Quiz 4
MATH 621

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Problem 1 (Problem 2(d), Chapter VI.2, p. 176)**.**
Find the radius of convergence of the power series for the function

$$f(z) = \frac{z^2}{\sin^3 z}$$

about $z = \pi i$.

Solution. The function $f(z)$ is meromorphic and has only isolated singularities at the integer multiples of π on the real line. The radius of convergence will be the distance to the nearest pole. The nearest non-removeable singularity to $i\pi$ is 0. Therefore, the radius of convergence will be π.

Indeed, the singularity at $z = 0$ is not a removeable singularity. From the power series expansion,

$$\begin{aligned}\frac{z^2}{\sin^3 z} = \frac{z^2}{\left(z + \frac{1}{3!}z^3 + \mathcal{O}(z^5)\right)^3} &= \frac{z^2}{z^3 + \frac{1}{2}z^4 + \mathcal{O}(z^6)} \\ &= \frac{1}{z}\frac{1}{1 + \frac{1}{2}z + \mathcal{O}(z^3)} = \frac{1}{z}\left(1 - \frac{1}{2}z + \mathcal{O}(z^2)\right) = \frac{1}{z} - \frac{1}{2} + \mathcal{O}(z^2),\end{aligned}$$

so the singularity at $z = 0$ is not removeable and is a pole.

Problem 2 (Problem 6, Chapter VII.2, p. 202)**.**
Show that

$$\int_{-\infty}^{\infty} \frac{x}{(x^2+2x+2)(x^2+4)}dx = -\frac{\pi}{10}.$$

Solution. Since the degree of the polynomial in the denominator is at least 2 larger than that of the numerator, we can evaluate this integral by noting that the limit of the integrals

$$\int_{\Gamma_R} \frac{z}{(z^2+2z+2)(z^2+4)}dz$$

tend to zero as $R \to \infty$, where Γ_R is the semi-circular path from R to $-R$ in the upper half-plane. The integrand has simle poles at $\pm 2i$ and $-1 \pm i$. We only need to worry about the poles on the upper half-plane, i.e. $2i$ and $-1+i$. The residues of the integrand at these points are

$$\operatorname{Res}\left[\frac{z}{(z^2+2z+2)(z^2+4)}, 2i\right] = \left(\frac{z}{(z^2+2z+2)(2z)}\right)\bigg|_{z=2i} =$$
$$= \frac{2i}{(2i)^2+2(2i)+2}\frac{1}{2(2i)} = \frac{1}{4(2i-1)} = -\frac{1}{20}(1+2i)$$

and

$$\operatorname{Res}\left[\frac{z}{(z^2+2z+2)(z^2+4)}, -1+i\right] = \left(\frac{z}{(2z+2)(z^2+4)}\right)\bigg|_{z=-1+i} =$$
$$= \frac{-1+i}{(2(-1+i)+2)((-1+i)^2+4)} = \frac{1}{20}(1+3i).$$

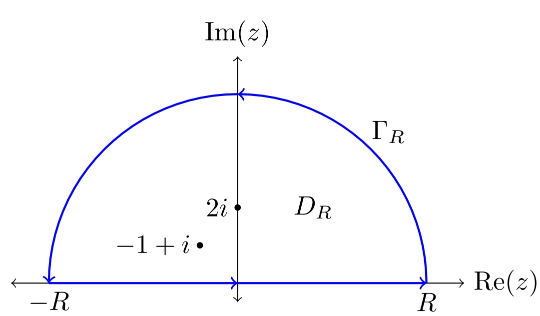
Let D_R denote the semi-circular disk of radius R in the upper half-plane. For R large enough, integrating along ∂D_R (i.e. the path that consists of Γ_R and the interval from $-R$ to R) yields

$$\begin{aligned}
&\int_{\partial D_R} \frac{z}{(z^2+2z+2)(z^2+4)}dz = \\
&\quad = 2\pi i \left(\operatorname{Res}\left[\frac{z}{(z^2+2z+2)(z^2+4)}, 2i\right] + \operatorname{Res}\left[\frac{z}{(z^2+2z+2)(z^2+4)}, -1+i\right]\right) \\
&\quad = \frac{2\pi i}{20}\left(1+3i-(1+2i)\right) \\
&\quad = -\frac{\pi}{10}.
\end{aligned}$$

So the integral in question evaluates to

$$\lim_{R\to\infty}\int_{\partial D_R} \frac{z}{(z^2+2z+2)(z^2+4)}dz - \underbrace{\lim_{R\to\infty}\int_{\partial \Gamma_R} \frac{z}{(z^2+2z+2)(z^2+4)}dz}_{=0} = -\frac{\pi}{10}$$

as desired.



