Quiz 3

MATH 621

Mark Girard

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Problem 1 (Problem 4, Chapter IV.5, p. 119)**.**
Suppose that f is an entire function such that $f(z)/z^n$ is bounded for $|z| \geq R$. Show that $f(z)$ is a polynomial of degree at most n. What can be said if $f(z)/z^n$ is bounded on the entire complex plane?

Solution. By assumption, there is an $R > 0$ and $M \in \mathbb{R}$ such that $\left|\frac{f(z)}{z^n}\right| \leq M$ for all z with $|z| > R$. Let $z_0 \in \mathbb{C}$ be arbitrary. Then $\left|\frac{f(z)}{z^n}\right| \leq M$ for all z such that $|z - z_0| > |z_0| + R$. For any r with $r \geq |z_0| + R$, and $|z - z_0| = r$, we have

$$|f(z)| \leq M\,|z - z_0|^n = Mr^n.$$

By the theorem of Cauchy estimates, this implies that

$$\left|f^{(m)}(z_0)\right| \leq \frac{m!}{r^m} Mr^n = m! M r^{n-m}.$$

If $m > n$, taking the limit as $r \to \infty$, we see that $f^{(m)}(z_0) = 0$. Hence, all of the derivatives higher than the n^{th} one will vanish on the entire complex plane. So $f(z)$ must be a polynomial of degree at most n.

Finally, if $\frac{f(z)}{z^n}$ is bounded on the entire complex plane, by the above result f must be a polynomial of degree at most n, so

$$f(z) = c_0 + c_1 z + \cdots + c_n z^n$$

for some $c_0, \ldots, c_n \in \mathbb{C}$. But for $\frac{f(z)}{z^n}$ to be bounded near zero requires that $c_0 = \cdots = c_{n-1} = 0$. Hence $f(z) = c_n z^n$.

Problem 2 (Problem 5, Chapter V.2, p. 137).

For which real numbers x does $\sum \frac{1}{k} \frac{x^k}{1+x^{2k}}$ converge?

Solution. The series converges for all $x \in \mathbb{R} \setminus \{1\}$, but diverges at $x = 1$.

Proof. Let $S(x) = \sum \frac{1}{k} \frac{x^k}{1+x^{2k}}$ and $T(x) = \sum \left| \frac{1}{k} \frac{x^k}{1+x^{2k}} \right|$.

- If $x = 0$, then $S(0) = 0$ which converges.
- If $x = 1$, then $S(1) = \sum \frac{1}{2k}$ which diverges.
- If $x = -1$, then $S(-1) = \sum \frac{(-1)^k}{2k}$ which converges by the alternating series test.
- If $0 < |x| < 1$, then
$$T(x) = \sum \left| \frac{1}{k} \frac{x^k}{1+x^{2k}} \right| = \sum \frac{1}{k} \frac{|x|^k}{1+|x|^{2k}} < \sum |x|^k$$
which converges, so $T(x)$ converges by comparison.
- If $|x| > 1$, then
$$T(x) = \sum \left| \frac{1}{k} \frac{x^k}{1+x^{2k}} \right| = \sum \frac{1}{k} \frac{|x|^k}{1+|x|^{2k}} = \sum \frac{1}{k} \frac{1}{|x|^{-k}+|x|^k} < \sum \frac{1}{|x|^k}$$
which converges, so $T(x)$ converges by comparison.

Since $T(x)$ converges whenever $|x| \neq 1$, we have that $S(x)$ converges absolutely for all $|x| \neq 1$. But $S(x)$ converges when $x = -1$ and diverges when $x = +1$. $\square$

Problem 3 (Problem 1 (i), Chapter V.3, p. 143)**.**
Find the radius of convergence of the following power series:

$$\sum_{k=1} \frac{k! z^k}{k^k}$$

Solution. The radius of convergence for the series is $R = e$.

