Quiz 2

MATH 621

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Problem 1 (Problem 4, Chapter II.7, p. 68)**.**
Consider the fractional linear transformation that maps -1 to $-i$, 1 to $2i$, and i to 0. Determine the image of the unit circle $\{|z| = 1\}$, the image of the open unit disk $\{|z| \leq 1\}$, and the interval $[-1, +1]$ on the real axis. Illustrate with a sketch.

Solution. The linear transformation will have the form $f(z) = \frac{az+b}{cz+d}$. Since $f(i) = 0$, we have $ai + b = 0$, so we may take $a = 1$ and $b = -i$. From the other two conditions, we have

$$\frac{-1-i}{-c+d} = -i \quad \text{and} \quad \frac{i-i}{c+d} = 2i.$$

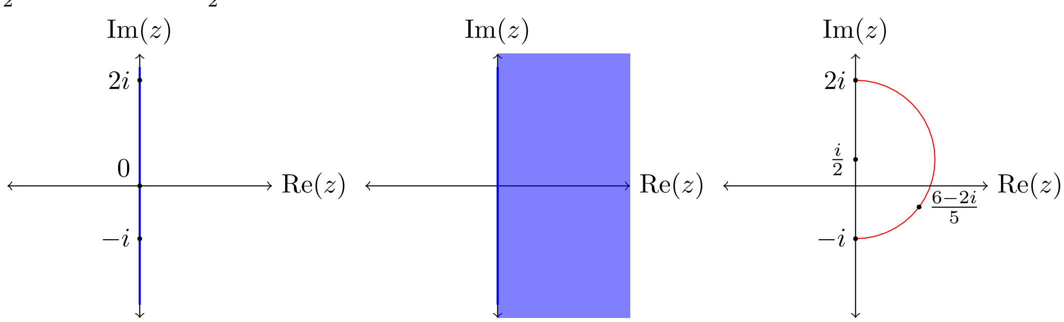
Solving this for c and d yields $c = \frac{1}{4}(i-3)$ and $d = \frac{1}{4}(1-3i)$. Hence

$$f(z) = 4\frac{z-i}{(i-3)z+1-3i}.$$

Note that f maps three different points on the unit circle to three points on the imaginary line, so the rest of the unit circle will also be mapped to this line. The image of the unit circle under f is thus $\{ai \,|\, a \in \mathbb{R}\}$, i.e. the imaginary line.

Note that, since the unit circle gets mapped to the imaginary axis, the unit disk will either be mapped to the right or left half-plane. The point $z = 0$ gets mapped to $\frac{-4i}{1-3i} = \frac{6}{5} - i\frac{2}{5}$, which is in the right half-plane. So image of the unit disk under f is the right half-plane.

The interval $I = [-1, +1]$ will get mapped to the arc of a circle with end-points $2i$ and $-i$ and passes through $f(0) = \frac{6}{5} - i\frac{2}{5}$. Since the interval I is part of the real line, which comprises a 'circle' through infinity, and this is perpendicular to the unit circle, the image of the real line must be perpendicular to the image of the unit circle wherever they intersect. Hence, the image of I consists of a half circle. This circle has center at $\frac{i}{2}$ and has radius $\frac{3}{2}$.



Problem 2 (Problem 4, Chapter III.3, p.84)**.**
Let $u(z)$ be harmonic on the annulus $A = \{a < |z| < b\}$. Show that there is a constant C such that $u(z) - C \log |z|$ has a harmonic conjugate on the annulus. Show that C is given by

$$C = \frac{r}{2\pi} \int_0^{2\pi} \frac{\partial u}{\partial r} \left(re^{i\theta}\right) d\theta,$$

where r is any fixed radius, $a < r < b$.

Solution (Following the hints in the back of the book)**.**
Let $A = \{z \,|\, a < |z| < b\}$. Note that from problem 3, in Chapter III.3, p. 84, any harmonic function on a slit annulus has a harmonic conjugate. Then u has a harmonic conjugate, say v_1, on the the slit annulus $A_1 = A \setminus (-b, -a)$. Similarly, u has a harmonic conjugate, say v_2, on the slit annulus $A_2 = A \setminus (a, b)$. Since $v1 - v2$ must be constant above and below the slit, the value of $v_1 - v_2$ jumps by a constant across the slit from the lower to the upper half-plane. Call this jumping constant c. The function $\operatorname{Arg} z$ also jumps by a constant across the slit. In particular, $\operatorname{Arg} z$ jumps by 2π across the slit from the lower to the upper half-plane. By choosing $C = \frac{c}{2\pi}$, the function $v_1 - C \operatorname{Arg} z$ is continuous across the slit $(-b, -a)$.

