Algebra III

Final Exam Review

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All exercises from Aluffi's *Algebra: Chapter 0*.

Chapter VI questions

Section 6

Exercise 6.9

Prove the Cayley-Hamilton theorem as follows. Recall that every square matrix M has an *adjoint* matrix, denoted by $\operatorname{adj}(M)$, such that $\operatorname{adj}(M)\cdot M = \det(M)I$. Applying this to $M = tI - A$ (with A a matrix realizaion of $\alpha \in \operatorname{End}_{R\text{-}\mathsf{Mod}}(F)$ gives

$$\operatorname{adj}(tI - A)\cdot(tI - A) = P_\alpha(t)I \tag{$*$}$$

where $P_\alpha(t)$ is the characteristic polynomial of α. Prove that there exist matrices $B_k \in \mathcal{M}_n(R)$ such that

$$\operatorname{adj}(tI - A) = \sum_{k=0}^{n-1} B_k t^k,$$

then use $(*)$ to obtain $P_\alpha(A) = 0$, proving Cayley-Hamilton.

Solution. Recall from the definition of the adjoint of a matrix that

$$\operatorname{adj}(tI - A) = \begin{bmatrix} (tI-A)^{(11)} & \cdots & (tI-A)^{(n1)} \\ \vdots & \ddots & \vdots \\ (tI-A)^{(1n)} & \cdots & (tI-A)^{(nn)} \end{bmatrix}$$

where each cofactor $(tI - A)^{(ij)}$ is equal to $(-1)^{i+j}$ times the determinant of the matrix of the matrix produced from deleting the i^{th} row and j^{th} column of $(tI - A)$. Note that each cofactor is an $(n-1)$-degree polynomial in t, which we may write as

$$(tI - A)^{(ij)} = b_0^{(ij)} + b_1^{(ij)}t + \cdots + b_{n-1}^{(ij)}t^{n-1}$$

and thus we can write the adjoint matrix as a sum of matrices $\operatorname{adj}(tI - A) = \sum_{k=0}^{n-1} B_k t^k$ where

$$B_k = \begin{bmatrix} b_k^{(11)} & \cdots & b_k^{(n1)} \\ \vdots & \ddots & \vdots \\ b_k^{(1n)} & \cdots & b_k^{(nn)} \end{bmatrix}.$$

We can also write the characteristic polynomial as $P_\alpha(t) = c_0 + c_1 t + \cdots + c_{n-1}t^{n-1} + t^n$, so expanding both sides of $(*)$ yields

$$\sum_{k=0}^{n-1} B_k t^k (tI - A) = -B_0 A + t(B_2 - B_1 A) + t^2(B_3 - B_2 A) + \cdots + t^{n-1}(B_{n-2} - B_{n-3}A) + t^n B_{n-1}$$

$$= c_0 I + c_1 tI + c_2 t^2 I + \cdots + c_{n-1}t^{n-1}I + t^n I. \qquad (\dagger)$$

Hence we have the relations $B_k = c_{k+1}I + B_{k-1}A$ for $1 \leq k \leq n-2$, as well as $B_{n-1} = I$ and $B_0 A = -c_0 I$. Clearly, B_0 and B_{n-1} commutes with A, and B_k commutes with A for $k = 1, \ldots, n-1$ if B_{k-1} does.

Finally, we see that plugging in A in the polynomial in $(\dagger)$ yields the zero matrix. Thus $P_\alpha(A) = 0$.

Exercise 6.10

Let F_1, F_2 be free R-modules of finite rank and let α_1 resp. α_2 be linear transformations of F_1 resp. F_2. Let $F = F_1 \oplus F_2$ and let $\alpha = \alpha_1 \oplus \alpha_2$.

- Prove that $P_\alpha(t) = P_{\alpha_1}(t)P_{\alpha_2}(t)$.
- Find an example showing that the minimal polynomial is not multiplicative over sums.

Solution. .

- Note that if α_1 is represented in some basis $\{e_1, \ldots, e_{n_1}\}$ by a matrix A_1 and α_2 is represented in some basis $\{f_1, \ldots, f_{n_2}\}$ a matrix by A_2, then a matrix representation of $\alpha = \alpha_1 \oplus \alpha_2$ in the basis $\{e_1, \ldots, e_{n_1}, f_1, \ldots, f_{n_2}\}$ is the block matrix

  $$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}.$$

  Then α has characteristic polynomial given by $P_\alpha(t) = \det(tI_{n_1+n_2} - A)$, or

  $$P_\alpha(t) = \det \begin{bmatrix} tI_{n_1} - A_1 & 0 \\ 0 & tI_{n_2} - A_2 \end{bmatrix},$$

  but the determinant of a block matrix of this form is the product of the determinants of the blocks. So $P_\alpha(t) = \det(tI_{n_1} - A_1)\det(tI_{n_2} - A_2) = P_{\alpha_1}(t)P_{\alpha_2}(t)$.
- For a counter example, consider the zero morphism on $\mathbb{Z}$ as a $\mathbb{Z}$-module, which has minimal polynomial $p(t) = t$. Then the direct sum of the zero morphism with itself acting on $\mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^2$ also has minimal polynomial $p(t) = t \neq t^2$.

