Problem Set
MATH 621

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Problem 1 (Problem 6, Chapter I.3, p. 14).
We define the *chordal distance* $d(z,w)$ between two points $z,w \in \mathbb{C}_\infty$ to be the length of the straight line segment joining the points P and Q on the unit sphere whose stereographic projections are z and w respectively.

(a) Show that the chordal distance is a metric. That is, it is symmetric, $d(z,w) = d(w,z)$; it satisfies the triangle inequality $d(z,w) \le d(z,\zeta) + d(\zeta,w)$; and $d(z,w) = 0$ if and only if $z = w$.

(b) Show that the chordal distance from z to w is given by

$$d(z,w) = \frac{2\,|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}, \qquad z,w \in \mathbb{C}$$

(c) What is $d(z,\infty)$?

Solution. Recall that the stereographic projection (from the north pole) maps points on the unit sphere $P = (x_1, x_2, x_3)$ to points on the extended complex plane according to

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \frac{x_1 + i x_2}{1 - x_3} \qquad \text{with inverse map} \qquad z \mapsto \begin{pmatrix} \frac{2\operatorname{Re} z}{1+|z|^2} \\ \frac{2\operatorname{Im} z}{1+|z|^2} \\ \frac{|z|^2-1}{1+|z|^2} \end{pmatrix}$$

(a) These facts follow trivially from the standard distance metric in $\mathbb{R}^3$.

(b) Let P and Q be the points on the sphere corresponding to z and w respectively. Set $P = (x_1, x_2, x_3)$ and $Q = (x_1', x_2', x_3')$, then

$$\begin{aligned}
[d(z,w)]^2 = \|P-Q\|^2 &= (x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_2 - x_2')^2 \\
&= \underbrace{x_1^2 + x_2^2 + x_3^2}_{} = 1 + \underbrace{x_1'^2 + x_2'^2 + x_3'^2}_{=1} - 2\,(x_1 x_1' + x_2 x_2' + x_3 x_3') \\
&= 2\,(1 - x_1 x_1' + x_2 x_2' + x_3 x_3') \\
&= 2\left(1 - \frac{4\operatorname{Re}(z)\operatorname{Re}(w) + 4\operatorname{Im}(z)\operatorname{Im}(w) + (|z|^2-1)(|w|^2-1)}{(1+|z|^2)(1+|w|^2)}\right) \\
&= 2\frac{(1+|z|^2)(1+|w|^2) - 4\operatorname{Re}(z)\operatorname{Re}(w) - 4\operatorname{Im}(z)\operatorname{Im}(w) - (1-|z|^2)(1-|w|^2)}{(1+|z|^2)(1+|w|^2)} \\
&= \frac{4}{(1+|z|^2)(1+|w|^2)}\Big(|z|^2 + |w|^2 - 2\big(\operatorname{Re}(z)\operatorname{Re}(w) + \operatorname{Im}(z)\operatorname{Im}(w)\big)\Big)
\end{aligned}$$

$$= \frac{4}{(1+|z|^2)(1+|w|^2)} \underbrace{\Big(|z|^2 + |w|^2 - (z\overline{w} + \overline{z}w)\Big)}_{=|z-w|^2}$$

$$= \frac{4\,|z-w|^2}{(1+|z|^2)(1+|w|^2)}.$$

Taking the square root yields the desired equality.

(c) The point at infinity corresponds to the north pole $(1, 0, 0)$ in the unit sphere. For $z \in \mathbb{C}$,

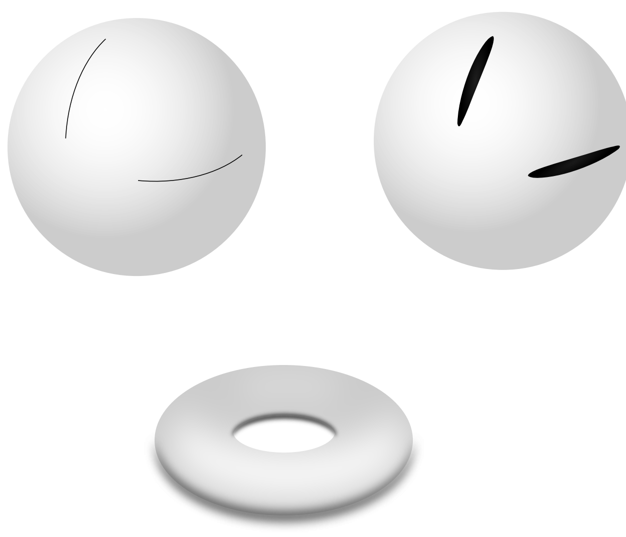
$$\begin{aligned} d(z,\infty) = \left\| \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\| &= \sqrt{x_1^2 + x_2^2 + (x_3 - 1)^2} \\ &= \sqrt{x_1^2 + x_2^2 + x_3^2 - 2x_3 + 1} \\ &= \sqrt{2(1 - x_3)} = \sqrt{2\frac{1 + |z|^2 - (|z|^2 - 1)}{|z|^2 + 1}} \\ &= \frac{2}{\sqrt{|z|^2 + 1}}. \end{aligned}$$

Problem 2 (Problem 7, Chapter I.7, p. 28).
Let $x_1 < x_2 < \cdots < x_n$ be n consecutive points on the real axis. Describe the Riemann surface of $\sqrt{(z-x_1)\cdots(z-x_n)}$. Show that for $n=1$ and $n=2$ the surface is topologically a sphere with certain punctures corresponding to the branch points and ∞. What is it when $n=3$ or $n=4$? Can you say anything for general n? (Note: Any compact Riemann surface is topologically a sphere with handles. Thus a torus is topologically a sphere with one handle. For a given n, how many handles are there and where do they come from?)

Solution.

All of the points x_k will be branch points. If $n=1$, we can take a branch cut along the interval $[x_1,\infty)$. If n is even, we can take the cuts along the intervals $[x_1,x_2]$, $[x_2,x_3]$, $\ldots$, $[x_{n-1},x_n]$. If $n>1$ and n is odd, we take cuts along the intervals $[x_1,x_2]$, $\ldots$, $[x_{n-2},x_{n-1}]$, and $[x_n,\infty)$. Hence, for any $n\in\mathbb{N}$, there will be a total of $\left\lfloor \frac{n+1}{2}\right\rfloor$ cuts on the plane. We can construct the Riemann surface by taking two sheets of the complex plane, each with slits as indicated above, and gluing the two sheets along the slits. If $n=1$, the result is a sphere with two punctures, one for the branch point and one for the point at infinity (there is only one, since the infinities of the two different sheets get glued together). If $n=2$, the result is still a sphere, but with 4 punctures: one for each branch point and two for the infinities on each of the sheets. If $n=3$ or 4, the result is a torus (since in either case there are only two cuts). In general, for a given n, the resulting surface will be a sphere with $n-1$ handles.

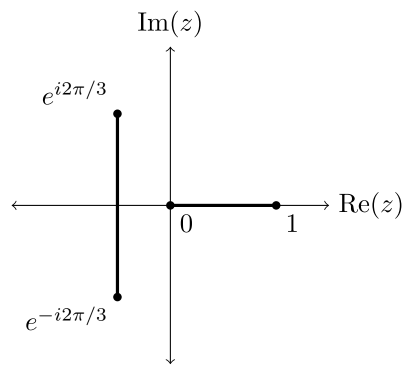
Indeed, we can visualize the construction of these Riemann surfaces in the following manner. Suppose k branch cuts are needed to define the function. We can identify the complex plane with a sphere with one point removed (the point at infinity). Branch cuts on the plane are carried to cuts on the sphere. Take two copies of the sphere with k cuts. We can visualize 'opening' each of the slits to a form a small hole in the sphere, and 'gluing' the edges of the holes from one sphere to the corresponding hole on the other sphere. If there is only one cut, gluing two spheres together after removing one small hole from each produces a surface that is topologically equivalent to a sphere. If there are $k>1$ cuts, each cut beyond the first forms a 'handle' to the resulting surface. For $k=2$ this is visualized in the following figure.



If none of the cuts extend to infinity, then the resulting surface will have $n+2$ punctures: one from each branch point and one from the puncture at infinity from each of the initial sheets. If one of these cuts extends to infinity, there are $n+1$ punctures: one from each branch point and only one additional puncture from the points at infinity, since the two infinities become identified.

Problem 3 (Problem 8, Chapter I.7, p. 28)**.**
Show that $\sqrt{z^2 - 1/z}$ can be defined as a (single-valued) continuous function outside the unit disk. Draw branch cuts so that the function can be defined continuously off the branch cuts. Describe the Riemann surface of the function.

Solution. We can rewrite the function as $\frac{\sqrt{z^3-1}}{\sqrt{z}}$, which has branch points at each of the cubed roots of unity and at zero. We can branch cuts connecting pairs of the 4 branch points as shown in the following diagram.



Using these branch cuts that are all confined to be inside the unit disk, we see that the function can be defined to be single-valued outside of the unit disk.

Construct the Riemann surface of this function by taking two copies of the complex plane with slits as in the above figure. As in the previous problem, the resulting surface will be a torus with two points removed, one for the point at infinity for each sheet.

Problem 4 (Problem 9, Chapter I.7, p. 28)**.**
Consider the function $\sqrt{z(z^3-1)(z+1)^3}$ that is positive at $z=2$. Draw branch cuts so that this branch of the function can be defined continuously off the branch cuts. Describe the Riemann surface of the function. To what value at $z=2$ does this branch return if it is continued continuously once counterclockwise around the circle $\{|z|=2\}$?

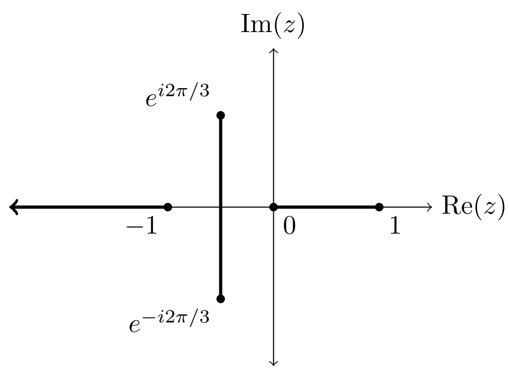
Solution. We can rewrite the square root as

$$\begin{aligned}\sqrt{z(z^3-1)(z+1)^3} &= \sqrt{z\cdot z^3\left(1-\tfrac{1}{z^3}\right)z\left(1+\tfrac{1}{z}\right)(z+1)^2} \\ &= \sqrt{z^4(z+1)^2}\sqrt{z\left(1-\tfrac{1}{z^3}\right)z\left(1+\tfrac{1}{z}\right)},\end{aligned}$$

where the first square root can be taken to be continuously single-valued everywhere on the complex plane. This yields

$$z^2(z+1)^2\sqrt{z}\sqrt{(1-\tfrac{1}{z^3})(1+\tfrac{1}{z})}, \tag{4.1}$$

which has five branch points at 0, -1, and at the cube roots of unity $\{1, e^{i2\pi/3}, e^{-i2\pi/3}\}$. We can draw two branch cuts between two pairs of these points, and one branch cut from -1 to ∞ as shown in the following figure.



The resulting Riemann surface is sphere with two handles, and no points removed.

Note that $\sqrt{(1-\frac{1}{z^3})(1+\frac{1}{z})}$ (i.e. the second square root in (4.1)) can be taken to be single-valued outside of the unit disk. Hence, starting at $z=2$ going around the circle of radius 2, we pick up a phase factor of -1 only from the $\sqrt{z}$ term in (4.1), so we end up at the negative of the initial starting value.

Problem 5 (Problem 4, Chapter I.8, p. 31)**.**
Show that

$$\tan^{-1} z = \frac{1}{2i} \log\left(\frac{1+iz}{1-iz}\right),$$

where both sides of the identity are to be interpreted as subsets of the complex plane. In other words, show that $\tan w = z$ if and only if $2iw$ is one of the values of the logarithm featured on the right.