Note that two functions f and g are harmonic conjugates if they satisfy the Cauchy-Riemann equations. Recall the polar form of the Cauchy-Riemann equations:

$$\frac{\partial f}{\partial r} = \frac{1}{r} \frac{\partial g}{\partial \theta} \quad \text{and} \quad \frac{\partial f}{\partial \theta} = -r \frac{\partial g}{\partial r}.$$

Then $f(z) = u(z) - C \log |z|$ has a harmonic conjugate $g(z) = v_1(z) - C \operatorname{Arg} z$ on the entire annulus. Indeed, note that

$$\frac{\partial f}{\partial r} = \frac{\partial}{\partial r} \left(u(z) - C \log r\right) = \frac{\partial u}{\partial r} - \frac{C}{r}.$$

Since v_1 is the harmonic conjugate to u, we have $\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v_1}{\partial \theta}$ and thus

$$\frac{\partial f}{\partial r} = \frac{\partial u}{\partial r} - \frac{C}{r} = \frac{1}{r} \frac{\partial v_1}{\partial \theta} - \frac{C}{r} \frac{\partial}{\partial \theta}(\operatorname{Arg} z) = \frac{1}{r} \frac{\partial}{\partial \theta}(v_1 + C \operatorname{Arg} z) = \frac{1}{r} \frac{\partial g}{\partial \theta}$$

so the first Cauchy-Riemann equation is satisfied for f and g. Similarly, since $\frac{\partial u}{\partial \theta} = -r \frac{\partial v_1}{\partial r}$, we have

$$\frac{\partial f}{\partial \theta} = \frac{\partial}{\partial \theta} \left(u + C \log r\right) = \frac{\partial u}{\partial \theta} = -r \frac{\partial v_1}{\partial r} = -r \frac{\partial}{\partial r}(v_1 + C \operatorname{Arg} z) = -r \frac{\partial g}{\partial r}$$

so the second Cauchy-Riemann equation is satisfied for f and g. Hence f and g are harmonic conjugates.

It remains to calculate the constant C. Note that recall that $C = \frac{c}{2\pi}$, where c is the jumping constant of v_1 from the lower to the upper half-plane. This will be the same as the jumping constant of v_2 from the upper to the lower half-plane. Since this does not depend on r, for any $r \in (a, b)$ we have

$$\begin{aligned} c &= \int_0^{2\pi} \frac{\partial v_2}{\partial \theta} d\theta \\ &= r \int_0^{2\pi} \frac{\partial u}{\partial r} d\theta \end{aligned}$$

where we make use of the Cauchy-Riemann relation $\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v_2}{\partial \theta}$ since v_2 is a harmonic conjugate to u.

Problem 3 (Problem 3, Chapter III.4, p. 87)**.**
A function $f(t)$ on an interval $I = (a, b)$ has the *mean value property* if

$$f\left(\frac{s+t}{2}\right) = \frac{f(s)+f(t)}{2}, \quad s, t \in I.$$

Show that any affine funcion $f(t) = At+B$ had the mean value property. Show that any continuous function on I with the mean value property is affine.

Solution. Let f be affine. Then $f(t) = At + B$ for some $A, B \in \mathbb{R}$ and thus

$$f\left(\frac{s+t}{2}\right) = A\frac{s+t}{2} + B = \frac{As+B+At+B}{2} = \frac{f(s)+f(t)}{2}$$

so f has the mean value property.

Now suppose that f is continuous and has the mean value property. We claim that f is linear. That is,

$$f(pt + (1-p)s) = pf(t) + (1-p)f(s) \qquad \text{for all } s, t \in I \text{ and } p \in [0, 1]. \tag{1}$$

To prove this, we make use of the following lemma.

Lemma. *For all $t, s \in I$ and for all nonnegative integers $k \in \mathbb{N}$, we have*

$$f\left(\frac{m}{2^k}t + \left(1 - \frac{m}{2^k}\right)s\right) = \frac{m}{2^k}f(t) + \left(1 - \frac{m}{2^k}\right)f(s) \quad \text{for all } m \in \{0, 1, \ldots, 2^k\}. \tag{2}$$