Exercise 6.12

Let α be a linear transformation of a finite dimensional $\mathbb{C}$-vector space V. Prove the identity of formal power series with coefficents in $\mathbb{C}$:

$$\frac{1}{\det(1 - \alpha t)} = \exp\left(\sum_{k=1}^{\infty} \operatorname{Tr} \alpha^k \frac{t^k}{k}\right).$$

Solution. Since $\mathbb{C}$ is an algebraically closed field, every polynomial factors completely into linear terms. Hence, every linear transformation of a complex vector space has a Jordan canonical form and instead of considering a linear transformation α we may consider its Jordan form

$$A = \begin{bmatrix} J_{\lambda_1, r_1} & & & \\ & J_{\lambda_2, r_2} & & \\ & & \ddots & \\ & & & J_{\lambda_m, r_m} \end{bmatrix},$$

where each Jordan block is of the form

$$J_{\lambda_i,r_i} = \underbrace{\begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}}_{r_i}$$

Note that $\operatorname{Tr} A^k = \sum_i \operatorname{Tr} J_{\lambda_i,r_i}^k = \sum_i r_i \lambda_i^k$. That is, the trace of A^k is the sum of the traces of the Jordan blocks to the power k. Noting that $\sum_{k=1}^{\infty} \frac{\lambda^k t^k}{k}$ is the Taylor expansion for $-\ln(1-\lambda t)$, this gives us

$$\begin{aligned} \exp\left(\sum_{k=1}^{\infty} \operatorname{Tr} A^k \frac{t^k}{k}\right) &= \exp\left(\sum_i r_i \sum_{k=1}^{\infty} \frac{t^k \lambda_i^k}{k}\right) \\ &= \exp\left(\sum_i r_i\big(-\ln(1-t\lambda_i)\big)\right) \\ &= \prod_i \frac{1}{(1-t\lambda_i)^{r_i}} \\ &= \frac{1}{\prod_i \det(I - tJ_{\lambda_i,r_i})} \\ &= \frac{1}{\det(I-At)} \end{aligned}$$

as desired.

Exercise 6.14

Let λ be an eigenvalue of two similar transformations α, β. Prove that the geometric multiplicities of λ with respect to α and β coincide.

Solution. Since α and β are similar, there is an automorphism π such that $\beta = \pi \circ \alpha\pi^{-1}$. Suppose that $\dim\big(\ker(\lambda I - \alpha)\big) = n$ such that there is a basis $\{e_1, \dots, e_n\}$ of $\ker(\lambda I - \alpha)$. Then the set $\{\pi(e_1), \dots, \pi(e_n)\}$ is a linearly independent set in $\ker(\lambda I - \beta)$. Indeed, we have

$$0 = r_1\pi(e_1) + \cdots + r_n\pi(e_n) = \pi(r_1 e_1 + \cdots r_n e_n)$$

implies $r_1 e_1 + \cdots r_n e_n = 0$ and thus $r_i = 0$ for all i. So the dimension of $\ker(\lambda I - \beta)$ is at least n. However, we may analogously suppose that $\dim\big(\ker(\lambda I - \alpha)\big) = m$ such that there is a basis $\{f_1, \dots, f_m\}$ of $\ker(\lambda I - \beta)$. Then $\{\pi^{-1}(f_1), \dots, \pi^{-1}(f_n)\}$ is a linearly independent set in $\ker(\lambda I - \alpha)$. So the dimension of $\ker(\lambda I - \beta)$ is at most n, thus the dimensions coincide. Hence the geometric multiplicities of λ with respect to α and β coincide.

Exercise 6.15

Let α be a linear transformation on a free R-module F and let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be eigenvectors corresponding to pairwise distinct eigenvalues $\lambda_1, \dots, \lambda_n$. Prove that $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly independent.

Solution. Suppose otherwise that $\mathbf{v}_1, \dots, \mathbf{v}_n$ are not linearly independent. Then there is a shortest linear combination of elements of $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ given by $\mathbf{v} = r_1\mathbf{v}_{i_1} + \cdots + r_m\mathbf{v}_{i_m} = 0$ with $r_j \in R$ and all $r_j \neq 0$. Then application of α yields

$$\alpha(\mathbf{v}) = \lambda_{i_1} r_1 \mathbf{v}_{i_1} + \lambda_{i_2} r_2 \mathbf{v}_{i_2} + \cdots + \lambda_{i_m} r_m \mathbf{v}_{i_m} = 0.$$

