Quiz 5

MATH 621

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Problem 1 (Problem 1, Chapter XVI.1, p. 423)**.**
Define the Riemann surface R of $\log z$ in terms of explicit coordinate patches. Define explicitly the function on R determined by $\log z$. Show that it is a one-to-one analytic map of R onto the complex plane $\mathbb{C}$.

Solution. The Riemann surface of R can be viewed as the graph $R = \{(z, w) \,|\, z \neq 0, w = \log z\}$. To cover this surface with coordinate patches, for each $k \in \mathbb{Z}$ choose the corresponding branch of the logarithm function where $U_k \subset R$ consists of the points $(z, w) \in R$ such that

$$(2k - 1)\pi < \operatorname{Im} w < (2k + 1)\pi.$$

Define the coordinate maps to be the projection onto the first coordinate. Specifically, $z_k \colon U_k \to \mathbb{C}$ where

$$z_k(z, w) = z.$$

We need to define the additional coordinate patches $U'_k \subset R$ to consist of the points $(z, w) \in R$ such that

$$2k\pi < \operatorname{Im} w < 2(k + 1)\pi,$$

and define the corresponding coordinate maps to be $z'_k \colon U'_k \to \mathbb{C}$ where

$$z'_k(z, w) = z.$$

The atlas for R is $\mathcal{A}_R = \{(U_k, z_k)\}_{k\in\mathbb{Z}} \cup \{(U'_k, z'_k)\}_{k\in\mathbb{Z}}$. Note that each U_k has nonempty intersection only with U'_k and U'_{k+1}, and the corresponding transition maps are the identity.

We can define the logarithm function from R to $\mathbb{C}$ as follows:

$$\log(z, w) = w.$$

On each of the coordinate patches, this map just reduces to the corresponding analytic branch of the logarithm, so this function is analytic. Furthermore, it is also one-to-one and onto. Indeed, let $\zeta \in \mathbb{C}$ and set $z = e^{\zeta}$ such that $(z, \zeta) \in R$ and $\log(z, \zeta) = \zeta$, so the function is onto. To show that it is one-to-one, suppose (z_1, w_1) and (z_2, w_2) are two points in R such that $\log(z_1, w_1) = \log(z_2, w_2)$. Then $w_1 = w_2$, and $z_1 = e^{w_1} = e^{w_2} = z_2$.

Problem 2 (Problem 4, Chapter XVI.1, p. 423)**.**
Show that the analytic maps from a Riemann surface R to the Riemann sphere $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ are the meromorphic functions on R and the constant function ∞.

Solution. Recall that the Riemann sphere $\mathbb{C}_\infty$ can be given the structure of a Riemann surface with the coordinate charts

$$\zeta \mapsto \zeta \qquad \text{for } \zeta \in \mathbb{C}_\infty \setminus \{\infty\}, \qquad \text{and} \qquad \zeta \mapsto \frac{1}{\zeta} \qquad \text{for } \zeta \in \mathbb{C}_\infty \setminus \{0\}.$$

Let $f\colon R \to \mathbb{C}_\infty$ be an analytic map. For each coordinate patch U_α containing a point $p \in R$,

i) if $f(p) \neq \infty$ then the map $f\left(z_\alpha^{-1}(\zeta)\right)$ is analytic at $z_\alpha(p)$,

ii) and if $f(p) \neq 0$ then the map $\frac{1}{f\left(z_\alpha^{-1}(\zeta)\right)}$ is analytic at $z_\alpha(p)$.

If $f(p) \neq \infty$ for all $p \in R$, then f is just an analytic function $f\colon R \to \mathbb{C}$, which is clearly meromorphic. So suppose that $f(p_0) = \infty$ for some $p_0 \in R$. These points in R are precisely the singularities of f when viewing it as a map to $\mathbb{C}$. Furthermore, the map $\frac{1}{f(p)}$ must be analytic at each such singularity.