Proof (*of lemma*). Let $t, s \in I$. We proceed by induction on k. If $k = 0$, then there is nothing to prove. Suppose that there is some $\ell \in \mathbb{N}$ such that (2) holds for $k = \ell$. Let $m \in \{0, 1, \ldots, 2^{\ell+1}\}$. If $m = 2n$ for some $n \in \{0, 1, \ldots, 2^\ell\}$ then this reduces to the $k = \ell$ case, so suppose that $m = 2n+1$ for some $n \in \{0, 1, \ldots, 2^\ell\}$. By the mean value property of f, we have

$$\begin{aligned}
f\left(\frac{m}{2^{\ell+1}}t + \left(1 - \frac{m}{2^{\ell+1}}\right)s\right) &= f\left(\frac{2n+1}{2 \cdot 2^\ell}t + \left(1 - \frac{2n+1}{2 \cdot 2^\ell}\right)s\right) \\
&= f\left(\frac{\left[\frac{n}{2^\ell}t + \left(1 - \frac{n}{2^\ell}\right)s\right] + \left[\frac{n+1}{2^\ell}t + \left(1 - \frac{n+1}{2^\ell}\right)s\right]}{2}\right) \\
&= \frac{f\left(\frac{n}{2^\ell}t + \left(1 - \frac{n}{2^\ell}\right)s\right) + f\left(\frac{n+1}{2^\ell}t + \left(1 - \frac{n+1}{2^\ell}\right)s\right)}{2} \\
&= \frac{\frac{n}{2^\ell}f(t) + \left(1 - \frac{n}{2^\ell}\right)f(s) + \frac{n+1}{2^\ell}f(t) + \left(1 - \frac{n+1}{2^\ell}\right)f(s)}{2} \\
&= \frac{2n+1}{2 \cdot 2^\ell}f(t) + \left(1 - \frac{2n+1}{2 \cdot 2^\ell}\right)f(s) \\
&= \frac{m}{2^{\ell+1}}f(t) + \left(1 - \frac{m}{2^{\ell+1}}\right)f(s),
\end{aligned}$$

where the third line is due to the mean value property of f and the fourth line follows from the induction hypothesis. Hence, the claim holds for $k = \ell + 1$. By induction, the claim holds for all $k \in \mathbb{N}$. □

By continuity of f, it follows from the Lemma that $f(pt + (1-p)s) = pf(t) + (1-p)f(s)$ for al $p \in [0, 1]$, since for all such $p \in [0, 1]$ there is a sequence $\frac{m_k}{2^k}$ such that $\frac{m_k}{2^k} \to p$ as $k \to \infty$.

It now remains to show that f is affine. Let $s \in I$ and let $t \in I$ with $s \neq t$. By the property in equation (1), we have

$$f'(s) = \lim_{p \to 0} \frac{f(s + p(t-s)) - f(s)}{p(t-s)} = \lim_{p \to 0} \frac{f(s) + p(f(t) - f(s)) - f(s)}{p(t-s)} = \frac{f(t) - f(s)}{t - s}.$$

But this holds for any $t \neq s$, so it cannot depend on t. Analogously, $f'(t) = \frac{f(t)-f(s)}{t-s}$, and so $f'(t) = f'(s)$. But s and t were chosen arbitrarily in (a, b) Hence $f'(x)$ is constant on the interval (a, b). Call this constant $f'(x) = A$. Then $f(x) = Ax + B$, where B is the constant $B = f(x) - Ax$.

Problem 4 (Problem 10, Chapter III.5, p. 89)**.**
Let D be a bounded domain, and let $z_0 \in \partial D$. Let $u(z)$ be a harmonic function on D that extends continuously to each boundary point of D except possibly z_0. Show that $u(z)$ is bounded below on D, and that $u(z) \geq 0$ for all $z \in \partial D$, $z \neq z_0$. Show that $u(z) \geq 0$ on D.

Solution. (Following the hints in the book to solve problem 9, Chapter III.5, p. 89). Let $z_1 \in \partial D$ be a boundary point with $z_0 \neq z_1$. Consider the fractional linear transformation defined by $f(z) = \frac{z_1 - z}{z - z_0}$. This maps z_0 to ∞, and D to an unbounded domain $D' = f(D)$. Note that $0 \in \partial D'$, since $z_1 \in \partial D$ and thus $0 = f(z_1) \in \partial D'$. Now we can consider the function $v\colon D' \to \mathbb{R}$ defined by

$$v(w) = u(f^{-1}(w)) \quad \text{ or } \quad u(z) = v(f(z)).$$

Since D' is an unbounded domain and v can be extended continuously to every point on its boundary, we may make use of the result of problem 9. Note that $v(w) = u(f^{-1}(w)) \geq 0$ for $w \in \partial D'$. So, by problem 9, $u(f^{-1}(w)) = v(w) \geq 0$ on D', and thus $u(z) \geq 0$ on D.