Solution. Set $z = \tan w$ and note that

$$z = \tan w = \frac{1}{i} \frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} = \frac{1}{i} \frac{e^{2iw} - 1}{e^{2iw} + 1}.$$

Rewriting yields

$$iz\left(1 + e^{2iw}\right) = e^{2iw} - 1 \qquad \text{and thus} \qquad e^{2iw} = \frac{1+iz}{1-iz}$$

Hence $z = \tan w$ implies that $2iw$ is a logarithm of $\frac{1+iz}{1-iz}$.

On the other hand, if $2iw$ is a logarithm of $\frac{1+iz}{1-iz}$, the same arguments in reverse show that $z = \tan w$.

Problem 6 (Problem 13, Chapter II.1, p. 41).
For which complex values of α does the principal value of z^α have a limit as z tends to 0? Justify your answer.

Solution. The principal value of z^α have a limit as z tends to 0 if $\alpha = 0$ or $\operatorname{Re}\alpha > 0$.

Proof. If $\alpha = 0$, then $z^\alpha = 1$ for every nonzero z. So $\lim_{z\to 0} z^0 = 1$. If $\alpha \neq 0$, then

$$\begin{aligned} |z^\alpha| &= \left|e^{\alpha(\log|z| + i\operatorname{Arg} z)}\right| = \left|e^{(\operatorname{Re}\alpha\log|z| - \operatorname{Im}\alpha\operatorname{Arg} z)}\right| \underbrace{\left|e^{i(\operatorname{Im}\alpha\log|z| + \operatorname{Re}\alpha\operatorname{Arg} z)}\right|}_{=1} \\ &= e^{\operatorname{Re}\alpha\log|z|}e^{-\operatorname{Im}\alpha\operatorname{Arg} z}. \end{aligned}$$

Note that $e^{-\operatorname{Im}\alpha\operatorname{Arg} z}$ is bounded by $e^{-\pi\operatorname{Im}\alpha}$ and $e^{\pi\operatorname{Im}\alpha}$. We split the remainder of the into the following cases: $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\alpha < 0$, and $\operatorname{Re}\alpha = 0$ with $\operatorname{Im}\alpha \neq 0$. Note that $\log|z| \to -\infty$ as $z \to 0$.

- If $\operatorname{Re}\alpha > 0$, then $e^{\operatorname{Re}\alpha\log|z|} \to 0$ as $z \to 0$. So $|z^\alpha| \to 0$.
- If $\operatorname{Re}\alpha < 0$, then $e^{\operatorname{Re}\alpha\log|z|} \to \infty$ as $z \to 0$. So $|z^\alpha|$ is unbounded as $z \to 0$ and the limit does not exist.
- If $\operatorname{Re}\alpha = 0$ but $\operatorname{Im}\alpha \neq 0$, then $|z^\alpha| = e^{-\operatorname{Im}\alpha\operatorname{Arg} z}$, which varies between $e^{-\pi\operatorname{Im}\alpha}$ and $e^{\pi\operatorname{Im}\alpha}$ in any neighborhood around $z = 0$. So the limit does not exist.

□

Problem 7 (Problem 19, Chapter II.1, p.41)**.**
Give a proof of the fundamental theorem of algebra along the following lines. Show that if $p(z)$ is a non-constant polynomial, then $|p(z)|$ attains its minimum at some point $z_0 \in \mathbb{C}$. Assume that the minimum is attained at $z_0 = 0$ and that $p(z) = 1 + az^m + \cdots$, where $m \geq 1$ and $a \neq 0$. Contradict the minimality by showing that $\left|p(\varepsilon e^{i\theta})\right| < 1$ for an appropriate choice of ε and θ.

Solution. Let $p(z) = c_0 + c_1 z + \cdots + c_n z^n$ be a non-constant polynomial with $c_n \neq 0$. Suppose that $p(z)$ does not have any zeros. Then $c_0 \neq 0$. Since p is non-constant, we can find a disk D_r of radius r centered at the origin such that $|p(z)| > |p(0)|$ for all $|z| > r$. Since D_r is compact, the minimum of $|p(z)|$ is achieved at some point $z_0 \in D_r$. Consider the polynomial $p'(z) = \frac{1}{c_0} p(z_0 - z)$, which has no zeros since $p(z)$ has no zeros. Then $|p'(z)|$ achieves its minimum value of 1 at $z = 0$. We may write $p'(z)$ as

$$p'(z) = 1 + a_m z^m + \cdots + a_n z^n,$$

where $1 \leq m \leq n$ is such that $a_m \neq 0$. Let $a_m = |a_m| e^{i\phi}$ and choose $\theta = \frac{\pi - \phi}{m}$ such that $a_m e^{i\theta} = -|a_m|$. Then

$$\begin{aligned} \left|p'(\varepsilon e^{i\theta})\right| &= \left|1 - |a_m| \varepsilon^m + a_{m+1} \varepsilon^{m+1} e^{i\theta(m+1)} + \cdots + a_n \varepsilon^n e^{i\theta n}\right| \\ &\leq \left|1 - \varepsilon^m |a_m|\right| + \varepsilon^{m+1} |a_{m+1}| + \cdots \varepsilon^n |a_n| . \end{aligned}$$

For small $\varepsilon > 0$,

$$\left|p'(\varepsilon e^{i\theta})\right| \leq 1 - \varepsilon^m |a_m| + \mathcal{O}(\varepsilon^{m+1}),$$

which is less than 1 for some ε small enough. This yields the desired contradiction.

Problem 8 (Problem 6, Chapter II.3, p. 50).
If $f = u + iv$ is analytic on D, then ∇v is obtained by rotating ∇u by $90°$. In particular, ∇u and ∇v are orthogonal.

Solution. Since $f = u + iv$ is analytic, the functions u and v fulfill the Cauchy-Riemann relations. Then the gradients ∇u and ∇v are given by

$$\nabla u = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} \qquad \text{and} \qquad \nabla v = \begin{pmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} -\frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial x} \end{pmatrix}.$$

Recall that the 2×2-rotation matrix that rotates vectors by an angle of θ is given by

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Applying the rotation matrix with $\theta = 90°$ to ∇u, we obtain

$$R_{90°} \nabla u = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} -\frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial x} \end{pmatrix} = \nabla v$$

as desired. To show that ∇u and ∇v are orthogonal, note that

$$\nabla u^T \nabla v = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{pmatrix} \begin{pmatrix} -\frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial x} \end{pmatrix} = -\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} = 0.$$

Problem 9 (Problem 8, Chapter II.3, p. 50)**.**
Derive the polar form of the Cauchy-Riemann equations for u and v:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}.$$

Check that for any integer m, the functions $u(re^{i\theta}) = r^m \cos(m\theta)$ and $v(re^{i\theta}) = r^m \sin(m\theta)$ satisfy the Cauchy-Riemann equations.

Solution. With the change of variables $y = x + iy = re^{i\theta}$, we have $x = r\cos\theta$ and $y = r\sin\theta$. Hence

$$\begin{aligned} \frac{\partial x}{\partial r} &= \cos\theta = \frac{1}{r}\frac{\partial y}{\partial \theta}, \\ \frac{\partial y}{\partial r} &= \sin\theta = -\frac{1}{r}\frac{\partial x}{\partial \theta}, \end{aligned} \qquad \text{and} \qquad \begin{aligned} \frac{\partial x}{\partial \theta} &= -r\sin\theta = -r\frac{\partial y}{\partial r}, \\ \frac{\partial y}{\partial \theta} &= r\cos\theta = r\frac{\partial x}{\partial r}. \end{aligned}$$

Suppose that u and v satisfy the Cauchy-Riemann equations. From the Cauchy-Riemann relations, and the equalities above, we have

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \\ &= \frac{\partial v}{\partial y}\left(\frac{1}{r}\frac{\partial y}{\partial \theta}\right) - \frac{\partial v}{\partial x}\left(-\frac{1}{r}\frac{\partial x}{\partial \theta}\right) \\ &= \frac{1}{r}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta}\right) = \frac{1}{r}\frac{\partial v}{\partial \theta} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} \\ &= \frac{\partial v}{\partial y}\left(-r\frac{\partial y}{\partial r}\right) - \frac{\partial v}{\partial x}\left(r\frac{\partial x}{\partial r}\right) \\ &= -r\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}\right) = -r\frac{\partial v}{\partial \theta}, \end{aligned}$$

which yields the desired Cauchy-Riemann relations in polar coordinates.

Now let m be an integer and consider the functions $u(re^{i\theta}) = r^m \cos(m\theta)$ and $v(re^{i\theta}) = r^m \sin(m\theta)$. To check that these functions satisfied the Cauchy-Riemann equations, note that

$$\begin{aligned} \frac{\partial u}{\partial r} &= mr^{m-1}\cos(m\theta) \\ &= \frac{1}{r}\left(mr^m \cos(m\theta)\right) \\ &= \frac{1}{r}\frac{\partial v}{\partial \theta} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial u}{\partial \theta} &= -mr^m \sin(m\theta) \\ &= -r\left(mr^{m-1}\sin(m\theta)\right) \\ &= -r\frac{\partial v}{\partial r}. \end{aligned}$$

Problem 10 (Problem 5, Chapter II.5, p. 57)**.**
Show that Laplace's equation in polar coordinates is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

Solution. Suppose a function $u(x, y)$ satisfies Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in Cartesian coordinates. Changing variables into polar coordinates, we have $x = r\cos\theta$ and $y = r\sin\theta$.

$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{dx}{dr} + \frac{\partial u}{\partial y}\frac{dy}{dr}\\ &= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta,\end{aligned}$$

where $\frac{dx}{dr} = \cos\theta$ and $\frac{dy}{dr} = \sin\theta$. Taking the second derivative yields

$$\begin{aligned}\frac{\partial^2 u}{\partial r^2} &= \frac{\partial^2 u}{\partial x^2}\left(\frac{dx}{dr}\right)^2 + 2\frac{\partial^2 u}{\partial x \partial y}\left(\frac{dx}{dr}\right)\left(\frac{dy}{dr}\right) + \frac{\partial^2 u}{\partial y^2}\left(\frac{dy}{dr}\right)^2\\ &= \frac{\partial^2 u}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 u}{\partial x \partial y}\cos\theta\sin\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta\\ &= \frac{\partial^2 u}{\partial x^2}(\cos^2\theta - \sin^2\theta) + 2\frac{\partial^2 u}{\partial x \partial y}\cos\theta\sin\theta,\end{aligned}$$

where the last line follows since u satisfies Laplace's equation in Cartesian coordinates, i.e. $\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 u}{\partial x^2}$. Similarly, taking the second derivative of u with respect to θ yields

$$\begin{aligned}\frac{\partial^2 u}{\partial \theta^2} &= \frac{\partial^2 u}{\partial x^2}\left(\frac{dx}{d\theta}\right)^2 + \frac{\partial u}{\partial x}\frac{d^2 x}{d\theta^2} + \frac{\partial^2 u}{\partial y^2}\left(\frac{dy}{d\theta}\right)^2 + \frac{\partial u}{\partial y}\frac{d^2 y}{d\theta^2} + 2\frac{\partial^2 u}{\partial x \partial y}\left(\frac{dx}{d\theta}\right)\left(\frac{dy}{d\theta}\right)\\ &= r^2\underbrace{\left[\frac{\partial^2 u}{\partial x^2}\left(\sin^2\theta - \cos^2\theta\right) - 2\frac{\partial^2 u}{\partial x \partial y}\sin\theta\cos\theta\right]}_{=-\frac{\partial^2 u}{\partial r^2}} - r\underbrace{\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right)}_{=\frac{\partial u}{\partial r}}\\ &= -r^2\frac{\partial^2 u}{\partial r^2} - r\frac{\partial u}{\partial r}.\end{aligned}$$

Hence $\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -r^2\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r}$, which is the desired identity.

Problem 11 (Problem 8, Chapter II.5, p. 58)**.**
Show using Laplace's equation in polar coordinates that $u(re^{i\theta}) = \theta \log r$ is harmonic. Use the polar form of the Cauchy-Riemann equations to find a harmonic conjugate v for u. What is the analytic function $u + iv$?