Suppose that there is at least one such singularity of f that is not isolated at some point $p_0 \in R$. Then the analytic function $\frac{1}{f(p)}$ has a non-isolated zero at $p = p_0$, and thus $\frac{1}{f(p)}$ must be identically zero. That is, $f(p)$ is identically ∞.

Now suppose that all of the singularities or f are isolated. Since the function $\frac{1}{f(p)}$ is analytic at each such singularity, and these singularity points are isolated zeros of $\frac{1}{f(p)}$, these zeros must be zeros of $\frac{1}{f(p)}$ with some finite order. (The only analytic function with a zero of infinite order is the function that is constant at 0.) Hence, each singularity of $f(p)$ must be a pole of finite order, where the order of the pole at some singularity p_0 is the same as the order of the zero of $\frac{1}{f(p)}$ at p_0. We conclude that the function f is meromorphic, since it is analytic on R at all but some isolated singularities, each of which is a pole.

Finally, note that any meromorphic function $f\colon R \to \mathbb{C}$ can clearly be seen as an analytic map from R to $\mathbb{C}_\infty$. Indeed, since f is meromorphic, it is analytic at all non-singularity points, and the map $\frac{1}{f(p)}$ must be analytic at each of the isolated singularities. This is exactly the definition for a map $f\colon R \to \mathbb{C}_\infty$ to be analytic.

Problem 3 (Problem 5, Chapter XVI.1, p. 423)**.**
Let $\omega \neq 0$ and let $\mathbb{Z}\omega$ be the integral multiples of ω. Let R be the set of congruence classes $z + \mathbb{Z}\omega$, $z \in \mathbb{C}$. Show that R is a Riemann surface that is conformally equivalent to the punctured plane $\mathbb{C}\backslash\{0\}$.

Solution.
The set R (which can be viewed as a 'cylinder') can be given the structure of a Riemann surface in the following manner. Choose $\varepsilon > 0$ such that $\varepsilon < |\omega|$. For each complex number $\lambda \in \mathbb{C}$, let U_λ be the subset of R defined by

$$U_\lambda = \big\{\zeta + \mathbb{Z}\omega \;\big|\; |\zeta - \lambda| < \varepsilon\big\},$$

and define the corresponding coordinate chart to be $z_\lambda \colon U_\lambda \to \mathbb{C}$ by

$$z_\lambda\left(\zeta + \mathbb{Z}\omega\right) = \zeta \qquad \text{for } \; |\zeta - \lambda| < \varepsilon.$$

This chart maps U_λ onto the disk $\{|z - \lambda| < \varepsilon\} \subset \mathbb{C}$, and the condition that $\varepsilon < |\omega|$ guarantees that this map is one-to-one. Transition maps $z_\mu \circ z_\lambda^{-1}$ are the identity wherever defined, hence analytic.

Consider the map $f \colon R \to \mathbb{C}\backslash\{0\}$ given by $f(\lambda + \mathbb{Z}\omega) = e^{\lambda\omega^{-1}2\pi i}$. This map is well-defined, since

$$\begin{aligned} \lambda + m\omega \mapsto e^{(\lambda + m\omega)\omega^{-1}2\pi i} &= e^{\lambda\omega^{-1}2\pi i + m2\pi i} \\ &= e^{\lambda\omega^{-1}2\pi i}. \end{aligned}$$

It is also one-to-one and onto. Indeed, consider $\lambda, \mu \in \mathbb{C}$ such that $f(\lambda) = f(\mu)$. Then

$$e^{\lambda\omega 2\pi i} = e^{\mu\omega 2\pi i} \qquad \Longleftrightarrow \qquad e^{(\lambda-\mu)\omega 2\pi i} = 1 \qquad \Longleftrightarrow \qquad \lambda - \mu = m\omega \quad \text{ for some } m \in \mathbb{Z},$$

and thus $\lambda + \mathbb{Z}\omega = \mu + \mathbb{Z}$. So f is one-to-one. Furthermore, to show that f is onto, for any $z \in \mathbb{C}\backslash\{0\}$, take $\lambda = \frac{\omega}{2\pi i}\log z$ where we take the any branch of $\log z$. Then $f(\lambda) = z$. Finally, we also note that f is analytic since it is analytic in every coordinate chart. The map f yields the desired conformal equivalence between R and $\mathbb{C}\backslash\{0\}$.