It now remains to prove the statement of problem 9. Let us recall and prove the statement of this problem:

Proposition (Problem 9, Chapter III.5, p. 89)**.** *Let D be an unbounded domain, $D \neq \mathbb{C}$, with $0 \in \partial D$. Let $v(w)$ be a harmonic function on D that extends continuously to all points on the boundary ∂D. Suppose that $v(w)$ is bounded below on D, and that $v(w) \geq 0$ on ∂D. Then $v(w) \geq 0$ on D.*

Proof. Let $\varepsilon > 0$. Since v extends continuously to 0 and $v(0) \geq 0$, there is a $\delta > 0$ such that $|v(w) - v(0)| < \varepsilon$ for any $w \in D$ with $|w| < \delta$. But $v(0) \geq 0$ by assumption since $0 \in \partial D$. Hence $v(w) > -\varepsilon$ for all $w \in D$ with $0 < |w| < \delta$. Let $\rho > 0$ such that $v(w) + \rho \log \delta \geq -\varepsilon$ for $|w| = \delta$, or equivalently

$$-v(w) - \rho \log |w| \leq \varepsilon \qquad \text{for } w \in D \quad \text{with } |w| = \delta$$

Since $v(w)$ is bounded below on D, there is an $M \in \mathbb{R}$ such that $v(w) \geq -M$ for all $w \in D$. Then there is an R large enough such that $\rho \log |w| > M$ for all $|w| > R$, and thus $v(w) + \rho \log |w| > 0$ for all $|w| > R$. Equivalently

$$-v(w) - \rho \log |w| < 0 \qquad \text{for } w \in D \quad \text{with } |w| > R$$

Applying the maximum principle to the function $-v(w) - \rho \log |w|$ on the region $D \cap \{\delta < |w| < R\}$, we have that $v(w) + \rho \log |w| > 0$. Since we may choose R arbitrarily large, this applies for all $w \in D \cap \{0 < |w|\}$. Since ε and ρ were chosen arbitrarily, we may take $\varepsilon \to 0$ and $\rho \to 0$ to obtain $v(w) \geq 0$ for all $w \in D$. □

Problem 5 (Problem 9, Chapter IV.1, p. 107)**.**
Suppose $h(z)$ is a continuous function on a curve γ. Show that

$$H(w) = \int_\gamma \frac{h(z)}{z-w}dz, \qquad w \in \mathbb{C} \setminus \gamma,$$

is analytic on the complement of γ and find $H'(w)$.

Solution. To show analyticity, we differentiate. Note that for Δw such that $w + \Delta w \in \mathbb{C} \setminus \gamma$ we have

$$\begin{aligned}
\frac{H(w+\Delta w) - H(w)}{\Delta w} &= \frac{\displaystyle\int_\gamma \left(\frac{h(z)}{z-w-\Delta w} - \frac{h(z)}{z-w}\right) dz}{\Delta w} \\
&= \frac{\displaystyle\int_\gamma h(z)\left(\frac{1}{z-w-\Delta w} - \frac{1}{z-w}\right) dz}{\Delta w} \\
&= \int_\gamma h(z)\left(\frac{\Delta w}{\Delta w(z-w-\Delta w)(z-w)}\right) dz \\
&= \int_\gamma \frac{h(z)}{(z-w-\Delta w)(z-w)}dz.
\end{aligned}$$

Note that $h(z)$ is bounded on γ, since γ is bounded and $h(z)$ is analytic on γ. So there is some $M > 0$ such that $|h(z)| < M$ for all $z \in \gamma$. For a fixed $w \in \mathbb{C} \setminus \gamma$, let $d = \operatorname{dist}(w, \gamma)$. Then

$$\begin{aligned}
\left|\int_\gamma \frac{h(z)}{(z-w-\Delta w)(z-w)}dz - \int_\gamma \frac{h(z)}{(z-w)^2}dz\right| &= \left|\int_\gamma \frac{h(z)}{(z-w)^2}\frac{\Delta w}{(z-w-\Delta w)}dz\right| \\
&\le \int \frac{|h(z)||\Delta w|}{|z-w|^2\big(|z-w| - |\Delta w|\big)}dz \\
&\le \int \frac{|h(z)||\Delta w|}{d^2\big(d - |\Delta w|\big)}dz \\
&\le \operatorname{Length}(\gamma)\frac{M\Delta w}{d^2(d-\Delta w)},
\end{aligned}$$

which goes to zero as $\Delta w \to 0$. Hence, for any $w \in \mathbb{C} \setminus \gamma$, we have

$$\lim_{\Delta w \to 0} \int_\gamma \frac{h(z)}{(z-w-\Delta w)(z-w)}dz = \int_\gamma \frac{h(z)}{(z-w)^2}dz.$$

Therefore $H(w)$ is analytic on $\mathbb{C} \setminus \gamma$ with derivative

$$H'(w) = \int_\gamma \frac{h(z)}{(z-w)^2}dz.$$