Quiz 1

MATH 621

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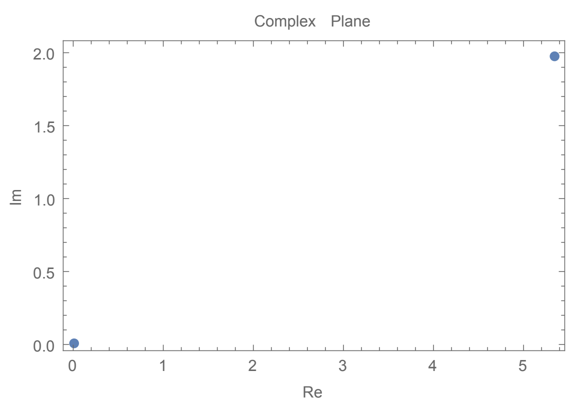
Problem 1 (Problem 1 (d), Chapter I.7, p. 27)**.**
Find all possible values and plot $(1+i\sqrt{3})^{(1-i)}$.

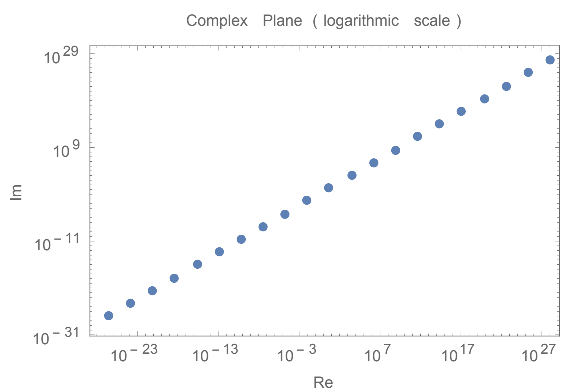
Solution. Set $z = 1+i\sqrt{3}$, which is $z = 2e^{i\frac{\pi}{3}}$ in polar form. We have $\operatorname{Log} z = \ln 2 + i\left(\frac{\pi}{3} + 2\pi n\right)$ for all $n \in \mathbb{Z}$. The expression in question becomes

$$z^{(1-i)} = e^{(1-i)\operatorname{Log} z} = e^{(1-i)\left(\ln 2 + i\frac{\pi}{3} + i2\pi n\right)}$$
$$= e^{\ln 2 + \frac{\pi}{3} + 2\pi n} e^{i\left(\frac{\pi}{3} - \ln 2\right)}.$$

See the figure below for the plot.



(a) Points with $n = 0$ and $n = -1$ shown on the complex plane.



(b) Points with $n = -10, \ldots, 10$ shown on the first quadrant of the complex plane with logarithmic axes scales.

Figure 1: Plots of the values of $(1+i\sqrt{3})^{(1-i)}$ on the complex plane.

Problem 2 (Problem 7, Chapter I.8, p. 32).
Set $w = \cos z$ and $\zeta = e^{iz}$. Show that $\zeta = w \pm \sqrt{w^2 - 1}$. Show that

$$\cos^{-1} w = -i \log \left[w \pm \sqrt{w^2 - 1} \right]$$

where both sides of the identity are to be interpreted as subsets of the complex plane.

Solution. With $\zeta = e^{iz}$, we have

$$w = \cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{\zeta + \zeta^{-1}}{2}$$

or $\zeta^2 - 2w\zeta + 1 = 0$. Solving the quadratic equation for ζ yields

$$\zeta = w \pm \sqrt{w^2 - 1}.$$

Taking the logarithm of both sides, we have $\operatorname{Log} \zeta = \operatorname{Log} e^{iz} = iz + i2\pi m$ and thus

$$iz + i2\pi m = \log \left[w \pm \sqrt{w^2 - 1} \right] + i2\pi n.$$

Multiplying by $-i$ and setting $k = n - m$ yields

$$z = -i \log \left[w \pm \sqrt{w^2 - 1} \right] + 2\pi k.$$

Replacing z with $\cos^{-1} w$ yields the desired result.

Problem 3 (Problem 5, Chapter II.1, p. 40)**.**
Show that the sequence

$$b_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n, \qquad n \geq 1,$$

is decreasing, while the sequence $a_n = b_n - \frac{1}{n}$ is increasing. Show that the sequences both converge to the same limit γ. Show that $\frac{1}{2} < \gamma < \frac{3}{5}$.