Problem 4 (Problem 11(a-c), Chapter XVI.1, p. 424)**.**
Let R be a finite bordered Riemann surface with border ∂R. Let $\tilde{R}$ be a duplicate copy of R, and denote by $\tilde{p}$ the point in $\tilde{R}$ corresponding to $p \in R$. Let $S = R \cup \tilde{R} \cup \partial R$. Define $\tau\colon S \to S$ by $\tau(p) = \tilde{p}$ if $p \in \partial R$, and by $\tau(p) = \tilde{p}$, $\tau(\tilde{p}) = p$ if $p \in R$.

(a) Show that S can be made into a compact Riemann surface with R as a subsurface so that τ is anticonformal, that is, $f(p)$ is analytic on an open set U if and only if $\overline{f(\tau(p))}$ is analytic on $\tau(U)$. (*Remark*: The surface S is called the **doubled surface** of R, and τ is the reflection in ∂R.)

(b) Show that the doubled surface of the unit disk is the Riemann sphere $\mathbb{C}_\infty$.

(c) What is the doubled surface of an annulus?

Solution.
Since R is a finite bordered Riemann surface, by definition $R \cup \partial R$ is a compact subsurface of some other Riemann surface M, and $\partial R \subset M$ consists of a finite number of disjoint simple closed analytic curves in M. Recall that a subset K of a Riemann surface is compact if and only if it can be expressed as a finite union $K = K_1 \cup \cdots \cup K_m$, where each K_j is a compact subset contained in a single coordinate patch. Finally, recall that a function $g\colon D \to \mathbb{C}$ is analytic on a domain $D \subset \mathbb{C}$ if and only if the function $\overline{g(\overline{z})}$ is analytic on the reflected domain $D^* = \{z = \overline{w} \,|\, w \in D\}$.

(a) Let (U_α, z_α) be an atlas of M. We can give S the structure of a Riemann surface in the following manner. For each α, define to coordinate patches (V_α, w_α) and $(\tilde{V}, \tilde{w}_\alpha)$ on S by $V_\alpha = U_\alpha \cap R$ with corresponding coordinate chart $w_\alpha(p) = z_\alpha(p)$, and $\tilde{V}_\alpha = \tau(U_\alpha \cap R)$ with coordinate chart $\tilde{w}_\alpha(\tilde{p}) = \overline{z_\alpha(\tau(\tilde{p}))}$. Note that the image $\tilde{w}_\alpha(\tilde{V}_\alpha)$ is just the reflected domain of the image $w_\alpha(V_\alpha)$.

For each intersection of coordinate patches, the analyticity of the transition maps is guaranteed by the analyticity of the transition maps for the parent Riemann surface M. Indeed, for $V_\alpha \cap V_\beta \neq \emptyset$, we have

$$w_\beta \circ w_\alpha^{-1}(\zeta) = z_\beta \circ z_\alpha^{-1}(\zeta) \qquad \text{and} \qquad \tilde{w}_\beta \circ \tilde{w}_\alpha^{-1}(\zeta) = \overline{z_\beta \circ z_\alpha^{-1}(\overline{\zeta})},$$

which are both analytic. The intersections $V_\alpha \cap \tilde{V}_\beta$ are all empty. This covers all of S except ∂R. It remains to construct patches across the boundary of R and $\tilde{R}$ in S.