However, multiplication of $\mathbf{v}$ by λ_{i_1} yields

$$\lambda_{i_1} r_1 \mathbf{v}_{i_1} + \lambda_{i_1} r_2 \mathbf{v}_{i_2} + \cdots + \lambda_{i_1} r_m \mathbf{v}_{i_m} = 0$$

and subtracting the first equation from the second yields a shorter linear combination

$$(\lambda_{i_1} - \lambda_{i_2}) r_2 \mathbf{v}_{i_2} + \cdots + (\lambda_{i_1} - \lambda_{i_m}) r_m \mathbf{v}_{i_m} = 0.$$

Hence we must have $(\lambda_{i_1} - \lambda_{i_k}) r_{i_k} = 0$ for some $k \in \{2, \ldots, m\}$. Since we are working over integral domains in this chapter, we must have $\lambda_{i_1} - \lambda_{i_k} = 0$, a contradiction to the assumption that all of the eigenvalues are distinct.

Section 7

Exercise 7.9

What is the number of distinct similarity classes of linear transformations on an n-dimensional vector space, with one fixed eigenvalues λ with algebraic multiplicity n?

Solution. This is equal to the number of partitions of n. There is no closed formula for this (as far as I know).

Exercise 7.10

Let k be a field. Classify all matrices $A \in \mathcal{M}_n(k)$ such that $A^2 = A$, up to similarity. Describe the action of such matrices 'geometrically'.

Solution. Note that we have $A^2 - A = 0$, so the minimal polynomial must divide the polynomial $t^2 - t$, which factors completely into $t(t-1)$ since this is a field. So the eigenvalues of A can only be 0 or 1. Furthermore, the blocks of the Jordan canonical form can have size no greater than 1. So the Jordan canonical form of A must have only 1s and 0s on its diagonal. Thus there are $n-1$ similarity classes of matrices such that $A^2 = A$, where the similarity class is determined by the number of 1s on the diagonal of its Jordan canonical form.

Geometrically, such an automorphism of a vector space is a projection onto a subspace.

Exercise 7.11

A square matrix $A \in \mathcal{M}_n(k)$ is *nilpotent* if $A^r = 0$ for some integer r.

- Characterize nilpotent matrices in terms of their Jordan canonical form.
- Prove that if $A^r = 0$ for some integer r, then $A^{r'} = 0$ for some integer r' no larger than n.
- Prove that the trace of a nilpotent matrix is zero.

Solution. Note that the minimal polynomial of A must divide the polynomial t^r. That is, the minimial polynomial must be t^m for some integer $m \leq r$. Furthermore, the only eigenvalue of A must be zero. So the characteristic polynomial of A is $p_A(t) = t^n$. Since the minimal polynomial must divide the characteristic polynomial, we must have $m \leq n$. Putting this all together, the Jordal normal form of A must be

$$PAP^{-1} = \begin{bmatrix} J_{0,r_1} & & & \\ & J_{0,r_2} & & \\ & & \ddots & \\ & & & J_{0,r_l} \end{bmatrix}$$

where the $r_i's$ are the size of the Jordan blocks

$$J_{0,r_i} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{r_i}$$

and the degree of the minimal polynomial is $r = \max_i r_i$, i.e. the size of the largest Jordan block. Hence, we see from the previous problem that the number of similarity classes of nilpotent matrices of size n is equal to the number of partitions of n. Two nilpotent matrices are similar if the sizes of all of their Jordan blocks are the same.

Finally, for each block we see that $\operatorname{Tr} J_{0,r_i} = 0$, since all of the elements on the diagonal are zero. So $\operatorname{Tr} A = 0$ for any nilpotent matrix, since the trace of A will be equal to the sum of the traces of its Jordan blocks.

Exercise 7.15

A *complte flag* of subspaces of a vector space V of dimension n is a sequence of nested subspaces

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V_n = V$$

with $\dim V_i = i$.

Let V be a finite dimensional vector space over an algebraically closed field. Prove that every linear transformation α of V preserves a complete flag. That is, there is a complete flag such that $\alpha(V_i) \subset V_i$. Find a linear transformation that does not preserve a complete flag.

Solution. Given a basis $\{e_1, \dots, e_n\}$ of a vector space we can define a complete flag of V by $V_i = \operatorname{span}\{e_1, \dots, e_i\}$. Any matrix that is upper triangular in this basis will preserve this complete flag. Hence, if a linear transformation can be represented by a matrix that is upper triangular in some basis, then it preserves some complete flag (where the flag is defined by the basis in which it is upper triangular). Since we are working over an algebraically closed field, every polynomial factors completely. So every linear transformation as a Jordan canonical representation in some basis and this matrix is upper triangular.