Problem 3 (Problem 7, Chapter VII.2, p. 202)**.**
Show that

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^4+1}dx = \frac{\pi}{\sqrt{2}}e^{-a/\sqrt{2}}\left(\cos\frac{a}{\sqrt{2}}+\sin\frac{a}{\sqrt{2}}\right)$$

for $a > 0$.

Solution. We can evaluate this integral as the real part of the integral $\int_{-\infty}^{\infty} \frac{e^{iax}}{x^4+1}dx$, and proceed by integrating the function $\frac{e^{iaz}}{z^4+1}$ around the semi-circular contour Γ_R of radius R in the upper half-plane. Note that $\left|e^{iaz}\right| = \left|e^{-a\,\mathrm{Im}(z)}e^{ia\,\mathrm{Re}(z)}\right| \le 1$ since $\mathrm{Im}(z) \ge 0$ on the upper half-plane. Thus, for R large enough,

$$\left|\int_{\Gamma_R} \frac{e^{iaz}}{z^4+1}dz\right| \le \frac{\pi R}{R^4-1}$$

by the ML-estimate, which tends to zero as $R \to \infty$. So we can instead evaluate the integrals around the semi-circular disks D_R of radius R in the upper half-plane.

The integrand has only simple poles at $e^{i\left(\frac{2\pi k}{4}+\frac{\pi}{4}\right)}$ for $k = 0, 1, 2, 3$. We only need to worry about the ones in the upper half-plane. These are $e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}}(1+i)$ and $e^{i\frac{3\pi}{4}} = \frac{1}{\sqrt{2}}(-1+i)$. The residues of the integrand at these points are

$$\begin{aligned}\mathrm{Res}\left[\frac{e^{iaz}}{z^4+1}, e^{i\frac{\pi}{4}}\right] = \left.\frac{e^{iaz}}{4z^3}\right|_{z=e^{i\frac{\pi}{4}}} = \frac{1}{4}e^{i\frac{a}{\sqrt{2}}(1+i)}e^{-i\frac{3\pi}{4}} &= \frac{e^{-\frac{a}{\sqrt{2}}}}{4\sqrt{2}}\left(\cos\tfrac{a}{\sqrt{2}} + i\sin\tfrac{a}{\sqrt{2}}\right)(-1-i)\\ &= -\frac{e^{-\frac{a}{\sqrt{2}}}}{4\sqrt{2}}\left[\left(\cos\tfrac{a}{\sqrt{2}} - \sin\tfrac{a}{\sqrt{2}}\right) + i\left(\cos\tfrac{a}{\sqrt{2}} + \sin\tfrac{a}{\sqrt{2}}\right)\right]\end{aligned}$$

and

$$\begin{aligned}\mathrm{Res}\left[\frac{e^{iaz}}{z^4+1}, e^{i\frac{3\pi}{4}}\right] = \left.\frac{e^{iaz}}{4z^3}\right|_{z=e^{i\frac{3\pi}{4}}} = \frac{1}{4}e^{i\frac{a}{\sqrt{2}}(i-1)}e^{-i\frac{\pi}{4}} &= \frac{e^{-\frac{a}{\sqrt{2}}}}{4\sqrt{2}}\left(\cos\tfrac{a}{\sqrt{2}} - i\sin\tfrac{a}{\sqrt{2}}\right)(-i+1)\\ &= -\frac{e^{-\frac{a}{\sqrt{2}}}}{4\sqrt{2}}\left[\left(-\cos\tfrac{a}{\sqrt{2}} + \sin\tfrac{a}{\sqrt{2}}\right) + i\left(\cos\tfrac{a}{\sqrt{2}} + \sin\tfrac{a}{\sqrt{2}}\right)\right].\end{aligned}$$