Recall that ∂R consists of disjoint analytic curves in M. Hence, for each $p \in \partial R$, there is a coordinate disk $U_p \subset M$ containing p with coordinate chart $z_p\colon U_p \to D_p$ such that $D_p \subset \mathbb{C}$ is a disk centered on the real line, and the boundary $V_p \cap \partial R$ is mapped to $D_p \cap \mathbb{R}$. For each $p \in \partial R$, we can define a coordinate patch (V_p, w_p) on S as follows. Let $V_p \subset S$ be the image of the one-to-one map

$$w_p^{-1}\colon \zeta \mapsto \begin{cases} z_p^{-1}(\zeta), & \zeta \in D_p \cap z_p\big(U_p \cap (R \cup \partial R)\big) \\ \tau\big(z_{\alpha_p}^{-1}(\overline{\zeta})\big), & \zeta \in D_p \setminus z_p\big(U_p \cap (R \cup \partial R)\big), \end{cases}$$

with corresponding coordinate chart $w_p\colon V_p \to U_p$ to be the inverse of this map. The map w_p^{-1} takes the disk D_p, which is centered on the real line, to the set V_p, which 'straddles' the boundary between R and $\tilde{R}$ in S. The resulting coordinate patch $V_p = w_p^{-1}(U_p)$ contains the point $p \in \partial R$ and intersects both R and $\tilde{R}$.

The family of coordinate patches $\{(V_\alpha, w_\alpha)\} \cup \{(\tilde{V}_\alpha, \tilde{w}_\alpha)\} \cup \{(V_p, w_p) \,|\, p \in \partial R\}$ is in fact a conformal atlas of $S = R \cup \tilde{R} \cup \partial R$. Indeed, the transition maps $w_\beta \circ w_p^{-1}$ are

$$w_\beta \circ w_p^{-1}(\zeta) = z_\beta \circ z_p^{-1}(\zeta) \qquad \text{for } \zeta \in w_p(V_\beta \cap V_p),$$

which are analytic, and the transition maps $\tilde{w}_\beta \circ w_p^{-1}$ are

$$\tilde{w}_\beta \circ w_p^{-1}(\zeta) = \overline{z_\beta \circ z_p^{-1}(\overline{\zeta})} \qquad \text{for } \zeta \in w_p(\tilde{V}_\beta \cap V_p),$$

which is also analytic. This completes the proof that S is a Riemann surface.

By definition, $R \subset M$ is compact, so there are finitely many compact subsets $K_j \subset M$ such that $R \cup \partial R = K_1 \cup \cdots \cup K_m$. For each $j = 1, \ldots, m$, let $\tilde{K}_j = \tau(K_j)$. Then each $\tilde{K}_j$ is compact, and S can be written as a union of finitely many compact subsets

$$S = K_1 \cup \cdots \cup K_m \cup \tilde{K}_1 \cup \cdots \cup \tilde{K}_m,$$

so S is a compact surface.

Let $f\colon U \to \mathbb{C}$ be an analytic map on an open subset $U \subset S$. This is equivalent to the statement that $f \circ w^{-1}$ is analytic on $w(U \cap V)$ for each coordinate patch (V, w) of S. I.e., the following are analytic functions:

$$f \circ w_\alpha^{-1}(\zeta) = f \circ z_\alpha^{-1}(\zeta) \qquad \text{for } \zeta \in w_\alpha(U \cap V_\alpha), \tag{1}$$

$$f \circ \tilde{w}_\alpha^{-1}(\zeta) = f \circ \tau \circ \tilde{z}_\alpha^{-1}(\bar{\zeta}) \qquad \text{for } \zeta \in \tilde{w}_\alpha(U \cap \tilde{V}_\alpha), \tag{2}$$

$$f \circ w_p^{-1}(\zeta) = \begin{cases} f \circ z_p^{-1}(\zeta), & \zeta \in w_p(U \cap V_p \cap R) \\ f \circ \tau \circ z_p^{-1}(\bar{\zeta}), & \zeta \in w_p(U \cap V_p \cap \tilde{R}) \end{cases} \qquad \text{for } \zeta \in w_p(U \cap V_p). \tag{3}$$