Chapter VIII questions

Section 3

Exercise 3.1

Verify that a combination of pure tensors $\sum_i (m_i \otimes n_i)$ is zero in the tensor product $M \otimes_R N$ if and only if $\sum_i (m_i, n_i) \in \mathbb{Z}^{\oplus(M \times N)}$ is a combination of the elements

$$(m, n_1 + n_2) - (m, n_1) - (m, n_2)$$
$$(m_1 + m_2, n) - (m_1, n) - (m_2, n)$$
$$(rm, n) - (m, rn)$$

with $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$ and $r \in R$.

Solution. ???

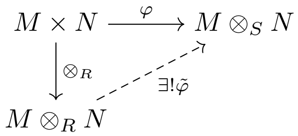
Exercise 3.2

If $f\colon R \to S$ is a ring homomorphism and M and N are S-modules (hence R-modules by restrction of scalars), prove that there is a canonical homomorphism of R-modules $M \otimes_R N \to M \otimes_S N$.

Solution. Let $\varphi\colon M \times N \longrightarrow M \otimes_S N$ be the map $\varphi(m,n) = m \otimes n$. This is R-biliear since

$$\begin{aligned}\varphi(r_1 \cdot m_1 + r_2 \cdot m_2, n) &= (r_1 \cdot m_1 + r_2 \cdot m_2) \otimes n \\ &= (f(r_1)m_1 + f(r_2)m_2) \otimes n \\ &= f(r_1)(m_1 \otimes n) + f(r_2)(m_2 \otimes r) \\ &= r_1\varphi(m_1, n) + r_2\varphi(m_2, n),\end{aligned}$$

and analogously in the second spot. Then there is a unique R-module homomorphism given by the universal property of the tensor product



such that the above diagram commutes.

Exercise 3.3

Let R and S be commutative rings and let M be an R-module, N an (R,S)-bimodule and P an S-module. Prove that there is an isomorphism of R-modules

$$M \otimes_R (N \otimes_S P) \cong (M \otimes_R N) \otimes_S P.$$

Solution. First note that $(M \otimes_R N) \otimes_S P$ is an R-module with R-action given by

$$r \cdot \left[\left(\sum_i m_i \otimes n_i\right) \otimes p\right] = \left(\sum_i rm_i \otimes n_i\right) \otimes p.$$

Let $\varphi\colon M \otimes_R (N \otimes_S P) \longrightarrow (M \otimes_R N) \otimes_S P$ be the map defined on pure tensors as

$$m \otimes \left(\sum_i n_i \otimes p_i\right) \overset{\varphi}{\longmapsto} \sum_i (m \otimes n_i) \otimes p_i,$$

and we can extend this to all of $(M \otimes_R N) \otimes_S P$ by R-linearity. Similarly, we can define another map $\psi\colon (M \otimes_R N) \otimes_S P \longrightarrow M \otimes_R (N \otimes_S P)$ be the map defined on pure tensors as

$$\left(\sum_i m_i \otimes n_i\right) \otimes p \overset{\psi}{\longmapsto} \sum_i m_i \otimes (n_i \otimes p)$$

and extend by R-linearity.

Composing $\varphi \circ \psi$ and acting on pure tensors yields

$$\begin{aligned}\varphi\left(\psi\left(\left(\sum_i m_i \otimes n_i\right) \otimes p\right)\right) &= \varphi\left(\sum_i m_i \otimes (n_i \otimes p)\right) \\ &= \sum_i \varphi\big(m_i \otimes (n_i \otimes p)\big) \\ &= \sum_i (m_i \otimes n_i) \otimes p \\ &= \left(\sum_i m_i \otimes n_i\right) \otimes p\end{aligned}$$

and thus $\varphi \circ \psi = \mathrm{id}_{(M \otimes_R N) \otimes_S P}$. Similarly, composing $\psi \circ \varphi$ and acting on pure tensors yields

$$\begin{aligned}
\psi\left(\varphi\left(m \otimes \left(\sum_i n_i \otimes p_i\right)\right)\right) &= \psi\left(\sum_i (m \otimes n_i) \otimes p_i\right) \\
&= \sum_i \psi\big((m \otimes n_i) \otimes p_i\big) \\
&= \sum_i m \otimes (n_i \otimes p_i) \\
&= m \otimes \left(\sum_i n_i \otimes p_i\right).
\end{aligned}$$

Hence $\psi \circ \varphi = \mathrm{id}_{M \otimes_R (N \otimes_S P)}$. Thus, these maps give us the desired isomorphism.

Exercise 3.4

Use the associativity of the tensor product to prove the formula below. Let R be a commutative ring, S a multiplicative subset of R, and M an R-module.

- Let N be an $S^{-1}R$-module. Prove that $(S^{-1}M) \otimes_{S^{-1}R} N \cong M \otimes_R N$.
- Let A be an R-module. Prove that $(S^{-1}A) \otimes_R M \cong S^{-1}(A \otimes_R M)$

Solution. Recall that, for any R-module M, we have $M \otimes_R R \cong M$. Also, given a multiplicative subset of R, we have $M \otimes_R (S^{-1}R) \cong S^{-1}M$.