Solution. The derivatives we need to compute are

$$\frac{\partial u}{\partial r} = \frac{\theta}{r}, \qquad \frac{\partial^2 u}{\partial r^2} = -\frac{\theta}{r^2}, \qquad \text{and} \qquad \frac{\partial^2 u}{\partial \theta^2} = 0.$$

Putting these into Laplace's equation in polar coordinates, we see that

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{\theta}{r^2} + \frac{\theta}{r} = 0,$$

so u is harmonic.

We need to find a function $v(re^{i\theta})$ such that $u + iv$ is harmonic. From the polar form of the Cauchy-Riemann equations, we see that

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\ln r}{r} \qquad \text{and} \qquad \frac{\partial v}{\partial \theta} = r\frac{\partial r}{\partial \theta} = r\frac{\theta}{r} = \theta.$$

Taking the integral of these respect to θ and r, we find

$$v = \int \frac{\partial v}{\partial r} dr + \int \frac{\partial v}{\partial \theta} d\theta = -\int \frac{\ln r}{r} dr + \int \theta \, d\theta = -\frac{1}{2}(\ln r)^2 + \frac{1}{2}\theta^2.$$

Putting u and v together to form an analytic function, we see that

$$\begin{aligned} u + iv = \frac{i}{2}\left(\theta^2 - 2i\theta \ln r - (\ln r)^2\right) &= \frac{i}{2}\left(\theta - i \ln r\right)^2 \\ &= -\frac{i}{2}\left(\ln r + i\theta\right)^2 \\ &= -\frac{i}{2}\left[\log\left(re^{i\theta}\right)\right]^2 \end{aligned}$$

where we choose the principal branch of the logarithm.

Problem 12 (Problem 9, Chapter II.6, p. 62)**.**
Let $f = u+iv$ be a continuously differentiable complex-valued function on a domain D such that the Jacobian matrix of f does not vanish at any point of D. Show that if f maps orthogonal curves to orthogonal curves, then either f or $\bar{f}$ is analytic, with non-vanishing derivative.

Solution. It suffices to show that either u and v or u and $-v$ satisfy the Cauchy-Riemann equations. Let $z_0 \in D$ and write $z_0 = x_0 + iy_0$. We may consider families of orthogonal curves at z_0 parameterized by s. For $s \in \mathbb{R}$, consider the two orthogonal curves at z_0 given by

$$\gamma_s(t) = z_0 + t(s+i) \qquad \text{and} \qquad \eta_s(t) = z_0 + t(1-is).$$

These curves are indeed orthogonal at z_0, since the vectors $(s, 1)$ and $(1, -s)$ are orthogonal for each for each s. The function f maps these curves to the curves given by

$$f(\gamma_s(t)) = f(x_0 + ts + i(y_0 + t)) = u(x_0 + ts, y_0 + t) + iv(x_0 + ts, y_0 + t)$$

and

$$f(\eta_s(t)) = f(x_0 + t + i(y_0 - ts)) = u(x_0 + t, y_0 - ts) + iv(x_0 + t, y_0 - ts)$$

The tangent vectors of the curves $f(\gamma_s(t))$ and $f(\eta_s(t))$ at z_0 are

$$\lim_{t\to 0} \frac{f(\gamma_s(t)) - f(\gamma_s(0))}{t} = su_x + u_y + i(sv_x + v_y)$$

and

$$\lim_{t\to 0} \frac{f(\eta_s(t)) - f(\eta_s(0))}{t} = u_x - su_y + i(v_x - sv_y).$$

By assumption, these vectors must be orthogonal, so we have

$$\begin{aligned} 0 &= \begin{pmatrix} u_y + su_x \\ v_y + sv_x \end{pmatrix}^T \begin{pmatrix} u_x - su_y \\ v_x - sv_y \end{pmatrix} \\ &= (u_y + su_x)(u_x - su_y) + (v_y + sv_x)(v_x - sv_y) \\ &= u_x u_y + v_x v_y + s\left(u_x^2 + v_x^2 - u_y^2 - v_y^2\right) - s^2\left(u_x u_y + v_x v_y\right). \end{aligned}$$

These must be orthogonal for all s. Therefore it must vanish for all s, which yields the relations

$$u_x u_y + v_x v_y = 0 \tag{12.1}$$

(which is seen by setting $s = 0$) and

$$u_x^2 + v_x^2 - u_y^2 - v_y^2 = 0. \tag{12.2}$$

Since the Jacobian is non-vanishing, we have that $u_x^2 + u_y^2 + v_x^2 + v_y^2 \neq 0$, so at least one of u_x, u_y, v_x, and v_y is nonzero. Suppose that $u_x \neq 0$. Multiplying (12.2) by u_x^2, and using (12.1) to note that $u_x^2 u_y^2 = v_x^2 v_y^2$, we see that

$$\begin{aligned} 0 = u_x^2\left(u_x^2 + v_x^2 - u_y^2 - v_y^2\right) &= u_x^4 + u_x^2 v_x^2 - u_x^2 u_y^2 - u_x^2 v_y^2 \\ &= u_x^4 + u_x^2 v_x^2 + v_x^2 v_y^2 - u_x^2 v_y^2 \\ &= u_x^2\left(u_x^2 - v_y^2\right) + v_x^2\left(u_x^2 - v_y^2\right) = \left(u_x^2 + v_x^2\right)\left(u_x^2 - v_y^2\right), \end{aligned}$$

and thus $u_x^2 = v_y^2$, or $u_x = \pm v_y$. Putting this back into (12.2), we see that $u_y^2 = v_x^2$. If $u_y \neq 0$, then from (12.1) we have $u_y = \mp v_x$. Hence, either f or $\bar{f}$ satisfies the Cauchy-Riemann equations at z_0. If $u_y = 0$, then we also have $v_x = 0$. If $u_x = 0$ then $v_y = 0$, and a similar analysis shows that $u_y = \pm v_x$, so we still have that either f or $\bar{f}$ satisfies the Cauchy-Riemann equations at z_0.

Finally, since the partial derivatives are continuous, the same choice of sign for $u_x = \pm v_y$ and $u_y = \mp v_x$ must hold over all D. Hence either f or $\bar{f}$ is analytic on D.

Problem 13 (Problem 12, Chapter II.7, p. 69)**.**
Classify the conjugacy classes of fractional linear transformations by establishing the following:

(a) A fractional linear transformation that is not the identity has either 1 or 2 fixed points. That is, points satisfying $f(z_0) = z_0$.

(b) If a fractional linear transformation $f(z)$ has two fixed points, then it is conjugate to the dilation $z \mapsto az$ with $a \neq 0$, $a \neq 1$. That is, there is a fractional linear transformation $h(z)$ such that $h(f(z)) = ah(z)$. Is a unique? (Hint: Consider a fractional linear transformation that maps the fixed points to 0 and ∞.)

(c) If a fractional linear transformation $f(z)$ has exactly one fixed point, then it is conjugate to the transformation $\zeta \mapsto \zeta + 1$. In other words, there is a fractional linear transformation $h(z)$ such that $h(f(h^{-1}(\zeta))) = \zeta + 1$, or equivalently, such that $h(f(z)) = h(z) + 1$. (Hint: Consider a fractional linear transformation that maps the fixed point to ∞.)

Solution. Recall that a fractional linear transformation is a map on the extended complex plane $f\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ of the form $f(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$, where $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. We also recall the definition of conjugacy.

Definition. Two maps f and g are *conjugate* if there is a one-to-one map h such that $g = h \circ f \circ h^{-1}$, where h has the appropriate domain and range.

a) For this part, we have to prove the following proposition.

Proposition 1. *A fractional linear transformation that is not identity has exactly 1 or 2 fixed points.*

Proof. Let $f(z) = \frac{az+b}{cz+d}$ be a fractional linear transformation. A fixed point of f is a point $z_0 \in \mathbb{C}_\infty$ such that $z_0 = \frac{az_0+b}{cz_0+d}$. Consider the following cases.

- Case 1: $c = 0$ and $d = a$, then $a, d \neq 0$. Furthermore $b \neq 0$, otherwise f is the identity transformation. So z_0 is a fixed point of f if $z_0 + \frac{b}{d} = z_0$, so f has exactly one fixed point at ∞.
- Case 2: $c = 0$ and $d \neq a$. Then $d \neq 0$ and z_0 is a fixed point if $z_0 = \frac{a}{d} z_0 + \frac{b}{d}$, in which case either $z_0 = \infty$ or $z_0 = \frac{b}{d-a}$. So f has exactly two fixed points.
- Case 3: $c \neq 0$. Then all fixed points z_0 satisfy

$$\frac{az_0 + b}{cz_0 + d} = z_0 \qquad \text{which is equivalent to} \qquad cz_0^2 + (d-1)z_0 - b = 0.$$

  So the fixed points are $z_0 = \frac{a-d}{+}\sqrt{(d-a)^2 + 4bc}2c$.
  - If $(d-a)^2 + 4bc = 0$, then f has exactly one fixed point at $z_0 = \frac{d-a}{2c}$.
  - If $(d-a)^2 + 4bc \neq 0$, then f has exactly two fixed point at $z_0 = \frac{a-d+\sqrt{(d-a)^2+4bc}}{2c}$, one for each of the two values of $\sqrt{(d-a)^2 + 4bc}$.

So each non-identity fractional linear transformation has exactly one or two fixed points, as desired. □

b) Let f be a fractional linear transformation that fixes exactly two points, say z_0 and z_1. Let h be a fractional linear transformation that maps z_0 to 0 and z_1 to ∞. Then the fractional linear transformation given by the composition $h \circ f \circ h^{-1}$ also has exactly two fixed points, namely ∞ and 0. Indeed, we have

$$h(f(h^{-1}(0))) = h(f(z_0)) = h(z_0) = 0 \qquad \text{and} \qquad h(f(h^{-1}(\infty))) = h(f(z_1)) = h(z_1) = \infty.$$

The only fractional linear transformations that fix both ∞ and 0 are just the dilation maps, so the composition map is just $h \circ f \circ h^{-1}(z) = az$ for some $a \neq 0$ and $a \neq 1$ (if $a = 1$, then it fixes every point).

Note that the map $f'(z) = az$ is conjugate to the map $z \mapsto \frac{1}{a}z$. Indeed, consider the transformation $h'(z) = \frac{1}{z}$, then

$$h' \circ f' \circ h'^{-1}(z) = \frac{1}{f\left(\frac{1}{z}\right)} = \frac{1}{a\frac{1}{z}} = \frac{1}{a}z.$$

Hence, any fractional linear transformation that fixes exactly two points is conjugate to the maps $z \mapsto az$ and $z \mapsto \frac{1}{a}z$ for some $a \in \mathbb{C}\backslash\{0, 1\}$ (i.e. the a is not unique).

c) Let f be a fractional linear transformation that fixes exactly one point, say z_0. Let h be a fractional linear transformation that maps z_0 to ∞. Furthermore, let $z_1 \in \mathbb{C}$ be any point $z_1 \neq z_0$, and we can take h to be the unique transformation that takes z_1 to 0 and $f(z_1) \neq z_1$ to 1. Then the composition map $h \circ f \circ h^{-1}$ has exactly one fixed point at ∞, since

$$h \circ f \circ h^{-1}(\infty) = h(f(z_0)) = h(z_0) = \infty,$$

and maps 0 to 1, since

$$h \circ f \circ h^{-1}(0) = h(f(h^{-1}(0))) = h(f(z_1)) = 1.$$

Every fractional linear transformation that fixes ∞ must be of the form $z \mapsto az + b$ for some $a \neq 0$. Furthermore, since this transformation maps 0 to 1, we have $b = 1$. Since $h \circ f \circ h^{-1}$ has exactly one fixed point, we must have $a = 1$ so that $az + 1 = z$ has no finite solutions. Hence f is conjugate to the transformation $z \mapsto z + 1$.

Problem 14 (Problem 3, Chapter IV.3, p. 112).
Let $f(z) = c_0 + c_1 z + \cdots + c_n z^n$ be a polynomial.