Proof. Let $a_k = \frac{k!}{k^k}$. Then we have

$$\begin{aligned}
\left|\frac{a_k}{a_{k+1}}\right| &= \frac{k!}{(k+1)!}\frac{(k+1)^{k+1}}{k^k} \\
&= \frac{1}{k+1}\left(\frac{k+1}{k}\right)^k (k+1) \\
&= \left(1+\frac{1}{k}\right)^k.
\end{aligned}$$

The limit of this as $k \to \infty$ exists. From the ratio test, we have

$$\begin{aligned}
R &= \lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right| \\
&= \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k \\
&= e
\end{aligned}$$

as desired. □

Problem 4 (Problem 4, Chapter V.5, p. 151).
Let E be a bounded subset of the complex plane $\mathbb{C}$ over which area integrals can be defined, and set

$$f(w) = \iint_E \frac{dx\,dy}{w-z}, \qquad w \in \mathbb{C} \setminus E,$$

where $z = x + iy$. Show that $f(w)$ is analytic at ∞, and find a formula for the coefficients of the power series of $f(w)$ at ∞ in decending powers of w. (Hint: use a geometric series expansion.)

Solution.
Recall that a function $f(w)$ is said to be analytic at infinity if the function $f(\frac{1}{u})$ is analytic at $u = 0$.

Since E is bounded, there is an $M \in \mathbb{R}$ such that $|z| \leq M$ for all $z \in E$. Let $u \in \mathbb{C}$ be close enough to zero such that $g(z) = \frac{1}{|u|} > M$ and thus $|uz| < 1$. Note that

$$\frac{1}{\frac{1}{u} - z} = \frac{u}{1 - uz} = u \sum_{k=0}^{\infty} (uz)^k$$

which is convergent since $|uz| < 1$. Note that the sequence of functions $g_n(z) = u \sum_{k=0}^{n} (uz)^k$ converges uniformly to $g(z) = \frac{1}{\frac{1}{u} - z}$ on E whenever $\frac{1}{|u|} > M$. So we can exchange the limits such that

$$\iint_E \lim_{n\to\infty} g_n(z)\,dxdy = \lim_{n\to\infty} \iint_E g_n(z)\,dxdy,$$

since the set E is bounded. Then

$$\begin{aligned} f\left(\frac{1}{u}\right) &= \iint_E \frac{dx\,dy}{\frac{1}{u} - z} \\ &= \sum_{k=0}^{\infty} u \iint_E (uz)^k dx\,dy \\ &= \sum_{k=0}^{\infty} u^{k+1} \iint_E z^k dx\,dy \end{aligned}$$

and this is convergent for $|u| < \frac{1}{M}$, so $f(\frac{1}{u})$ is analytic at $u = 0$. Hence $f(w)$ is analytic at infinity. In decending powers of w, the power series for $f(w)$ at infinity may be given by

$$f(w) = \sum_{k=1}^{\infty} w^{-k} b_k \qquad \text{where } b_k = \iint_E z^{k-1} dx\,dy.$$

Problem 5 (Problem 4, Chapter V.6, p. 154)**.**
Define the *Bernoulli numbers* B_n by

$$\frac{z}{2}\cot(z/2) = 1 - B_1\frac{z^2}{2!} - B_2\frac{z^4}{4!} - B_3\frac{z^6}{6!} - \cdots .$$

Explain why there are no odd terms in the series. What is the radius of convergence of the series? Find the first five Bernoulli numbers.

Solution. Note that the function in question, given by $f(\frac{z}{2}) = \frac{z}{2}\cot(z/2)$, is even. Indeed, the functions $\cos x$ and $\frac{\sin x}{x}$ are even, so their ratio $\cos x \frac{x}{\sin x} = f(x)$ will also be even. Hence, all of the exponents in the power series expansion will be even.

Claim. *The radius of convergence of the power series of $f(\frac{z}{2})$ given above is $R = 2\pi$.*

Proof. Since the power series for $\frac{\sin x}{x}$ is convergent for all x, the power series for $\frac{x}{\sin x}$ will be divergent when $\frac{\sin x}{x} = 0$. The first zero of this function occurs at $|x| = \pi$, so the power series for $\frac{x}{\sin x}$ will have a radius of convergence of π. Since the power series expansion for cos converges for all x, and $\cos(\pi) \neq 0$, this implies that the power series expansion for $x\cot x = \frac{x}{\sin x}\cos x$ will also have a radius of convergence $R = \pi$. Hence $f(\frac{z}{2})$ will have a radius of convergence of 2π. □