- First note that N is also an R-module with canonical R-action given by $r \cdot n = \frac{r}{1}n$. Noting that $(S^{-1}M) \cong (S^{-1}R) \otimes_R M$, by the associativity of the tensor product we have

$$\begin{aligned}
(S^{-1}M) \otimes_{S^{-1}R} N &\cong \big(M \otimes_R (S^{-1}R)\big) \otimes_{S^{-1}R} N \\
&\cong M \otimes_R \big((S^{-1}R) \otimes_{S^{-1}R} N\big) \\
&\cong M \otimes_R N.
\end{aligned}$$

- Similarly, we have

$$\begin{aligned}
(S^{-1}A) \otimes_R M &\cong \big(S^{-1}R \otimes_R A\big) \otimes_R M \\
&\cong (S^{-1}R) \otimes_R (A \otimes_R M) \\
&\cong S^{-1}(A \otimes_R M).
\end{aligned}$$

Exercise 3.5

(Limits and colimits?)

Exercise 3.6

Let $f \colon R \to S$ be a ring homomorphism and let $\varphi \colon N_1 \to N_2$ be a homomorphism of S-modules. Prove that φ is an isomorphism if and only if $f_*(\varphi)$ is an isomorphism. In fact, prove that f_* is faithfully exact: a sequence of S-modules

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

is exact if and only if the sequence of R-modules

$$0 \longrightarrow f_*(L) \longrightarrow f_*(M) \longrightarrow f_*(N) \longrightarrow 0$$

is exact.

Solution. First note that, for any S-module P, the R-module $f_*(P)$ and the S-module P have the same underlying sets and abelian group structure. Let $\varphi\colon P \longrightarrow Q$ be a homomorphism of S-modules. Then as abelian groups we have $\ker\varphi = \ker f_*(\varphi)$ and $\operatorname{im}\varphi = \operatorname{im} f_*(\varphi)$.

Hence, given the sequence of S-modules

$$0 \longrightarrow L \xrightarrow{\ \alpha\ } M \xrightarrow{\ \beta\ } N \longrightarrow 0 \tag{$*$}$$

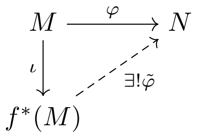
we have that $\ker\alpha = 0$ if and only if $\ker f_*(\alpha) = 0$. So the sequence $(*)$ is exact at L if and only if the sequence

$$0 \longrightarrow f_*(L) \xrightarrow{f_*(\alpha)} f_*(M) \xrightarrow{f_*(\beta)} f_*(N) \longrightarrow 0 \tag{$**$}$$

is exact at $f_*(L)$. Similarly, we have that $\operatorname{im}\beta = N$ if and only if $\operatorname{im} f_*(\beta) = f_*(N)$ and N and $f_*(N)$ are the same as abelian groups, so the sequence $(*)$ is exact at N if and only if the sequence $(**)$ is exact at $f_*(N)$. Finally, we have $\operatorname{im}\alpha = \ker\beta$ if and only if $\operatorname{im} f_*(\alpha) = \ker f_*(\beta)$, since we have $\operatorname{im}\alpha = \operatorname{im} f_*(\alpha)$ and $\ker\beta = \ker f_*(\beta)$ as abelian groups. Hence the sequence $(*)$ is exact at M if and only if the sequence $(**)$ is exact at $f_*(M)$.

Exercise 3.9

Let $f\colon R \to S$ be a ring homomorphism and let M be an R-module. Prove that the extension $f^*(M)$ satisfies the following universal property: if N is an S-module and $\varphi\colon M \to N$ is an R-linear map, then there is a unique S-linear map $\tilde\varphi\colon f^*(M) \to N$ making the diagram



commute, where $\iota\colon M \to f^*(M) = M \otimes_R S$ is defined by $m \mapsto m \otimes 1$.

Solution. Define $\tilde\varphi\colon f^*(M) \to N$ through $\tilde\varphi(m\otimes s) = m\varphi(s)$ end extend by S-linearity. This indeed makes the diagram commute. Suppose there is another S-linear map $\psi\colon f^*(M) \to N$ that makes the diagram commute. That is, $\psi(\iota(m)) = \psi(m\otimes 1) = \varphi(m)$ for all $m \in M$. Then for all $m \in M$ and $s \in S$ we have

$$\psi(m\otimes s) = \psi\big(s(m\otimes 1)\big) = s\psi(m\otimes 1) = s\varphi(m) = s\tilde\varphi(m\otimes 1) = \tilde\varphi(m\otimes s)$$

and thus $\psi = \tilde\varphi$.

Exercise 3.11

Let $f\colon R \to S$ be a ring homomorphism and let M be a flat R-module. Prove that $f^*(M)$ is a flat S-module.