(a) If the c_k's are real, show that

$$\int_{-1}^{1} f(x)^2 dx \leq \pi \int_0^{2\pi} \left| f\left(e^{i\theta}\right) \right|^2 \frac{d\theta}{2\pi} = \pi \sum_{k=0} c_k^2.$$

Hint: For the first inequality, apply Cauchy's theorem to the function $f(z)^2$ separately on the top half and the bottom half of the unit disk.

(b) If the c_k's are complex, show that

$$\int_{-1}^{1} |f(x)|^2 dx \leq \pi \int_0^{2\pi} \left| f\left(e^{i\theta}\right) \right|^2 \frac{d\theta}{2\pi} = \pi \sum_{k=0} |c_k|^2.$$

(c) Establish the following variant of **Hilbert's inequality**, that

$$\left| \sum_{j,k=0}^{n} \frac{c_j c_k}{j+k+1} \right| \leq \pi \sum_{k=0}^{n} |c_k|^2.$$

Solution. Recall from Cauchy's theorem that $\int_{\partial D} f(z) dz = 0$ for any analytic function $f(z)$ on $D \cup \partial D$, where D is any bounded domain. Let Γ_+ denote the boundary of the unit disk in the upper half-plane. Integrating around the upper half of the unit disk, we see that

$$\int_{-1}^{1} f(x)^2 dx = -\int_{\Gamma_+} f(z)^2 dz = -i \int_0^{\pi} e^{i\theta} f\left(e^{i\theta}\right)^2 d\theta.$$

Similarly, integrating around the lower half of the unit disk, where Γ_- is the boundary of the unit disk in the lower half-plane, we find that

$$\int_{-1}^{1} f(x)^2 dx = \int_{\Gamma_-} f(z)^2 dz = -i \int_{\pi}^{2\pi} e^{i\theta} f\left(e^{i\theta}\right)^2 d\theta.$$

Putting these two integrals together, we see that

$$\int_{-1}^{1} f(x)^2 dx = \frac{1}{2}\left(2 \int_{-1}^{1} f(x)^2 dx\right) = -\frac{i}{2} \int_0^{2\pi} e^{i\theta} f\left(e^{i\theta}\right)^2 d\theta.$$

(a) Suppose that the c_j's are real, then $f(x)$ is real-valued on the real line. Since the integral $\int_{-1}^{1} f(x)^2 dx$ must be real-valued and non-negative, it is equal to its absolute value, and thus

$$\int_{-1}^{1} f(x)^2 dx = \frac{1}{2} \left| -i \int_0^{2\pi} e^{i\theta} f\left(e^{i\theta}\right)^2 d\theta \right| \leq \int_0^{2\pi} \left| f\left(e^{i\theta}\right) \right|^2 d\theta,$$

which is the desired inequality.

To show the equality, note that for integers j and k, we have $\frac{1}{2\pi} \int_0^{2\pi} e^{i\theta(j-k)} d\theta = \delta_{jk}$, where δ_{jk} is the Kronecker delta. Then

$$\int_0^{2\pi} \left| f\left(e^{i\theta}\right) \right|^2 \frac{d\theta}{2\pi} = \frac{1}{2\pi} \int_0^{2\pi} f\left(e^{i\theta}\right) \overline{f\left(e^{i\theta}\right)} d\theta$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_0^{2\pi} \left(c_0 + c_1 e^{i\theta} + \cdots + c_n e^{in\theta}\right)\left(c_0 + c_1 e^{-i\theta} + \cdots + c_n e^{-in\theta}\right) d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{j,k=0}^{n} c_k c_j e^{i\theta(k-j)} d\theta \\
&= \sum_{j,k=0}^{n} c_j c_k \frac{1}{2\pi}\int_0^{2\pi} e^{i\theta(k-j)} d\theta \\
&= \sum_{k=0}^{n} c_k^2,
\end{aligned}
$$

which yields the desired equality.

(b) Now suppose that the c_k's are complex. Then $c_k = a_k + ib_k$ for some $a_k, b_k \in \mathbb{R}$. For x real, note that

$$
\begin{aligned}
\left|f(x)^2\right| = \left|\left(\sum_{k=0}^{n} c_k x^k\right)^2\right| = \left|\left(\sum_{k=0}^{n} a_k x^k + i b_k x^k\right)\right|^2 &= \left|\sum_{k=0}^{n} a_k x^k + i\sum_{k=0}^{n} b_k x^k\right|^2 \\
&= \left(\sum_{k=0}^{n} a_k x^k\right)^2 + \left(\sum_{k=0}^{n} b_k x^k\right)^2.
\end{aligned}
$$

Making use of the result from part (a), we see that

$$
\begin{aligned}
\int_{-1}^{1} \left|f(x)^2\right| dx &= \int_{-1}^{1} \left(\sum_{k=0}^{n} a_k x^k\right)^2 dx + \left(\sum_{k=0}^{n} b_k x^k\right)^2 \\
&\leq \pi\left(\sum_{k=0}^{n} a_k^2 + \sum_{k=0}^{n} b_k^2\right) \\
&= \pi\sum_{k=0}^{n}\left(a_k^2 + b_k^2\right) = \pi\sum_{k=0}^{n}\left|c_k\right|^2,
\end{aligned}
$$

as desired.

(c) Integrating the polynomial from 0 to 1 along the real axis yields

$$
\int_0^1 \left(\sum_{k=0}^{n} c_k x^k\right)^2 dx = \sum_{j,k=0}^{n} c_j c_k \int_0^1 x^{j+k} dx = \sum_{j,k=0}^{n} \frac{c_j c_k}{j+k+1}.
$$

Taking the absolute value, we find that

$$
\begin{aligned}
\left|\sum_{j,k=0}^{n} \frac{c_j c_k}{j+k+1}\right| = \left|\int_0^1 \left(\sum_{k=0}^{n} c_k x^k\right)^2 dx\right| &\leq \int_0^1 \left|\sum_{k=0}^{n} c_k x^k\right|^2 dx \\
&\leq \int_{-1}^1 \left|\sum_{k=0}^{n} c_k x^k\right|^2 dx \\
&\leq \pi\sum_{k=0}^{n}\left|c_k\right|^2.
\end{aligned}
$$

Problem 15 (Problem 5, Chapter IV.3, p. 113)**.**
Suppose that D is a bounded domain with piecewise smooth boundary, and that $f(z)$ is analytic on $D \cup \partial D$. Show that

$$\sup_{z \in \partial D} |\bar{z} - f(z)| \geq 2 \frac{\text{Area}(D)}{\text{Length}(\partial D)}.$$

Show that this estimate is sharp, and that in fact there exist D and $f(z)$ for which equality holds. Hint: Consider $\int_{\partial D} [\bar{z} - f(z)] dz$, and use the fact that, if D is a bounded domain, then $\displaystyle\int_{\partial D} \bar{z} dz = 2i \,\text{Area}(D)$.

Solution. We first show that $\int_{\partial D} \bar{z} dz = 2i \,\text{Area}(D)$. Using $\bar{z} = x - iy$ and $dz = dx + idy$, we have

$$\bar{z} dz = xdx + ydy + i(ydx + xdx).$$

From Green's theorem, we have that

$$\int_{\partial D} xdx + ydy = \iint_D 0 dx dy = 0$$

and

$$\int_{\partial D} i(ydx + ydy) = i \iint_D (1+1) dx dy = 2i \,\text{Area}(D).$$

Using the ML-estimate for the integral $\int_{\partial D} \bar{z} dz$, we have

$$\begin{aligned} \frac{1}{\text{Length}(\partial D)} \left| \int_{\partial D} \bar{z} dz \right| &\leq \sup_{\partial D} |\bar{z}| \\ &\leq \sup_{\partial D} |\bar{z}| + \sup_{\partial D} |f(z)| \\ &\leq \sup_{\partial D} |\bar{z} - f(z)|. \end{aligned}$$

Putting this together with the fact that $\left| \int_{\partial D} \bar{z} dz \right| = 2 \,\text{Area}(D)$ yields the desired result.

To show that this bound is sharp, consider $f(z) = 0$ and let D be any disk of radius $r > 0$ centered at the origin. Then $\text{Length}(\partial D) = 2\pi r$ and $\text{Area}(D) = \pi r^2$. Note that

$$\sup_{z \in D} |\bar{z}| = r$$

and

$$2 \frac{\text{Area}(D)}{\text{Length}\, \partial D} = 2 \frac{\pi r^2}{2\pi r} = r,$$

so the estimate is indeed sharp.

Problem 16 (Problem 3, Chapter IV.5, p. 119).
A function $f(z)$ on the complex plane is doubly periodic if there are two periods ω_0 and ω_1 of $f(z)$ that do not lie on the same line through the origin (that is, ω_0 and ω_1 are linearly independent over the reals, and $f(z+\omega_0) = f(z+\omega_1) + f(z)$ for all complex numbers z). Prove that the only entire functions that are double periodic are the constants.

Solution. Consider the disk D_R with radius $R = |\omega_0| + |\omega_1|$. The closure of D_R is compact, so $f(z)$ is bounded on D_R since $f(z)$ is analytic everywhere. So there is an $M > 0$ such that $|f(z)| < M$ for all $z \in D_R$. For any point $z \in \mathbb{C}$, there are integers $m, n \in \mathbb{Z}$ such that $z + m\omega_0 + n\omega_1 \in D_R$, and thus

$$|f(z)| = |f(z + m\omega_0 + n\omega_1)| < M,$$

so $f(z)$ is bounded on the entire complex plane. By Liouville's theorem, f is constant.

Problem 17 (Problem 2, Chapter IV.6, p. 122)**.**
Let $h(t)$ be a continuous function on the interval $[a, b]$. Show that the Fourier transform

$$H(z) = \int_a^b h(t)e^{-itz}dt$$

is an entire function that satisfies

$$|H(z)| \leq Ce^{A|y|}, \qquad z = x + iy \in \mathbb{C},$$

for some constants $A, C \geq 0$. *Remark:* An entire function satisfying such a growth restriction is called an **entire function of finite type**.

Solution. Let $z = x + iy$ such that $e^{itz} = e^{itx-ty}$ for real t. Then $\left|e^{-itz}\right| = e^{ty}$ and

$$\begin{aligned}
|H(z)| = \left|\int_a^b h(t)e^{-itz}dt\right| &\leq \int_a^b \left|h(t)e^{-itz}\right| dt \\
&= \int_a^b |h(t)|\, e^{ty} dt \\
&\leq \max_{t\in[a,b]} |h(t)| \max_{t\in[a,b]} \left[e^{ty}\right] (b-a) \\
&\leq (b-a) \max_{t\in[a,b]} |h(t)| \max\{e^{|a||y|}, e^{|b||y|}\} \\
&\leq (b-a) \max_{t\in[a,b]} |h(t)|\, e^{\max\{|a|,|b|\}|y|} \\
&= Ce^{A|y|},
\end{aligned}$$

where $C = (b-a)\max_{t\in[a,b]} |h(t)|$ and $A = \max\{|a|, |b|\}$.

Problem 18 (Problem 3, Chapter IV.6, p. 123)**.**
Let $h(t)$ be a continuous function on a subinterval $[a, b]$ of $[0, \infty)$. Show that the Fourier transform $H(z)$, defined above, is bounded in the lower half-plane.

Solution. We use the same arguments as in the previous problem. For all $z = x + iy$ in the lower half-plane, with $y < 0$, and all t in $[a, b] \subset [0, \infty)$, note that $ty \geq 0$ and thus $e^{ty} \leq 1$. Then

$$\left|e^{-itz}\right| = \left|e^{-itx}\right| e^{ty} \leq 1.$$

Hence

$$|H(z)| \leq \int_a^b |h(t)| \left|e^{itz}\right| dt \leq \int_a^b |h(t)| \, dt \leq (b-a) \max_{t \in [a,b]} |h(t)| \,,$$

so $|H(z)| \leq M$ for all z in the lower half-plane, where $M = (b-a) \max_{t \in [a,b]} |h(t)|$.

Problem 19 (Problem 3, Chapter V.1, p. 132)**.**
Show that if $p > 1$, then the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges to S, where

$$\left| S - \sum_{k=1}^{n} \frac{1}{k^p} \right| < \frac{1}{(p-1)n^{p-1}}.$$

Hint: Use the estimate $\dfrac{1}{k^p} < \displaystyle\int_{k-1}^{k} \frac{dx}{x^p}$.