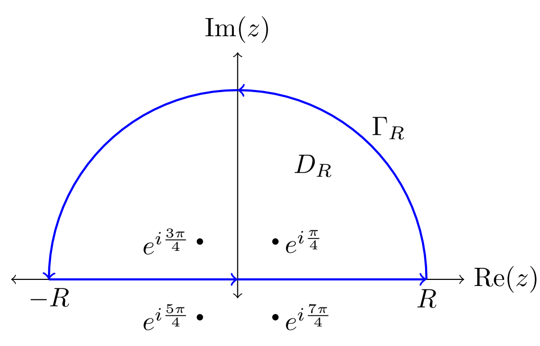
Hence, for R large enough, integrating over the boundary of D_R yields

$$\begin{aligned}\int_{\partial D_R} \frac{e^{iaz}}{z^4+1}dz = 2\pi i\left(\mathrm{Res}\left[\frac{e^{iaz}}{z^4+1}, e^{i\frac{\pi}{4}}\right] + \mathrm{Res}\left[\frac{e^{iaz}}{z^4+1}, e^{i\frac{3\pi}{4}}\right]\right) &= 2\pi i\frac{e^{-\frac{a}{\sqrt{2}}}}{4\sqrt{2}}(-i)2\left(\cos\tfrac{a}{\sqrt{2}} + \sin\tfrac{a}{\sqrt{2}}\right)\\ &= \frac{e^{-\frac{a}{\sqrt{2}}}}{\sqrt{2}}\left(\cos\frac{a}{\sqrt{2}} + \sin\frac{a}{\sqrt{2}}\right).\end{aligned}$$

Putting this together, the integral in question is

$$\int_{-\infty}^{\infty}\frac{\cos(ax)}{x^4+1}dx = \mathrm{Re}\left[\lim_{R\to\infty}\int_{\partial D_R}\frac{e^{iaz}}{z^4+1}dz - \underbrace{\lim_{R\to\infty}\int_{\Gamma_R}\frac{e^{iaz}}{z^4+1}dz}_{=0}\right] = \frac{e^{-\frac{a}{\sqrt{2}}}}{\sqrt{2}}\left(\cos\frac{a}{\sqrt{2}} + \sin\frac{a}{\sqrt{2}}\right)$$

as desired.



Problem 4 (Problem 2, Chapter VII.3, p. 205)**.**
Show using residue theory that

$$\int_0^{2\pi} \frac{1}{a+b\sin\theta}d\theta = \frac{2\pi}{\sqrt{a^2-b^2}}$$

for $a > b > 0$.

Solution.
First write $\sin\theta$ as $\frac{e^{i\theta}-e^{-i\theta}}{2i}$. With the change of variables $z = e^{1\theta}$, we have $d\theta = \frac{dz}{iz}$ and the integral becomes

$$\begin{aligned}\oint_{|z|=1} \frac{2i}{2ia+b(z+\frac{1}{z})}\frac{1}{iz}dz &= \frac{1}{i}\oint_{|z|=1}\frac{1}{\frac{b}{2i}z^2+az-\frac{b}{2i}}dz \\ &= \frac{2}{b}\oint_{|z|=1}\frac{1}{z^2+2i\frac{a}{b}z-1}dz.\end{aligned}$$

The polynomial in the denominator has roots at

$$z = \frac{-2i\frac{a}{b} \pm i\sqrt{4\frac{a^2}{b^2}-4}}{2} = i\frac{-a\pm\sqrt{a^2-b^2}}{b}.$$

Only one of these roots is in the unit disk. Indeed, we have

$$\left|i\frac{-a-\sqrt{a^2-b^2}}{b}\right| = \frac{a+\sqrt{a^2+b^2}}{b} > \frac{a}{b} > 1,$$

since $a > b > 0$, but

$$\left|i\frac{-a+\sqrt{a^2-b^2}}{b}\right| = \frac{a-\sqrt{a^2-b^2}}{b} < \frac{b}{b} = 1$$

since $a - b < \sqrt{a^2-b^2}$. Denote this root as $z_0 = i\frac{-a+\sqrt{a^2-b^2}}{b}$.