Solution. Recall that $f^*(M) = S \otimes_R M$ is an S-module and let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be an exact sequence of S-modules. Tensoring this sequence with $f^*(M)$ yields the sequence

$$0 \longrightarrow A \otimes_S (S \otimes RM) \longrightarrow B \otimes_S (S \otimes RM) \longrightarrow C \otimes_S (S \otimes RM) \longrightarrow 0.$$

Any S-module N is also an R-module by restriction of scalars, so by associativity of the tensor product we have $N \otimes_S (S \otimes_R M) \cong (N \otimes_S S) \otimes_R M \cong N \otimes_R M$. This gives the exact sequence of S-modules

$$0 \longrightarrow A \otimes_R M \longrightarrow B \otimes_R M \longrightarrow C \otimes_R M \longrightarrow 0.$$

Since each $N \otimes_R M$ is an R-module by restriction of scalars, this sequence is exact as S-modules if and only if it is exact as a sequence of R-modules (from the previous problem). But this is indeed exact since M is a flat R-module.

Chapter IX questions

Exercise 1.1

Prove that if $\psi \circ \varphi$ is an epimorphism, then ψ is an epimorphism. Prove that if $\psi \circ \varphi$ is a monomorphism, then φ is a monomorphism.

Solution. .

- Let α and β be morphisms such that $\alpha \circ \psi = \beta \circ \psi$. Then composition with φ yeilds that $\alpha \circ \psi \circ \varphi = \beta \circ \psi \circ \varphi$. But $\psi \circ \varphi$ is an epimorphism, so this implies $\alpha = \beta$.

$$A \xrightarrow{\varphi} B \xrightarrow{\psi} C \underset{\beta}{\overset{\alpha}{\rightrightarrows}} D$$

  Hence $\alpha \circ \psi = \beta \circ \psi$ implies $\alpha = \beta$, so ψ is an epimorphism.

- Let α and β be morphisms such that $\varphi \circ \alpha = \varphi \circ \beta$. Then composition with ψ yeilds that $\psi \circ \varphi \circ \alpha = \psi \circ \varphi \circ \beta$. But $\psi \circ \varphi$ is an monomorphism, so this implies $\alpha = \beta$.

$$Z \underset{\beta}{\overset{\alpha}{\rightrightarrows}} A \xrightarrow{\varphi} B \xrightarrow{\psi} C$$

  Hence $\varphi \circ \alpha = \varphi \circ \beta$ implies $\alpha = \beta$, so φ is an monomorphism.

Exercise 1.3

A *preaditive* category is a category in which each Hom set is endowed with a structure of abelian group in such a way that composition maps are bilinear. Prove that a ring is 'the same as' a preadditive category with a single object.

Solution. Let R be a ring. We can build a category with a single object $\mathsf{C} = \{*\}$ where there is only hom set we need to worry about, and we define it as $\mathrm{Hom}(*, *) = R$ where it retains the abelian group structure of R. Define composition of morphisms as $\alpha \circ \beta = \alpha\beta$ where multiplication is in R. Since multiplication distributes in R, we have

$$\alpha \circ (\beta + \gamma) = \alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma = \alpha \circ \beta + \alpha \circ \gamma$$

and similarly

$$(\alpha + \beta) \circ \gamma = (\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma = \alpha \circ \gamma + \beta \circ \gamma$$

for all $\alpha, \beta, \gamma \in \mathrm{Hom}(*, *)$. Hence, the composition maps are biliear with respect to the abeliean group structure. So this category is preadditive.

Exercise 1.4

Let A be an additive category and let A be an object of A. Show that $\mathrm{End}_{\mathsf{A}}(A)$ has the natural structure of a ring.

Solution. Since A is an additive category, we have that $\mathrm{End}_{\mathsf{A}}(A)$ is an abelian group with some binary operation $+$ such that $\alpha + \beta = \beta + \alpha \in \mathrm{End}_{\mathsf{A}}(A)$ whenever $\alpha, \beta \in \mathrm{End}_{\mathsf{A}}(A)$. As a hom set, there is a binary operation on $\mathrm{End}_{\mathsf{A}}(A)$ given by composition $\alpha \circ \beta \in \mathrm{End}_{\mathsf{A}}(A)$ for all $\alpha, \beta \in \mathrm{End}_{\mathsf{A}}(A)$, such that there is an identity morphism $\mathrm{id}_A \in \mathrm{End}_{\mathsf{A}}(A)$ with the property $\mathrm{id}_A \circ \alpha = \alpha \circ \mathrm{id}_A = \alpha$ and such that composition is associative, i.e. $\alpha \circ (\beta \circ \gamma) = (\alpha \circ \beta) \circ \gamma$ for all $\alpha, \beta, \gamma \in \mathrm{End}_{\mathsf{A}}(A)$. Furthermore, the fact that A is an additive category says that composition of maps is bilinear. That is, it distributes over $+$ such that $\alpha \circ (\beta + \gamma) = \alpha \circ \beta + \alpha \circ \gamma$ and $(\alpha + \beta) \circ \gamma = \alpha \circ \gamma + \beta \circ \gamma$ for all $\alpha, \beta, \gamma \in \mathrm{End}_{\mathsf{A}}(A)$.