Solution. Let k be a positive integer. Then the function $\frac{1}{x}$ is decreasing on the interval $(k-1, k)$, and $\frac{1}{k} < \frac{1}{x}$ for all x on this interval. Since $p > 1$, it follows that $\frac{1}{k^p} \leq \frac{1}{x^p}$ on this interval as well. This justifies the estimate

$$\frac{1}{k^p} < \int_{k-1}^{k} \frac{1}{x^p} dx.$$

Therefore

$$\sum_{k=1}^{n} \frac{1}{k^p} = 1 + \sum_{k=2}^{n} \frac{1}{k^p} < 1 + \int_{1}^{n} \frac{1}{x^p} dx = \frac{1}{(1-p)n^{p-1}} - \frac{p}{1-p}$$

and the limit of the right-hand side goes to $\frac{p}{p-1}$ as $n \to \infty$. So the series does indeed converge. Let $S = \sum_{k=1}^{\infty} \frac{1}{k^p}$, then for $n > 0$

$$\left| S - \sum_{k=1}^{n} \frac{1}{k^p} \right| = \sum_{k=n+1}^{\infty} \frac{1}{k^p} < \int_{n}^{\infty} \frac{1}{x^p} dx = \frac{1}{(1-p)n^{p-1}},$$

as desired.

Problem 20 (Problem 4, Chapter V.4, p. 148)**.**
Suppose $f(z)$ is analytic at $z = 0$ and satisfies $f(z) = z + f(z)^2$. What is the radius of convergence of the power series expansion of $f(z)$ about $z = 0$?

Solution. Note that $f(z)$ must satisfy $f(z)^2 - f(z) = z = 0$, and thus $f(z)$ must coincide with a branch of $\frac{1-\sqrt{1-4z}}{2}$. The radius of convergence of the power series of this function is the largest number R such that it is analytic on the disk of radius R around $z = 0$. Since this function is not analytic when $1 - 4z = 0$, or $z = \frac{1}{4}$, but it can be analytically defined everywhere else, the radius of convergence is $\frac{1}{4}$.

Problem 21 (Problem 4, Chapter VI.2, p. 176)**.**
Suppose that $f(z)$ is meromorphic on the disk $\{|z| < s\}$, with only a finite number of poles on the disk. Show that the Laurent decomposition of $f(z)$ with respect to the annulus $\{s - \varepsilon < |z| < s\}$ has the form $f(z) = f_0(z) + f_1(z)$, where $f_1(z)$ is the sum of the principal parts of $f(z)$ at its poles.

Solution. Since there are only finitely many poles, they must all be isolated. Let $z_1, \ldots, z_n$ be the poles of $f(z)$ on the disk $D_s = \{|z| < s\}$. Note that $\max_j |z_j| < s$, and pick $\varepsilon > 0$ such that $\varepsilon < s - \max_j |z_j|$. Then $f(z)$ is analytic on the annulus $\{s - \varepsilon < |z| < s\}$. At each of its poles z_k, $f(z)$ has Laurent decomposition

$$f(z) = f_{k,1}(z) + f_{k,0}(z),$$

where $f_{k,1}(z) = \frac{a_{k,1}}{z-z_k} + \cdots + \frac{a_{k,N_k}}{(z-z_k)^{N_k}}$ and N_k is the order of the pole at z_k. Each $f_{k,1}$ is meromorphic on D_s with a pole of order N_k at z_k. In particular, they are all analytic on $\{|z| > s - \varepsilon\}$. Define the function

$$g(z) = \sum_{k=1}^{n} f_{k,1}(z),$$

which is meromorphic, and has the same poles and principal parts of $f(z)$ at each of its poles and is analytic on $\{|z| > s - \varepsilon\}$. Furthermore, since each $f_{k,1}(z) \to 0$ as $|z| \to \infty$, we must have that the principal part of $f(z)$ is $f_1(z) = g(z)$ by the uniqueness of the Laurent decomposition.

Problem 22 (Problem 5, Chapter VI.2, p. 176)**.**
By estimating the coefficients of the Laurent series, prove that if z_0 is an isolated singularity of f, and if $(z-z_0)f(z) \to 0$ as $z \to z_0$, then z_0 is removable. Give a second proof based on Morera's theorem.

Solution. Suppose that f is analytic on the punctured disk $D_R \backslash \{z_0\} = \{0 < |z-z_0| < R\}$ for some R. For positive integers $k \geq 1$, the coefficients of the principal part of the Laurent series of $f(z)$ at $z = z_0$ are given by

$$a_{-k} = \frac{1}{2\pi i} \oint_{|z-z_0|=r} (z-z_0)^{k-1} f(z) dz.$$

for $0 < r < R$. We can find the ML-estimates of these coefficients:

$$\begin{aligned}
|a_{-k}| &\leq \frac{1}{2\pi} \oint_{|z-z_0|=r} \left| \underbrace{(z-z_0)}_{=r}^{k-1} \right| |f(z)| \, |dz| \\
&\leq \frac{1}{2\pi} 2\pi r \cdot r^{k-1} \cdot \max_{|z-z_0|=r} |f(z)| \\
&= r^k \max_{|z-z_0|=r} |f(z)| \,.
\end{aligned}$$

Let z_1 be an optimal point on the circle $|z-z_0| = r$ such that $|f(z_1)| = \max_{|z-z_0|=r} |f(z)|$. Then $r = |z_1 - z_0|$ and

$$|a_{-k}| \leq r^{k-1} \left| (z_1 - z_0) f(z) \right| .$$

If $k = 1$, then $r^{k-1} = 1$. If $k > 1$, then $r^{k-1} \to 0$ as $r \to 0$. In the limit as $r \to 0$, note that

$$|(z_1 - z_0) f(z_1)| \to 0$$

since $z_1 \to z_0$ as $r \to 0$. Hence $|a_{-k}| = 0$ for all $k \geq 1$. Hence the principal part of $f(z)$ is zero, so $f(z)$ has a removable singularity.

Alternatively, we can use Morera's theorem, which states that if the integral of a continuous function around any closed rectangle in the domain is zero, then the function is analytic. Define the function $g(z) = (z-z_0)f(z)$ on the same disk for $z \neq z_0$, and $g(z_0) = 0$. By assumption, $g(z)$ is continuous at z_0, and thus on all of D_R, so we may apply Morera's theorem. Let T be a rectangle in D_R with sides parallel to the axes. If $z_0 \notin T$, then $\int_{\partial T} g(z)dz = 0$, so suppose $z_0 \in T$. If z_0 is on a vertex of T, we can split T into smaller sub-rectangles, where only one sub-rectangle contains z_0. The integral around T is the sum of the integrals around all of the sub-rectangles, but all of these vanish except for the one containing z_0. Since $g(z) \to 0$ as $z \to z_0$, we can make our sub-rectangles as small as needed such that $\int_{\partial T} |g(z)| \, |dz|$ is small. Therefore $\int_{\partial T} g(z)dz = 0$. If z_0 is not on a vertex of T, we can split T into smaller rectangles such that z_0 is on a vertex of at most 4 of the sub-rectangles. This reduces to the previous case, where the integral over each sub-rectangle is zero and thus the integral over the whole rectangle is zero. Hence $g(z)$ is analytic on D_R.

Since $g(z)$ is analytic on D_R, it has the power series representation

$$g(z) = b_0 + b_1(z-z_0) + b_2(z-z_0)^2 + \cdots$$

on D_R, where $b_0 = 0$ since $g(z_0) = 0$. On the punctured disk, we can write $f(z)$ as

$$f(z) = \frac{g(z)}{z-z_0} = b_1 + b_2(z-z_0) + \cdots .$$

In particular, $f(z)$ has a removable singularity at z_0, since we can define $f(z_0) = b_1$.

Problem 23 (Problem 12, Chapter VII.1, p. 203).
Let $Q(z)$ be a polynomial of degree m with no zeros on the real line, and let $f(z)$ be a function that is analytic in the upper half-plane and across the real line. Suppose there is $b < m-1$ such that $|f(z)| \leq |z|^b$ for z in the upper half-plane, $|z| > 1$. Show that

$$\int_{-\infty}^{\infty} \frac{f(x)}{Q(x)} dx = 2\pi i \sum \operatorname{Res}\left[\frac{f(z)}{Q(z)}, z_j\right],$$

summed over the zeros z_j of $Q(z)$ in the upper half-plane.

Solution. From residue theory, integrating around the disk of radius R (for R large enough) in the upper half plane yields

$$\int_{-R}^{R} \frac{f(x)}{Q(x)} dx + \left|\int_{\Gamma_R} \frac{f(z)}{Q(z)} dz\right| = 2\pi i \sum_j \operatorname{Res}\left[\frac{f(z)}{Q(z)},\, z_j\right], \tag{23.1}$$

where the sum is taken over all zeros of $Q(z)$ in the upper half-plane. However, integrating along the semicircular path of radius R about the origin in the upper half-plane, we find

$$\begin{aligned} \left|\int_{\Gamma_R} \frac{f(z)}{Q(z)} dz\right| &\leq \int_0^{\pi} \frac{\left|f(Re^{i\theta})\right|}{|Q(Re^{i\theta})|} d\theta \\ &\leq \int_0^{\pi} \frac{R^b}{|Q(Re^{i\theta})|} d\theta, \end{aligned}$$

and the last integral tends to 0 as $R \to \infty$, since $\deg Q(z) = m \geq b+2$. Taking the limit of both sides of (23.1) as $R \to \infty$ yields the desired result.

Problem 24 (Problem 7, Chapter VII.3, p. 205).
Show using residue theory that

$$\int_0^{2\pi} \frac{1}{(w+\cos\theta)^2}d\theta = \frac{2\pi w}{(w^2-1)^{3/2}}, \qquad \text{for } w \in \mathbb{C} \setminus [-1,1]. \tag{24.1}$$

Specify the branch of the power function. Check the answer by differentiating the integral of $\frac{1}{w+\cos\theta}$ with respect to the parameter w.

Solution. Using the substitution $z = e^{i\theta}$ such that $d\theta = \frac{dz}{iz}$, and $\cos\theta = \frac{z+\frac{1}{z}}{2}$, the integral becomes

$$I = \int_0^{2\pi} \frac{1}{(w+\cos\theta)^2}d\theta = \frac{4}{i}\oint_{|z|=1} \frac{1}{(2w+z+\frac{1}{z})^2 z}dz = \frac{4}{i}\oint_{|z|=1} \frac{z}{(z^2+2wz+1)^2}dz.$$

The quadratic in the denominator has roots $\frac{-2w\pm\sqrt{4w^2-4}}{2} = -w \pm \sqrt{w^2-1}$, where we choose the branch of $\sqrt{w^2-1}$ that is positive for $w > 1$. Note that the function $\sqrt{w^2-1}$ has branch points at $z = \pm 1$, so it can be continuously defined on the domain $\mathbb{C}\backslash[-1,1]$. It suffices to show the identity in (24.1) for $w \in (1,\infty)$. Then, by the uniqueness principle, the two analytic functions in (24.1) will hold for all $\mathbb{C}\backslash[-1,1]$.

Call the roots of the polynomial $z_0 = -w - \sqrt{w^2-1}$ and $z_1 = -w + \sqrt{w^2-1}$. Since we may assume that $w > 1$, with this choice of branch, only z_1 sits inside the unit circle. So the integral can be written as

$$\begin{aligned} I &= \frac{4}{i}\oint_{|z|=1} \frac{z}{(z-z_0)^2(z-z_1)^2}dz \\ &= 8\pi \operatorname{Res}\left[\frac{z}{(z-z_0)^2(z-z_1)^2}, z_1\right]. \end{aligned}$$

Since the integrand has a double pole at $z = z_1$, this residue can be computed as

$$\operatorname{Res}\left[\frac{z}{(z-z_0)^2(z-z_1)^2}, z_1\right] = \frac{d}{dz}\left[\frac{z}{(z-z_0)^2}\right]\bigg|_{z=z_1} = -\left.\frac{z+z_0}{(z-z_0)^3}\right|_{z=z_1} = \frac{2w}{(2\sqrt{w^2-1})^3} = \frac{w}{4(w^2-1)^{3/2}},$$

where (as before) the branch of the function is chosen such that it is positive for $w > 1$. This yields the desired identity

$$\int_0^{2\pi} \frac{d\theta}{(w+\cos\theta)^2} = \frac{2\pi w}{(w^2-1)^{3/2}}$$

for $w > 1$. Since both sides of this identity are analytic on $\mathbb{C}\backslash[-1,1]$, by the uniqueness principle this identity holds for all $w \in \mathbb{C}\backslash[-1,1]$.