The integrand has a simple pole at this root, so the residue of the integrand at z_0 is

$$\operatorname{Res}\left[\frac{1}{z^2+2i\frac{a}{b}z-1}, z_0\right] = \left.\frac{1}{2z+2i\frac{a}{b}}\right|_{z=z_0} = \frac{1}{2i}\frac{b}{-a+\sqrt{a^2-b^2}+a} = \frac{b}{2i}\frac{1}{\sqrt{a^2-b^2}}.$$

Since this is the only pole in the unit disk, the integral in question evaluates to

$$\frac{2}{b}\oint_{|z|=1}\frac{1}{z^2+2i\frac{a}{b}z-1}dz = 2\pi i\frac{2}{b}\operatorname{Res}\left[\frac{1}{z^2+2i\frac{a}{b}z-1}, z_0\right] = \frac{4\pi i}{b}\frac{b}{2i\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

as desired.

Problem 5 (Problem 3, Chapter VII.7, p. 218).
Evaluate the limits

$$L(a) = \lim_{R\to\infty} \int_{-R}^{R} \frac{x\sin(ax)}{x^2+1}dx$$

for $-\infty < a < \infty$. Show that they do not depend continuously on the parameter a.

Solution. If $a = 0$, then the limit vanishes. So suppose that $a \neq 0$. It suffices to consider only $a > 0$, since

$$L(-a) = \lim_{R\to\infty} \int_{-R}^{R} \frac{x\sin(-ax)}{x^2+1}dx = -\lim_{R\to\infty} \int_{-R}^{R} \frac{x\sin(ax)}{x^2+1}dx = -L(a).$$

So suppose $a > 0$ and make the change of variables $u = ax$ such that $dx = \frac{1}{a}du$. The limit becomes

$$\lim_{R\to\infty} \int_{-Ra}^{Ra} \frac{u\sin(u)}{u^2+a^2}du.$$

We may evaluate this limit by taking the imaginary part of the limit

$$\lim_{R\to\infty} \int_{-Ra}^{Ra} \frac{ue^{iu}}{u^2+a^2}du.$$

Note that taking the integral around the semicircular path from Ra to $-Ra$ in the upper half-plane tends zero as $R \to \infty$, since, for $R > 1$,

$$\begin{aligned} \left|\int_{\Gamma_{Ra}} \frac{ze^{iz}}{z^2+a^2}dz\right| &\leq \int_{\Gamma_{Ra}} \frac{|z|\left|e^{iz}\right|}{|z^2+a^2|}|dz| \leq \int_{\Gamma_{Ra}} \frac{|z|\left|e^{iz}\right|}{|z^2|-a^2}|dz| \\ &= \frac{Ra}{(Ra)^2-a^2}\int_{\Gamma_{Ra}} \left|e^{iz}\right||dz| < \frac{Ra}{R^2a^2-a^2}\frac{\pi}{Ra} = \frac{1}{a^2}\frac{\pi}{R^2-1} \end{aligned}$$

which tends to zero as $R \to \infty$, where the final inequality is due to Jordan's lemma. So the integral in question may be evaluated by instead integrating along the path from $-Ra$ to $+Ra$, then around the path Γ_{Ra}. The integrand has only one pole in this region at $z = ia$. The residue of the integrand at this point is

$$\operatorname{Res}\left[\frac{ze^{iz}}{z^2+a^2}, ia\right] = \left.\frac{ze^{iz}}{2z}\right|_{z=ia} = \frac{e^{-a}}{2}.$$

Therefore, taking the integral around this path yields

$$\left(\int_{-Ra}^{Ra} + \int_{\Gamma_{Ra}}\right)\frac{ze^{iz}}{z^2+a^2}dz = 2\pi i \operatorname{Res}\left[\frac{ze^{iz}}{z^2+a^2}, ia\right] = i\pi e^{-a}.$$

Taking the imaginary part of this yields the result. Hence the limits evaluate to

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{x\sin(ax)}{x^2+1}dx = \begin{cases} \pi e^{-|a|}, & a > 0 \\ -\pi e^{-|a|}, & a < 0 \\ 0, & a = 0. \end{cases}$$

Clearly, this does not depend continuously on a, since $\lim_{a\to 0} e^{-|a|} = 1$. As a function of a, this is not continuous at $a = 0$.