These are all the requirements for $\big(\mathrm{End}_{\mathsf{A}}(A), +, \circ\big)$ to be a ring.

Exercise 1.5

Let A and B be objects of an additive category A with a zero-object 0. Since 0 is both final and initial, $\mathrm{Hom}_{\mathsf{A}}(A, 0)$ and $\mathrm{Hom}_{\mathsf{A}}(0, B)$ are both singletons. So the image of the composition

$$A \longrightarrow 0 \longrightarrow B$$

is a single element e of $\mathrm{Hom}_{\mathsf{A}}(A, B)$. Prove that this is the identity of the abelian group $\mathrm{Hom}_{\mathsf{A}}(A, B)$. (Hint: Prove $e + e = e$.) This is the zero element, denoted 0.

Prove that for every morphism φ in A, $\varphi \circ 0 = 0$ and $0 \circ \varphi = 0$.

Solution. Let A and B be objects in this category, and denote e_{AB} as the unique element in $\mathrm{Hom}_{\mathsf{A}}(A, B)$ given by the composition $A \xrightarrow{e_{A0}} 0 \xrightarrow{e_{0B}} B$. But $\mathrm{Hom}_{\mathsf{A}}(A, 0)$ is a singleton set, so e_{A0} is the only element of this group and thus $e_{A0} + e_{A0} = e_{A0}$. So we have

$$\begin{aligned} e_{AB} + e_{AB} &= e_{0B} \circ e_{A0} + e_{0B} \circ e_{A0} \\ &= e_{0B} \circ (e_{A0} + e_{A0}) \\ &= e_{0B} \circ e_{A0} \\ &= e_{AB}. \end{aligned}$$

Hence e_{AB} is the zero element in the abelian group $\mathrm{Hom}_{\mathsf{A}}(A, B)$, which we will denote $e_{AB} = 0_{AB}$.

Let C be another object in A and $\varphi \in \mathrm{Hom}_{\mathsf{A}}(B, C)$. Then

$$\varphi \circ 0_{AB} = \varphi \circ (0_{AB} + 0_{AB}) = \varphi \circ 0_{AB} + \varphi \circ 0_{AB}$$

and thus $\varphi \circ 0_{AB} = 0_{AC}$. Similarly, if Z is another object in the category and $\psi \in \mathrm{Hom}_{\mathsf{A}}(Z, A)$, then

$$0_{AB} \circ \psi = (0_{AB} + 0_{AB}) \circ \psi = 0_{AB} \circ \psi + 0_{AB} \circ \psi$$

and thus $0_{AB} \circ \psi = 0_{ZB}$.

Exercise 1.8

Let $\varphi\colon A \to B$ be a morphism in an additive category A. Prove that $\iota\colon K \to A$ is a kernel of φ if and only if for all objects Z the induced sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathsf{A}}(Z, K) \longrightarrow \mathrm{Hom}_{\mathsf{A}}(Z, A) \longrightarrow \mathrm{Hom}_{\mathsf{A}}(Z, B)$$

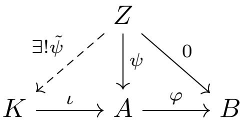
is exact. Formulate an analogous result for cokernels.

Solution. Note that, for all objects Z, the morphisms $K \xrightarrow{\iota} A \xrightarrow{\varphi} B$ induce a sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathsf{A}}(Z, K) \xrightarrow{\iota\circ} \mathrm{Hom}_{\mathsf{A}}(Z, A) \xrightarrow{\varphi\circ} \mathrm{Hom}_{\mathsf{A}}(Z, B)$$

if and only if $\varphi \circ \iota = 0$ (otherwise it is not a valid sequence), where the first map is post-composition with ι and the second map is post-composition with φ.

First suppose $\iota\colon K \to A$ is a kernel of φ. Then $\varphi \circ \iota = 0$ and for all other morphisms $\psi\colon Z \to A$ such that $\varphi \circ \psi = 0$ the universal property tells us there is a unique morphism $\tilde{\psi}\colon Z \to K$ such that the diagram



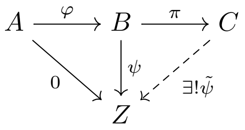
commutes. Suppose $f_1\colon Z \to K$ and $f_2\colon Z \to K$ are two morphisms such that $\iota \circ f_1 = \iota \circ f_2$. Then $\varphi \circ \iota \circ f_1 = \varphi \circ \iota \circ f_2 = 0$ and thus there is a unique morphism from Z to K such that this commutes. But both f_1 and f_2 make it commute, so $f_1 = f_2$. Hence $\iota\circ$ is injective and thus the sequence in question is exact at $\mathrm{Hom}_{\mathsf{A}}(Z, K)$. Now let $g \in \mathrm{Hom}\,\mathsf{A}(Z, A)$ such that $\varphi \circ g = 0$. The universal property says that there is a unique morphism $\tilde{g}\colon Z \to K$ such that $\iota \circ \tilde{g} = g$. Hence g in the kernel of the map $\varphi\circ$ is also in the image of map $\iota\circ$. So the sequence is exact at $\mathrm{Hom}\,\mathsf{A}(Z, A)$, and thus the whole sequence is exact.