We now compute the first five Bernoulli numbers by computing the series expansion of $\frac{z}{2}\cot\left(\frac{z}{2}\right)$. We first find the power series expansion for $\frac{x}{\sin x}$. Up to terms of the order x^{10}, this is

$$\begin{aligned}
\frac{x}{\sin x} &= \frac{x}{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \mathcal{O}(x^{13})} \\
&= \frac{1}{1 - \left(\frac{1}{3!}x^2 - \frac{1}{5!}x^4 + \frac{1}{7!}x^6 + \frac{1}{9!}x^8 - \frac{1}{11!}x^{10} + \mathcal{O}(x^{12})\right)} \\
&= \sum_{k=0}^{\infty}\left(\frac{1}{3!}x^2 - \frac{1}{5!}x^4 + \frac{1}{7!}x^6 + \frac{1}{9!}x^8 - \frac{1}{11!}x^{10} + \mathcal{O}(x^{12})\right)^k \\
&= 1 + \left(\frac{1}{3!}x^2 - \frac{1}{5!}x^4 + \frac{1}{7!}x^6 + \frac{1}{9!}x^8 - \frac{1}{11!}x^{10}\right) + \left(\frac{1}{3!}x^2 - \frac{1}{5!}x^4 + \frac{1}{7!}x^6 + \frac{1}{9!}x^8 + \mathcal{O}(x^{10})\right)^2 \\
&\quad + \left(\frac{1}{3!}x^2 - \frac{1}{5!}x^4 + \frac{1}{7!}x^6 + \mathcal{O}(x^8)\right)^3 + \left(\frac{1}{3!}x^2 - \frac{1}{5!}x^4 + \mathcal{O}(x^6)\right)^4 \\
&\quad + \left(\frac{1}{3!}x^2 - \mathcal{O}(x^4)\right)^5 + \mathcal{O}(x^{12}) \\
&= 1 + \frac{1}{3!}x^2 + \left(-\frac{1}{5!} + \frac{1}{(3!)^2}\right)x^4 + \left(\frac{1}{7!} - \frac{2}{3!5!} + \frac{1}{(3!)^3}\right)x^6 + \left(\frac{1}{9!} + \frac{1}{(5!)^2} + \frac{2}{3!7!} + \frac{1}{(3!)^4}\right)x^8 \\
&\quad + \left(-\frac{1}{11!} + \frac{2}{3!9!} + \frac{3}{3!(5!)^2} + \frac{3}{(3!)^2 7!} - \frac{4}{(3!)^3 5!} + \frac{2}{(3!)^5}\right)x^{10} + \mathcal{O}(x^{12}) \\
&= 1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \frac{31}{15120}x^6 + \frac{127}{604800}x^8 + \frac{73}{3421440}x^{10} + \mathcal{O}(x^{12}).
\end{aligned}$$

Multiplying this series with the one for $\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \mathcal{O}(x^{12})$, this yilds

$$\begin{aligned}
x\cot x = &\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \mathcal{O}(x^{12})\right) \\
&\left(1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \frac{31}{15120}x^6 + \frac{127}{604800}x^8 + \frac{73}{3421440}x^{10} + \mathcal{O}(x^{12})\right)
\end{aligned}$$

$$= 1 - \frac{1}{3}x^2 - \frac{1}{45}x^4 - \frac{2}{945}x^6 - \frac{1}{4735}x^8 - \frac{2}{9355}x^{10} + \left(\mathcal{O}(x^{12})\right).$$

Substituting $x = \frac{z}{2}$, we find that

$$\frac{z}{2}\cos\left(\frac{z}{2}\right) = 1 - \frac{1}{6}\frac{1}{2!}z^2 - \frac{1}{30}\frac{1}{4!}z^4 - \frac{1}{42}\frac{1}{6!}z^6 - \frac{1}{30}\frac{1}{8!}z^8 - \frac{5}{66}\frac{1}{10!}z^{10} + \mathcal{O}(z^{12}).$$

So the first five Bernoulli numbers are $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}$, $B_5 = \frac{5}{66}$.