The function (1) is analytic if and only if $\overline{f \circ w_\alpha^{-1}(\bar{\zeta})} = \overline{f \circ \tau \circ \tilde{w}_\alpha^{-1}(\zeta)}$ is analytic on the corresponding reflected domain, which is $\tilde{\omega}_\alpha(\tau(U \cap V_\alpha))$. Similarly, the function (2) is analytic if and only if $\overline{f \circ \tilde{w}_\alpha^{-1}(\bar{\zeta})} = \overline{f \circ \tau \circ w_\alpha^{-1}(\zeta)}$ is analytic on the corresponding reflected domain, $\omega_\alpha(\tau(U \cap \tilde{V}_\alpha))$. Finally, the function (3) is analytic if and only if

$$\overline{f \circ w_p^{-1}(\bar{\zeta})} = \begin{cases} \overline{f \circ z_p^{-1}(\bar{\zeta})} \\ \overline{f \circ \tau \circ z_p^{-1}(\zeta)} \end{cases} = \begin{cases} \overline{f \circ \tau \circ z_p^{-1}(\zeta)}, & \zeta \in w_p(\tau(U) \cap V_p \cap \tilde{R}) \\ \overline{f \circ z_p^{-1}(\bar{\zeta})}, & \zeta \in w_p(\tau(U) \cap V_p \cap R) \end{cases} = \overline{f \circ \tau \circ w_p^{-1}(\zeta)}$$

is analytic on the corresponding reflected domain $w_p(\tau(U \cap V_p))$.

Therefore $f\colon U \to \mathbb{C}$ is analytic if and only if $\overline{f(\tau(\tilde{p}))}$ is analytic on $\tau(U)$.

(b) We have $D = \{z \mid |z| < 1\}$ and $\tilde{D} = \{\tau(z) \mid |z| < 1\}$. For $z \in D$, we can identify $\tau(z)$ with the point $\frac{1}{\bar{z}} \in \mathbb{C}_\infty \setminus (D \cup \partial D)$. Then D, ∂D and $\tilde{D}$ can be viewed as subsets of the extended complex plane $\mathbb{C}_\infty$, and $\tau(z) = \frac{1}{\bar{z}}$ as an involution of $\mathbb{C}_\infty$ that fixes ∂D.

We can take a conformal atlas on $D \cup \partial D \cup \tilde{D}$ by using the same standard atlas on the extended complex plane.

(c) The doubled surface of an annulus is a torus.

Problem 5 (Problem 13, Chapter XVI.1, p. 425)**.**
For τ in the open upper half-plane $\mathbb{H}$, denote by L_τ the lattice $\mathbb{Z}+\mathbb{Z}\tau$ generated by 1 and τ, and denote the Riemann surface $\mathbb{C}/L_\tau$ by T_τ.

(a) Show that the Riemann surface $T = \mathbb{C}/L$ in this section is conformally equivalent to the Riemann surface T_τ for some $\tau \in \mathbb{H}$. (*Hint*: Take $\tau = \pm\omega_1/\omega_2$, where the sign is chosen so that $\operatorname{Im}\tau > 0$.)

(b) Show that T_τ is conformally equivalent to $T_{\tau'}$ if and only if there is a fractional linear transformation of the form $f(z) = \frac{az+b}{cz+d}$ where a, b, c, d are integers satisfying $ad - bc = 1$, such that $f(\tau) = \tau'$. (*Remark*: the matrix with entries a, b, c, d is called a unimodular matrix. The unimodular matrices form a group, which is the special linear group $\mathrm{SL}(2,\mathbb{Z})$.)