Solution. We first show that b_n is decreasing. Indeed, we have

$$\begin{aligned} b_{n+1} - b_n &= \frac{1}{n+1} - \log(n+1) + \log n \\ &= \frac{1}{n+1} + \log\left(1 - \frac{1}{n+1}\right) \\ &< 0 \end{aligned}$$

since the logarithm function is concave down and thus $\log(1+x) < x$ for all $x > -1$. Note that b_n is bounded below by 0, since

$$b_n = \sum_{k=1}^{n} \frac{1}{k} - \log n > \underbrace{\int_1^{n+1} \frac{1}{t} dt}_{\log(n+1)} - \log n = \log(n+1) - \log n = \log\left(1 + \frac{1}{n}\right) > 0.$$

Hence b_n converges since it is decreasing and bounded below.

We now show that a_n is increasing. Indeed,

$$\begin{aligned} a_{n+1} - a_n &= \frac{1}{n} - \log(n+1) + \log n \\ &= \frac{1}{n} - \log\left(1 + \frac{1}{n}\right) \\ &> 0 \end{aligned}$$

since $\log(1+x) < x$ for all $x > -1$. Note that a_n is bounded above by 1, since

$$a_n = 1 + \sum_{k=2}^{n-1} \frac{1}{k} - \log n > 1 + \underbrace{\int_1^{n-1} \frac{1}{t} dt}_{\log(n-1)} - \log n = 1 + \log(n-1) - \log n = 1 + \log\left(1 - \frac{1}{n}\right) < 1.$$

Hence a_n converges since it is increasing and bounded above.

Note that a_n and b_n converge to the same limit, since both sequences converge and

$$\lim_{n\to\infty} (b_n - a_n) = \lim_{n\to\infty} \frac{1}{n} = 0.$$

To show that $\frac{1}{2} < \gamma < \frac{3}{5}$, note that

$$a_7 - \frac{1}{2} = \frac{39}{20} - \log 7 = 0.00409\cdots > 0 \qquad \text{and} \qquad \frac{3}{5} - b_{22} = 0.000229\cdots > 0.$$

Problem 4 (Problem 5, Chapter II.2, p. 46).
Show that if f is analytic on D, then $g(z) = \overline{f(\bar{z})}$ is analytic on the reflected domain $D^* = \{\bar{z} \mid z \in D\}$, and $g'(z) = \overline{f'(\bar{z})}$.

Solution. From the definition of the derivative, we have

$$\begin{aligned}
\lim_{\Delta z \to 0} \frac{g(z+\Delta z) - g(z)}{\Delta z} &= \lim_{\Delta z \to 0} \frac{\overline{f(\overline{z+\Delta z})} - \overline{f(\bar{z})}}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{\overline{f(\bar{z} + \overline{\Delta z}) - f(\bar{z})}}{\Delta z} \\
&= \left(\lim_{\Delta z \to 0} \frac{f(\bar{z} + \overline{\Delta z}) - f(\bar{z})}{\overline{\Delta z}} \right)^* \\
&= \left(\lim_{\overline{\Delta z} \to 0} \frac{f(\bar{z} + \overline{\Delta z}) - f(\bar{z})}{\overline{\Delta z}} \right)^* \\
&= \overline{f'(\bar{z})}.
\end{aligned}$$

Hence the derivative $g'(z)$ exists for all $z \in D^*$, so g is analytic on D^*.

Problem 5 (Problem 4, Chapter II.3, p. 50)**.**
Show that if f is analytic on a domain D and $|f|$ is constant, then f is constant.

Solution. If $|f| = 0$, then there is nothing to show since in this case f is constant at zero. So suppose that $|f| \neq 0$. Hence f is nonzero on all of D, and we can write $\bar{f} = \frac{|f|^2}{f}$. Since $|f|$ is constant, it is differentiable with $(|f|)' = 0$. So we can use the quotient rule to take the derivative

$$(\bar{f})' = \left(\frac{|f|^2}{f}\right)' = -|f|^2 \frac{f'}{f^2}$$

so f' is also differentiable on all of D, and thus analytic. From the problem in class (see Problem 3, Chapter II.3 p. 50), if f and $\bar{f}$ are analytic, then f is constant.