Now suppose that the sequence in question is exact for all objects Z. Since it is exact, we must have $\varphi \circ \iota = 0$. Now for any morphism $\psi\colon Z \to A$ such that $\varphi \circ \psi = 0$ (that is, ψ is in the kernel of the second map), exactness tells us that it is in the image of the first map. So there is a morphism $\tilde{\psi}\colon Z \to K$ such that $\iota \circ \tilde{\psi} = \psi$. But injectivity of $\iota\circ$ in the sequence tells us, in addition, that this is the unique morphism that fulfils this property. This is exactly the universal property that we are looking for, so $\iota\colon K \to A$ is a kernel of φ.

The analogous statement for cokernels is: $\pi\colon B \to C$ is a cokernel of φ if and only if the induced sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathsf{A}}(C, Z) \xrightarrow{\circ\pi} \mathrm{Hom}_{\mathsf{A}}(A, Z) \xrightarrow{\circ\varphi} \mathrm{Hom}_{\mathsf{A}}(B, Z)$$

is exact for all objects Z. Indeed, recall the universal property of the cokernel: if $\psi\colon B \to Z$ is a morphism such that $\psi \circ \varphi = 0$, then there is a unique morphism $\tilde{\psi}\colon C \to Z$ such that the diagram



commutes.

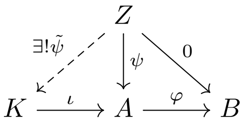
Exercise 1.9

Let A be an additive category.

- Let $\iota\colon K \to A$ be a kernel in A. Prove that ι is a monomorphism.
- Let $\varphi\colon A \to B$ be a morphism in A. If φ has a cokernel, prove that φ is an epimorphism if and only if $B \to 0$ is its cokernel.
- If A is abelian, prove that every kernel in A is the kernel of its own cokernel.

Solution. .

- Let Z be an object and $\alpha, \beta\colon Z \longrightarrow K$ be two morphisms such that $\psi = \iota \circ \alpha = \iota \circ \beta$. Then $\psi\colon Z \longrightarrow A$ is a morphism such that $\varphi \circ \psi = \varphi \circ \iota \circ \alpha = 0$ since $\varphi \circ \iota = 0$. By the universal property of the kernel, there is a unique map $\tilde{\psi}\colon Z \longrightarrow K$ such that the diagram

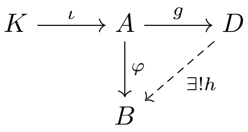


commutes. But both α and β make this diagram commute, so we must have $\alpha = \beta$.

- Let $\pi\colon B \longrightarrow C$ be a cokernel of φ. Then $\pi \circ \varphi = 0 = 0 \circ \varphi$ and thus $\pi = 0$ since φ is an epimorphism. Let ψ be any other morphism such that $\psi \circ \varphi = 0$. Then $\psi = 0$ and this must factor uniquely through $B \xrightarrow{0} C$. Hence $C = 0$.

  Conversely, suppose $B \longrightarrow 0$ is a cokernel of φ, and let α and β be two morphisms such that $\alpha \circ \varphi = \beta \circ \varphi$. Since this is an additive category, we have $(\alpha - \beta) \circ \varphi = 0$. Then $\alpha - \beta$ factors uniquely through 0, which implies $\alpha - \beta = 0$ and thus $\alpha = \beta$.

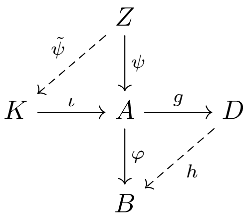
- Let $\iota\colon K \longrightarrow A$ be a kernel of some morphism $\varphi\colon A \longrightarrow B$, and let $g\colon A \longrightarrow D$ be a cokernel of ι. Then $g \circ \iota = 0$ and $\varphi \circ \iota = 0$. By the universal property of the cokernel, there is a unique morphism $h\colon D \longrightarrow B$ such that the diagram



  commutes, i.e. $h \circ g = \varphi$. Now let $\psi\colon Z \longrightarrow A$ be another morphism such that $g \circ \psi = 0$. Then

$$0 = h \circ g \circ \psi = \varphi \circ \psi.$$

  Since ι is a cokernel of φ, there is a unique morphism $\tilde{\psi}$ such that the diagram



  commutes, and thus ι is the kernel of g.