To check this identity, we differentiate both sides of the identity

$$\frac{2\pi}{\sqrt{w^2-1}} = \int_0^{2\pi} \frac{d\theta}{w+\cos\theta}$$

with respect to the parameter w. We are allowed to differentiate under the integral in the right-hand side of the above identity, since the integrand $\frac{1}{w+\cos\theta}$ and its partial derivative $-\frac{1}{(w+\cos\theta)^2}$ are continuous on $w \in \mathbb{C}\backslash[-1,1]$ and $\theta \in [0,2\pi]$. Differentiating, we have

$$-\frac{2\pi w}{(w^2-1)^{3/2}} = \frac{d}{dw}\int_0^{2\pi} \frac{d\theta}{w+\cos\theta} = -\int_0^{2\pi} \frac{1}{(w+\cos\theta)^2}d\theta$$

which yields the desired identity.

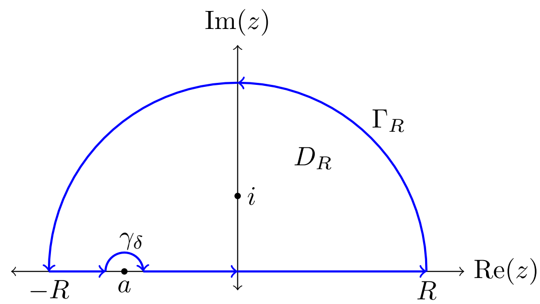
Problem 25 (Problem 3, Chapter VII.6, p. 215)**.**
By integrating around the boundary of an indented half-disk in the upper half-plane, show that

$$\mathrm{PV}\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x-a)}\,dx = -\frac{\pi a}{a^2+1} \qquad \text{for } -\infty < a < \infty.$$

Solution. The integral may be evaluated by taking the limit of the sum of the integrals

$$\int_{-R}^{a-\delta} \frac{1}{(x^2+1)(x-a)}\,dx + \int_{a+\delta}^{R} \frac{1}{(x^2+1)(x-a)}\,dx$$

as $R \to \infty$ and $\delta \to 0$. These can be evaluated by integrating around the boundaries of the indented half-disks D_R as shown in the following figure.



The indented half-disk contains only one pole at $z = i$, so integrating around the boundary of D_R yields

$$\begin{aligned}\int_{\partial D_R} \frac{1}{(z^2+1)(z-a)}\,dz &= 2\pi i \operatorname{Res}\left[\frac{1}{(z^2+1)(z-a)}, i\right] = 2\pi i \left.\frac{1}{2z(z-a)}\right|_{z=i} \\ &= \frac{2\pi i}{2i(i-a)} = \frac{\pi}{i-a} = \frac{-\pi(a+i)}{a^2+1}.\end{aligned}$$

The integral along the semicircular path Γ_R of radius R will tend to zero as $R \to \infty$, since the polynomial in the denominator has degree three. So we only need to worry about the integral along the semicircular path γ_δ in the clockwise direction around the point $z = a$. By the Fractional Residue Theorem, taking the limit of this as $\delta \to 0$ yields

$$\lim_{\delta\to 0}\int_{\gamma_\delta} \frac{1}{(z^2+1)(z-a)}\,dz = -\pi i \operatorname{Res}\left[\frac{1}{(z^2+1)(z-a)}, a\right] = -\pi i \left.\frac{1}{z^2+1}\right|_{z=a} = \frac{-\pi i}{a^2+1}.$$

Putting this together in the limit as $R \to \infty$ yields

$$\left(\int_{\partial D_R}\right) - \left(\lim_{\delta\to 0}\int_{\gamma_\delta}\right) = -\frac{\pi a}{a^2+1}$$

as desired.

Problem 26 (Problem 5, Chapter VII.7, p. 218)**.**
Show that

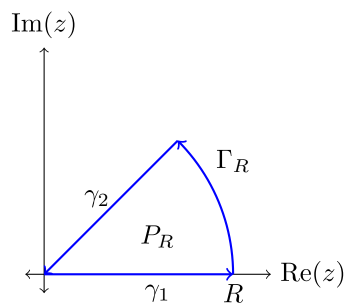
$$\lim_{R\to\infty}\int_0^R \sin(x^2)dx = \lim_{R\to\infty}\int_0^R \cos(x^2)dx = \frac{\sqrt{\pi}}{2\sqrt{2}},$$

by integrating e^{iz} around the boundary of the pie-slice domain determined by $0 < \arg z < \frac{\pi}{4}$ and $|z| < R$. (*Remark*: These improper integrals are called the **Fresnel integrals**.)

Solution. We can evaluate these integrals by examining the real and imaginary parts of the integrals

$$\int_0^R e^{ix^2}dx,$$

and integrating this around the boundary of the pie-slice domain P_R as shown in the following figure.



The integral around the outer arc of radius R tends to zero as $R \to \infty$. Indeed, making the change of variables $z = Re^{i\theta}$ and $dz = iRe^{i\theta}d\theta$ yields

$$\begin{aligned}\int_{\Gamma_R} e^{iz^2}dz = iR\int_0^{\frac{\pi}{4}} e^{iR^2e^{i2\theta}}e^{i\theta}d\theta &= iR\int_0^{\frac{\pi}{4}} e^{iR^2(\cos(2\theta)+i\sin(2\theta))}e^{i\theta}d\theta \\ &= iR\int_0^{\frac{\pi}{4}} e^{-R^2\sin(2\theta)}e^{i\left(R^2\cos(2\theta)+\theta\right)}d\theta.\end{aligned}$$

Along the lines of the proof of Jordan's lemma, note that $\sin(2\theta) \geq \frac{4\theta}{\pi}$ on the interval $0 \leq \theta \leq \frac{\pi}{4}$, and thus

$$\left|\int_{\Gamma_R} e^{iz^2}dz\right| \leq R\int_0^{\frac{\pi}{4}} e^{-R^2\sin(2\theta)}d\theta \leq R\int_0^{\frac{\pi}{4}} e^{-4R^2\theta/\pi}d\theta = R\frac{\pi}{4R^2}\int_0^{R^2} e^{-t}dt \leq \frac{\pi}{4R}\int_0^{\infty} e^{-t}dt = \frac{\pi}{4R}.$$

Hence $\int_{\Gamma_R} e^{iz^2}dz \to 0$ as $R \to \infty$.

Next, we evaluate the integral along the path γ_2 from the point $Re^{i\frac{\pi}{4}}$ to the origin. This may be parameterized as $z = re^{i\frac{\pi}{4}}$ where we take r from R to zero, and note that $dz = e^{i\frac{\pi}{4}}dr$. This becomes

$$\int_{\gamma_2} e^{iz^2}dz = e^{i\frac{\pi}{4}}\int_R^0 e^{ir^2e^{i\frac{\pi}{2}}}dr = -e^{i\frac{\pi}{4}}\int_0^R e^{-r^2}dr,$$

and taking the limit of this as $R \to \infty$ yields the well-known Gaussian integral $\int_0^\infty e^{-r^2}dr = \frac{\sqrt{\pi}}{2}$. Since there are no poles within the pie-slice domain, in the limit as $R \to \infty$ we have that

$$\int_{\gamma_1} e^{iz^2}dz = -\int_{\gamma_1} e^{iz^2}dz = e^{i\frac{\pi}{4}}\frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{2\sqrt{2}}\left(1+i\right).$$

Taking the real and imaginary parts of this integral yields the desired results.

Problem 27 (Problem 6, Chapter VIII.1, p. 228)**.**
For a fixed real number α, find the number of solutions of $z^5 + 2z^3 - z^2 + z = \alpha$ satisfying $\operatorname{Re} z > 0$.

Solution. To find the number of zeros of $f_\alpha(z) = z^5+2z^3-z^2+z-\alpha$, we can choose R big enough and consider the semicircular parth of radius R in the right half-plane so that it encloses any zeros, together with the path from iR to R on the imaginary axis. For $R \gg 0$, note that $f_\alpha(iR) \approx iR^5 + R^2$ and $f_\alpha(-iR) \approx -iR^5 + R^2$. Following the semicircular path from iR to R in the right half-plane, the path of $f_\alpha(Re^{i\theta})$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ goes almost two and a half times around the origin, so the change in argument is 5π.

We now need to check along the the imaginary axis. Setting $z = iy$, we have

$$f_\alpha(iy) = (iy)^5 + 2(iy)^3 - 2(iy)^2 + iy - \alpha = iy(y^4 - 2y^2 + 1) + y^2 - \alpha = iy(y^2 - 1)^2 + y^2 - \alpha. \qquad (27\text{-}1)$$

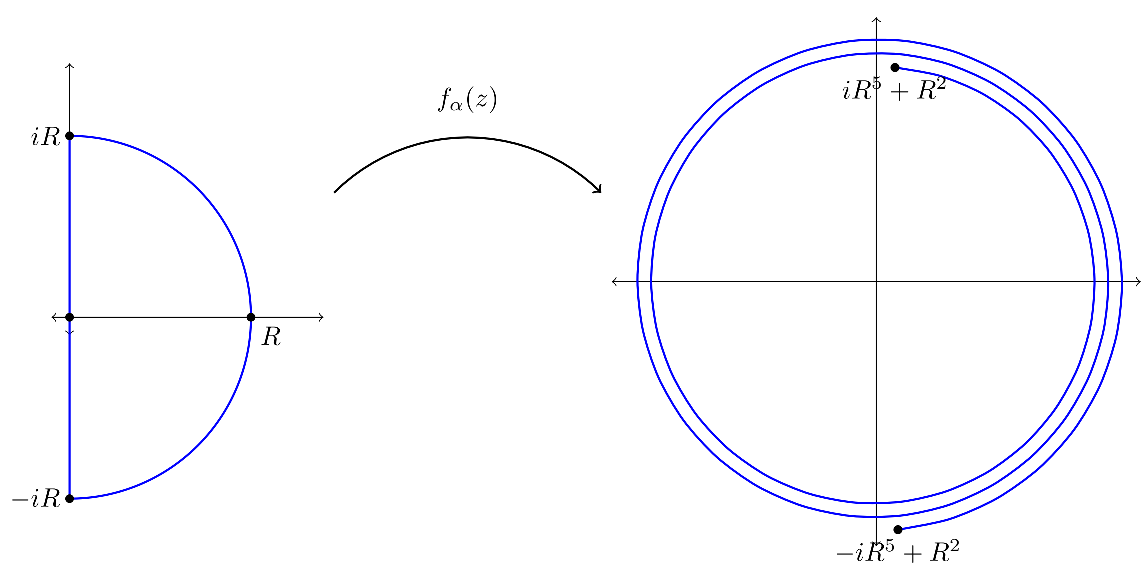
The imaginary part of this vanishes when $y = 0$ and $y = \pm 1$, wheras the real part vanishes when $y^2 = \alpha$.

- If $\alpha < 0$, then the real part never vanishes, so there are no roots on the imaginary axis. The path of $f_\alpha(iy)$ from $y = R$ to $y = -R$ never crosses the imaginary axis, hence the change in argument along this path is $-\pi$. The total change in argument around the entire right half-plane is 4π, and thus there are two zeros in the right half-plane.

- If $\alpha = 0$, then the path of $f_\alpha(iy)$ from $y = R$ to $y = -R$ only touches the imaginary axis when $y = 0$, but does not cross it. (In fact, $z = 0$ is a zero, but it does not have $\operatorname{Re} z > 0$ so we ignore it). We can slightly modify the path as to bypass crossing through the origin and staying in the right half-plane. Again, the change in argument along this path is $-\pi$, so the total change in argument is 4π and therefore there are two zeros in the right half-plane.

