oplus r} & \xrightarrow{\pi} & M \\ & {\scriptstyle\beta}\nwarrow & \uparrow{\scriptstyle \pi_M} \\ & & R^{\oplus n} \cong M \oplus N \end{array}$$

commutes (see Exercise III.6.9 in Aluffi), where $M \oplus N \xrightarrow{\pi_M} M$ is the projection that maps the direct sum onto M. Note that π_M has the natural right-inverse given by $M \xrightarrow{\iota_M} M \oplus N$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\pi & \longrightarrow & R^{\oplus r} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & & & & & {\scriptstyle\beta}\nwarrow & {\scriptstyle\iota_M}\downarrow\uparrow{\scriptstyle\pi_M} & & \\ & & & & & & R^{\oplus n} \cong M \oplus N & & \end{array}$$

commutes. Hence π has a right-inverse with $\pi \circ \beta \circ \iota_M = \mathrm{id}_M$ and thus the exact sequence

$$0 \longrightarrow \ker\pi \longrightarrow R^{\oplus r} \xrightarrow{\pi} M \longrightarrow 0 \tag{**}$$

splits. This is equivalent to saying that $R^{\oplus r} \cong M \oplus \ker\pi$.

iv) Tensoring $- \otimes_R R/\mathfrak{m}$ with $R^{\otimes r}$ and $M \oplus \ker\pi$ yields the R-modules

$$R^{\otimes r} \otimes_R R/\mathfrak{m} \cong (R/\mathfrak{m})^{\oplus r} \quad \text{and} \quad (M \oplus \ker\pi) \otimes_R R/\mathfrak{m} \cong (M/\mathfrak{m}M) \oplus (\ker\pi/\mathfrak{m}\ker\pi),$$

which all finite-dimensional as $R/\mathfrak{m}$-vector spaces. We have the exact sequence

$$0 \longrightarrow \ker\pi/\mathfrak{m}\ker\pi \longrightarrow (R/\mathfrak{m})^{\oplus r} \xrightarrow{\pi} M/\mathfrak{m}M \longrightarrow 0.$$

But $M/\mathfrak{m}M \cong (R/\mathfrak{m})^{\oplus r}$ as an $(R/\mathfrak{m})$-vector space, so we have that $\ker\pi/\mathfrak{m}\ker\pi = 0$. Thus $\ker\pi = 0$ by Nakayama's lemma.

v) Since $\ker\pi = 0$, we have the exact sequence

$$0 \longrightarrow R^{\oplus r} \xrightarrow{\pi} M \longrightarrow 0$$

and thus $M \cong R^{\oplus r}$ so M is free as desired.