- If $\alpha > 0$ and $\alpha \neq 1$, the real part of (27-1) vanishes at $y = \pm\sqrt{\alpha}$, while the imaginary part vanishes at $y = 0$ and $y = \pm 1$, so there are no zeros of $f = -\alpha$ along the path. The path of $f_\alpha(iy)$ crosses the imaginary axis at $y = \pm\sqrt{\alpha}$, going past the origin in the left half-plane. So the change in argument along the path is $+\pi$. The total change in argument along the whole path around the right half-plane is thus 6π, so there are three zeros in the right half-plane.

- If $\alpha = 1$, then $f_\alpha(z)$ has zeros at $z = \pm i$. We can circumnavigate these zeros by taking instead the path along $iy + \varepsilon$ from $y = R$ to $-R$ for some small $\varepsilon > 0$. Along this path, the real part of (27-1) becomes $y^2 - \alpha + \varepsilon$, and the path then goes around the origin in a clockwise manner. This gives a change in argument of -3π along this part of the path, resulting in a net change of argument of 2π. In this case, there is only one zero in the right-half plane.



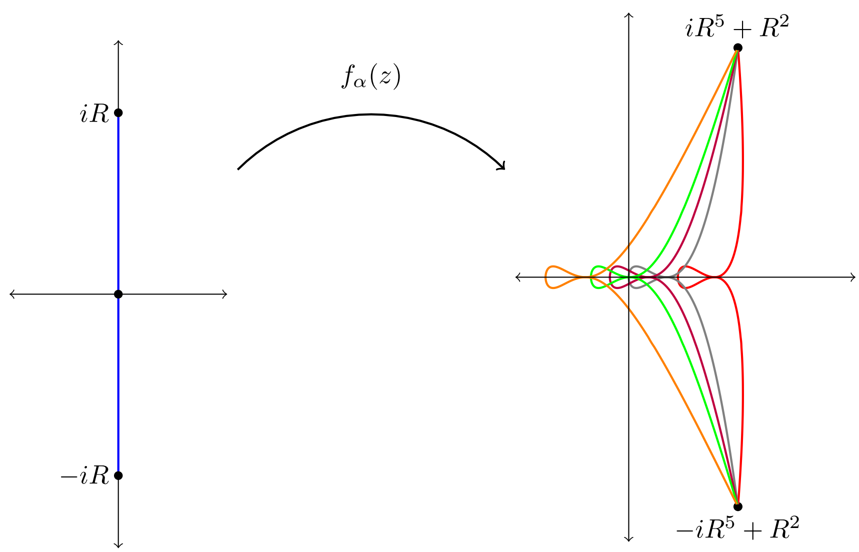


Figure 27-1: A plot of of the curve $f_\alpha(iy)$ for $y \in [R, -R]$, for various different $\alpha \in \mathbb{R}$. These curves are: red for $\alpha < 0$, gray for $\alpha = 0$, purple for $0 < \alpha < 1$, gray for $\alpha = 1$, and orange for $\alpha > 1$. (Not to scale.)

Problem 28 (Problem 6, Chapter VIII.1, p. 228-229)**.**
Show that if $f(z)$ is analytic in a domain D, and if γ is a closed curve in D such that the values of $f(z)$ on γ lie in the slit plane $\mathbb{C}\backslash(-\infty, 0]$, then the increase in the argument of $f(z)$ around γ is zero.

Solution. Since all of the values of $f(\gamma(z))$ lie in the slit plane $\mathbb{C}\backslash(-\infty, 0]$, and $f(z)$ is analytic on γ, we can define an analytic branch $\operatorname{Log} f(z)$ of $\log f(z)$ that is analytic (and single-valued) on γ. So the logarithmic integral $\int_\gamma d \log f(z) = \int_\gamma d \operatorname{Log} f(z)$ vanishes, and thus the increase in argument is zero, since

$$\int_\gamma d \log f(z) = \frac{1}{i} \int_\gamma d \arg(f(z))$$

for any closed curve γ.

Problem 29 (Problem 5, Chapter X.12, p. 279)**.**
Let $h\left(e^{i\theta}\right)$ be a piecewise continuous function (or an integrable function) on the unit circle. Show that the Poisson integral $\tilde{h}(z)$ tends to $h(\zeta)$ as $z \in \mathbb{D}$ tends to any point ζ of the unit circle at which $h\left(e^{i\theta}\right)$ is continuous.

Solution. Let $\zeta = e^{i\theta}$ be a point on the unit circle such that $h\left(e^{i\varphi}\right)$ is continuous. Since $h\left(e^{i\varphi}\right)$ is piecewise continuous on a bounded domain, $h\left(e^{i\varphi}\right)$ is bounded as well. Let $\varepsilon > 0$ and choose $M > 0$ such that $\left|e^{i\varphi}\right| \le M$ for all φ and choose $\pi > \delta > 0$ such that $|\theta - \varphi| < \delta$ implies $\left|h\left(e^{i\theta}\right) - h\left(e^{i\varphi}\right)\right| \le \varepsilon$. Now note that

$$
\begin{aligned}
\left|\tilde{h}(z) - h\left(e^{i\theta}\right)\right| &= \left|\int_{-\pi}^{\pi} \left[\tilde{h}\left(e^{i(\theta-\varphi)}\right) - h\left(e^{i\theta}\right)\right] P_r(\varphi)\frac{d\varphi}{2\pi}\right| \\
&\le \int_{-\pi}^{\pi} \left|\tilde{h}\left(e^{i(\theta-\varphi)}\right) - h\left(e^{i\theta}\right)\right| P_r(\varphi)\frac{d\varphi}{2\pi} \\
&\le \int_{-\delta}^{\delta} \underbrace{\left|\tilde{h}\left(e^{i(\theta-\varphi)}\right) - h\left(e^{i\theta}\right)\right|}_{\le\varepsilon} P_r(\varphi)\frac{d\varphi}{2\pi} + \int_{\delta\le|\varphi|\le\pi} \underbrace{\left|\tilde{h}\left(e^{i(\theta-\varphi)}\right) - h\left(e^{i\theta}\right)\right|}_{\le 2M} P_r(\varphi)\frac{d\varphi}{2\pi} \\
&\le \varepsilon \underbrace{\int_{-\delta}^{\delta} P_r(\varphi)\frac{d\varphi}{2\pi}}_{\le\int_{-\pi}^{\pi} P_r(\varphi)\frac{d\varphi}{2\pi}=1} + 2M \max_{\delta\le|\varphi|\le\pi} P_r(\varphi) \\
&\le \varepsilon + 2M \max_{\delta\le|\varphi|\le\pi} P_r(\varphi).
\end{aligned}
$$

But $\max\left\{P_r(\varphi) \,\middle|\, \delta \le |\varphi| \le \pi\right\} \to 0$ for any fixed $\delta > 0$. Hence, for fixed θ, the values of $\tilde{h}\left(re^{i\theta}\right)$ cluster to within ε of $h\left(e^{i\theta}\right)$ as $z \to e^{i\theta}$, and this is for any $\varepsilon > 0$. This result yields the desired limit.

Problem 30 (Problem 6, Chapter X.1, p. 279)**.**
A function $f(z)$, $z \in \mathbb{D}$, is said to have **radial limit** L at $\zeta \in \partial\mathbb{D}$ if $f(r\zeta) \to L$ as r increases to 1. Let $h\left(e^{i\theta}\right)$ be a piecewise continuous function on the unit circle. Show that $\tilde{h}(z)$ has a radial limit at each $\zeta \in \partial\mathbb{D}$, equal to the average of the limits of $h\left(e^{i\theta}\right)$ at ζ from each side.

Solution.
Let $\zeta \in \partial\mathbb{D}$ with $\zeta = e^{i\theta}$. Since h is piecewise continuous and bounded, the 'right'- and 'left'-sided limits (or perhaps we should call them the 'clockwise' and 'counterclockwise' limits) of h exist at each point $e^{i\theta}$ on the unit circle. We denote these one-sided limits as

$$h_-\left(e^{i\theta}\right) = \lim_{\varphi \searrow 0^+} h\left(e^{i(\theta-\varphi)}\right) \qquad \text{and} \qquad h_+\left(e^{i\theta}\right) = \lim_{\varphi \nearrow 0^-} h\left(e^{i(\theta-\varphi)}\right).$$

Hence for all $\varepsilon > 0$ there is a $\delta > 0$ such that $\left|h\left(e^{i(\theta-\varphi)}\right) - h_-\left(e^{i\theta}\right)\right| < \varepsilon$ whenever $\varphi \in (0,\delta)$, and $\left|h\left(e^{i(\theta-\varphi)}\right) - h_+\left(e^{i\theta}\right)\right| < \varepsilon$ whenever $\varphi \in (-\delta, 0)$.

The rest of the proof follows along the lines of the proof of the boundary-value problem in chapter X.1 of the book. Let $\varepsilon > 0$ and choose $\delta > 0$ as above. Since h is bounded, there is an $M > 0$ such that $\left|h\left(e^{i\theta}\right)\right| < M$ for all θ. Note that $\int_0^\pi P_r(\varphi)\frac{d\varphi}{2\pi} = \int_{-\pi}^0 P_r(\varphi)\frac{d\varphi}{2\pi} = \frac{1}{2}$, and thus comparing $h\left(e^{i(\theta-\varphi)}\right)$ to the left-sided limit of h at $e^{i\theta}$ for $\varphi > 0$ yields

$$\begin{aligned}\left|\int_0^\pi h\left(e^{i(\theta-\varphi)}\right) P_r(\varphi)\frac{d\varphi}{2\pi} - \frac{1}{2}h_-\left(e^{i\theta}\right)\right| &\leq \int_0^\pi \left|h\left(e^{i(\theta-\varphi)}\right) - h_-\left(e^{i\theta}\right)\right| P_r(\varphi)\frac{d\varphi}{2\pi} \\ &\leq \int_0^\delta \varepsilon P_r(\varphi)\frac{d\varphi}{2\pi} + M \max_{\delta\leq\varphi\leq\pi} P_r(\varphi) \\ &\leq \frac{\varepsilon}{2} + M \max_{\delta\leq\varphi\leq\pi} P_r(\varphi). \end{aligned} \tag{30.1}$$

Similarly, comparing $h\left(e^{i(\theta-\varphi)}\right)$ to the right-sided limit of h at $e^{i\theta}$ for $\varphi < 0$ yields

$$\begin{aligned}\left|\int_{-\pi}^0 h\left(e^{i(\theta-\varphi)}\right) P_r(\varphi)\frac{d\varphi}{2\pi} - \frac{1}{2}h_+\left(e^{i\theta}\right)\right| &\leq \int_{-\pi}^0 \left|h\left(e^{i(\theta-\varphi)}\right) - h_+\left(e^{i\theta}\right)\right| P_r(\varphi)\frac{d\varphi}{2\pi} \\ &\leq \int_{-\delta}^0 \varepsilon P_r(\varphi)\frac{d\varphi}{2\pi} + M \max_{-\pi\leq\varphi\leq-\delta} P_r(\varphi) \\ &\leq \frac{\varepsilon}{2} + M \max_{-\pi\leq\varphi\leq-\delta} P_r(\varphi). \end{aligned} \tag{30.2}$$

Note that we can consider $\tilde{h}\left(re^{i\theta}\right)$ by splitting the integral in two parts

$$\tilde{h}\left(re^{i\theta}\right) = \int_{-\pi}^0 h\left(re^{i(\theta-\varphi)}\right) P_r(\varphi)\frac{d\varphi}{2\pi} + \int_0^\pi h\left(re^{i(\theta-\varphi)}\right) P_r(\varphi)\frac{d\varphi}{2\pi}. \tag{30.3}$$

Putting together equations (30.1), (30.2), and (30.3), we see that

$$\begin{aligned}&\left|\tilde{h}\left(re^{i\theta}\right) - \frac{h_-\left(e^{i\theta}\right) + h_+\left(e^{i\theta}\right)}{2}\right| \\ &\qquad \leq \left|\int_0^\pi h\left(e^{i(\theta-\varphi)}\right) P_r(\varphi)\frac{d\varphi}{2\pi} - \frac{1}{2}h_-\left(e^{i\theta}\right)\right| + \left|\int_{-\pi}^0 h\left(e^{i(\theta-\varphi)}\right) P_r(\varphi)\frac{d\varphi}{2\pi} - \frac{1}{2}h_+\left(e^{i\theta}\right)\right| \\ &\qquad \leq \varepsilon + 2M \max_{\delta\leq|\varphi|\leq\pi} P_r(\varphi),\end{aligned}$$

and the value of the second summand in the last line tends to zero as $r \to 1$. Hence, for fixed θ, the values of $\tilde{h}\left(re^{i\theta}\right)$ cluster to within ε of $\frac{h_-\left(e^{i\theta}\right)+h_+\left(e^{i\theta}\right)}{2}$ as $r \to 1$, and this is for any $\varepsilon > 0$. This result yields the desired limit.

Problem 31 (Problem 7, Chapter X.1, p. 279-280)**.**
For each $t > 0$, define the kernel function $C_t(s)$ on the real line by

$$C_t(s) = \frac{t}{\pi}\frac{1}{s^2+t^2}, \qquad -\infty < s < \infty.$$

For $h(\xi)$ a bounded piecewise continuous function on the real line, define a function $\tilde{h}(s+it)$ on the open upper-half plane $\mathbb{H}$ by

$$\tilde{h}(s+it) = \int_{-\infty}^{\infty} C_t(s-\xi)h(\xi)d\xi, \qquad s+it \in \mathbb{H}.$$

(a) Sketch the graph of $C_t(s)$ for $t = 1$, $t = 0.1$, and $t = 0.01$.

(b) Show that $C_t(s) > 0$ and $\int_{-\infty}^{\infty} C_t(s)ds = 1$. (Thus $C_t(s)$ is a probability measure.)

(c) Show that for each $\delta > 0$, $\int_{\{|s|>\delta\}} C_t(s)ds \to 0$ as $t \to 0$. (Thus $C_t(s)ds$ is an approximate identity.)

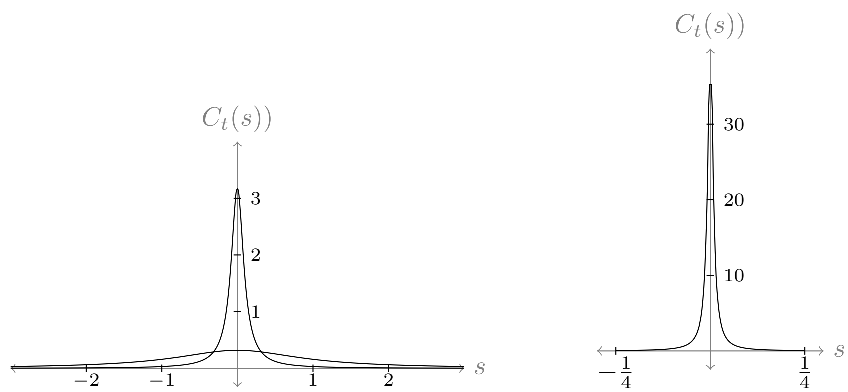
(d) Show that $\tilde{h}(s+it)$ is a bounded harmonic function in the upper half-plane.

(e) Show that $\tilde{h}(s+it) \to h(\xi_0)$ as $s+it \in \mathbb{H}$ tends to ξ_0, whenever $h(\xi)$ is continuous at ξ_0.

(*Remark*: The kernel function $C_t(s)$ is the **Poisson kernel** for the upper half-plane.)

Solution.

(a) See the following figure. The first plot shows $C_t(s)$ for $t = 1$ and $t = 0.1$, while the second shows $t = 0.01$ on a different scale.



(b) Since we have chosen $t > 0$, note that $\frac{t}{\pi}\frac{1}{s^2+t^2} > 0$ for any $s \in \mathbb{R}$. Note that $\frac{d}{ds}\arctan\frac{s}{t} = \frac{t}{s^2+t62}$, and thus

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{t}{\pi}\frac{1}{s^2+t^2}ds = \frac{1}{\pi}\lim_{r\to\infty}\left(\arctan\left(\frac{r}{t}\right) - \arctan\left(\frac{-r}{t}\right)\right) = \frac{1}{\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 1.$$

(c) Since the integrand $\frac{t}{\pi}\frac{1}{s^2+t^2}$ is symmetric, we have $\int_{|s|>\delta}\frac{t}{\pi}\frac{1}{s^2+t^2}ds = 2\int_{\delta}^{\infty}\frac{t}{\pi}\frac{1}{s^2+t^2}ds$, and

$$\frac{1}{\pi}\int_{\delta}^{\infty}\frac{t}{\pi}\frac{1}{s^2+t^2}ds = \frac{1}{\pi}\lim_{r\to\infty}\arctan\left(\frac{r}{t}\right) - \arctan\left(\frac{\delta}{t}\right) = \frac{1}{2} - \frac{1}{\pi}\arctan\left(\frac{\delta}{t}\right).$$

Taking the limit as $t \to 0$, we have $\arctan\left(\frac{\delta}{t}\right) \to \frac{\pi}{2}$, and thus

$$\int_{|s|>\delta}\frac{t}{\pi}\frac{1}{s^2+t^2}ds = 2\int_{\delta}^{\infty}\frac{t}{\pi}\frac{1}{s^2+t^2}ds = \frac{1}{2} - \frac{1}{\pi}\arctan\left(\frac{\delta}{t}\right) \to 0$$

as $t \to 0$ for any fixed $\delta > 0$.

(d) To check harmonicity, it suffices to check harmonicity of $C_t(s)$. Note that $C_t(s)$ is continuous for all $s \in \mathbb{R}$ when $t > 0$. Taking the first and second derivatives of $C_t(s)$ with respect to t, we find that

$$\frac{dC_t(s)}{dt} = \frac{1}{\pi}\frac{d}{dt}\left(\frac{t}{s^2+t^2}\right) = \frac{1}{\pi}\frac{s^2-t^2}{(s^2+t^2)^2}$$

and

$$\frac{d^2C_t(s)}{dt^2} = \frac{1}{\pi}\frac{d}{dt}\left(\frac{s^2-t^2}{(s^2+t^2)^2}\right) = \frac{1}{\pi}\frac{2t(t^2-3s^2)}{(s^2+t^2)^3}.$$

Taking the first and second derivatives of $C_t(s)$ with respect to s, we have

$$\frac{dC_t(s)}{ds} = \frac{1}{\pi}\frac{d}{ds}\left(\frac{t}{s^2+t^2}\right) = \frac{1}{\pi}\frac{-st}{(s^2+t^2)^2}$$

and

$$\frac{d^2C_t(s)}{ds^2} = -\frac{1}{\pi}\frac{d}{ds}\left(\frac{st}{(s^2+t^2)^2}\right) = -\frac{1}{\pi}\frac{2t(t^2-3s^2)}{(s^2+t^2)^3}.$$

Hence $\frac{d^2C_t(s)}{ds^2} = -\frac{d^2C_t(s)}{dt^2}$, so $C_t(s)$ is harmonic.

We now check harmonicity of $\tilde{h}(t)$. Since $C_t(s-\xi)$ is continuous and has continuous partial derivatives for any $s, \xi \in \mathbb{R}$ and $t > 0$, and $h(\xi)$ is continuous and does not depend on either t or s, we can pass differentiation under the integral sign. Hence

$$\frac{d^2}{ds^2}\tilde{h}(s+it) = \int_{-\infty}^{\infty}\frac{d^2}{ds^2}C_t(s-\xi)h(\xi)d\xi = -\int_{-\infty}^{\infty}\frac{d^2}{dt^2}C_t(s-\xi)h(\xi)d\xi = -\frac{d^2}{dt^2}\tilde{h}(s+it),$$

so $\tilde{h}(s+it)$ is harmonic on $s \in \mathbb{R}$ and $t > 0$.

Finally, to show that $\tilde{h}(s+it)$ is bounded, note that there is an $M > 0$ such that $|h(\xi)| \le M$ for all $\xi \in \mathbb{R}$, since $h(\xi)$ is bounded. Rewriting the integral and taking the absolute value yields

$$\left|\tilde{h}(s+it)\right| \le \int_{-\infty}^{\infty} C_t(\xi)\,|h(s-\xi)|\,d\xi \le M\underbrace{\int_{-\infty}^{\infty} C_t(\xi)d\xi}_{=1} = M,$$

so $\tilde{h}(s+it)$ is bounded on the upper half-plane.

(e) Since $h(\xi)$ is bounded, there is an $M > 0$ such that $|h(\xi)| \le M$ for all $\xi \in \mathbb{R}$. Let $\xi_0 \in \mathbb{R}$ be a point on the real line such that $h(\xi)$ is continuous at ξ_0. Let $\varepsilon > 0$, then there is a $\delta > 0$ such that $|h(\xi) - h(\xi_0)| < \varepsilon$ whenever $|\xi - \xi_0| < \delta$. For any $s + it$ in the upper half-plane such that $|s + it - \xi_0| < \delta$, we have $\sqrt{(s-\xi_0)^2 + t^2} < \delta$ and thus $|h(\xi) - h(\xi_0)| < \varepsilon$. To show the desired limit, we can split the integral and see that

$$\begin{aligned}
\left|\tilde{h}(s+it) - h(\xi_0)\right| &= \left|\int_{-\infty}^{\infty} C_t(\xi)\left(h(s-\xi) - h(x_0)\right)d\xi\right| \\
&\le \int_{-\delta}^{\delta} C_t(\xi)\,|h(s-\xi) - h(x_0)|\,d\xi + 2M\int_{\delta}^{\infty} C_t(\xi)d\xi \\
&\le \varepsilon\underbrace{\int_{-\delta}^{\delta} C_t(\xi)d\xi}_{<1} + 2M\int_{\delta}^{\infty} C_t(\xi)d\xi \\
&\le \varepsilon + +2M\int_{\delta}^{\infty} C_t(\xi)d\xi.
\end{aligned}$$

The value of the second summand in the last line tends to zero as $t \to 0$. Hence, for fixed ξ_0, the values of $\tilde{h}(s+it)$ cluster to within ε of $h(\xi_0)$ as $s + it \to \xi_0$, and this holds for any $\varepsilon > 0$.

Problem 32 (Problem 2, Chapter X.3, p. 287)**.**
Show that a reflection in a circle maps circles in the plane to circles.

Solution. Let $C_0 = \{|z - z_0| = r\}$ be a circle in the plane for some $r > 0$. For a point z_1 on the circle, denote the point on the opposite side of the circle as $z_2 = 2z_0 - z_1$. Define a fractional linear transformation $f(z)$ by

$$z_0 \mapsto 1, \qquad z_1 \mapsto 0, \qquad \text{and} \qquad z_2 \mapsto \infty.$$

Then $f^{-1}(\zeta)$ conformally maps $\mathbb{C}$ onto the domain $\mathbb{C}\backslash z_2$, and maps $\mathbb{R}$ onto $C\backslash\{z_2\}$. So C is an analytic arc, and we can define a reflection across C as $z^* = f^{-1}(\overline{f(z)})$. This reflection maps C to itself. For any other circle C' on the plane, this reflection will map it to another circle. To show this, we consider the following two cases: C' intersects C at z_2 or C' does not intersect C at z_2.

- If C' does not intersect C at z_2, then f maps C' to a circle in $\mathbb{C}$ and the reflection $\zeta \mapsto \overline{\zeta}$ maps $f(C')$ to that circle reflected across the real line. Finally, f^{-1} maps that circle back to another circle in $\mathbb{C}$.

- If C' intersects C at z_2, then f maps C' to a straight line and $\zeta \mapsto \overline{\zeta}$ maps the line $f(C')$ to that line reflected across the real line. Finally, f^{-1} maps that line back to another circle that intersects C at z_2.

Hence the composition $C' \mapsto f(C') \mapsto \overline{f(C')} \mapsto f^{-1}(\overline{f(C')})$ maps circles to circles.