Note: (I thought this was a really neat problem. Thanks for assigning it! I was really stumped by part (b) and had to go looking through a few books to help me solve it. I figured out one direction on my own, but I had to cite a resource (see my footnotes on the next page) that outlined a proof showing that equivalence of T_τ and T'_τ implies $\tau' = \frac{a\tau+b}{c\tau+d}$. But I didn't completely understand all of the proof. In particular, showing that conformal maps of tori can be lifted to a conformal maps of $\mathbb{C}$ onto itself was what confused me.)

Solution.

(a) Let ω_1 and ω_2 be two linearly independent points in $\mathbb{C}$. Consider the lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and define the torus $T = \mathbb{C}/L$ with the standard conformal structure. That is, set $\varepsilon = \min(|\omega_1|, |\omega_2|)$ such that $|m\omega_1 + n\omega_2| > \varepsilon$ for each nonzero lattice point in L. For each $\lambda \in \mathbb{C}$, define the coordinate patch

$$U_\lambda = \big\{\zeta + L \,\big|\, |\zeta - \lambda| < \varepsilon\big\}$$

with corresponding coordinate chart $z_\lambda(\zeta + L) = \zeta$ for $|\zeta - \lambda| < \varepsilon$.

Let $\tau = \pm\frac{\omega_1}{\omega_2}$ where the sign is chosen such that $\operatorname{Im}\tau > 0$. We can similarly give T_τ the structure of a Riemann surface by picking $\delta = \min\{|\tau|, 1\}$, and defining coordinate patches

$$V_\lambda = \big\{\zeta + L \,\big|\, |\zeta - \lambda| < \delta\big\}$$

with corresponding coordinate chart $w_\lambda(\zeta + L) = \zeta$ for $|\zeta - \lambda| < \delta$.

Define a map $f\colon T \to T_\tau$ by $f(\zeta + L) = \zeta\omega_2^{-1} + L_\tau$. Note that this map is well-defined, since

$$\zeta + m\omega_1 + n\omega_2 + L \mapsto \zeta\omega_2^{-1} + \underbrace{m\tfrac{\omega_1}{\omega_2} + n}_{\in L_\tau} + L_\tau = \zeta\omega_2^{-1} + L_\tau.$$

Furthermore, this map is both one-to-one and onto. Indeed, if $f(\zeta + L) = f(\xi + L)$ for some $\zeta, \xi \in \mathbb{C}$ implies that $(\zeta - \xi)\omega_2^{-1} = m\tau + n$ for some $m, n \in \mathbb{Z}$, and thus $\zeta - \xi = m\tau\omega_2 + n\omega_n \in L$, so f is one-to-one. For any $\zeta + L_\tau \in T_\tau$, we have $f(\zeta\omega_2 + L) = \zeta + L_\tau$, so f is onto.

This map is also analytic. Indeed, for any $\lambda \in \mathbb{C}$, the maps $w_{\lambda\omega_2^{-1}} \circ f \circ z_\lambda^{-1}\colon z_\lambda(U_\lambda) \to w_{\lambda\omega_2^{-1}}(V_{\lambda\omega_2^{-1}})$ defined by

$$w_{\lambda\omega_2^{-1}}\big(f(z_\lambda^{-1}(\zeta))\big) = \zeta\omega_2^{-1}$$

are analytic, since they are just multiplication by a constant.

(b) Let T_{ω_1,ω_2} be the torus defined above as $T_{\omega_1,\omega_2} = \mathbb{C}/L_{\omega_1,\omega_2}$ from the lattice $L_{\omega_1,\omega_2} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. For another pair of linearly independent complex numbers $\omega_1', \omega_2' \in \mathbb{C}$, we can define another torus $T_{\omega_1',\omega_2'} = \mathbb{C}/L_{\omega_1',\omega_2'}$, where the lattice $L_{\omega_1',\omega_2'}$ is defined equivalently.

Suppose $\omega_1' = a\omega_1 + b\omega_2$ and $\omega_2' = c\omega_1 + d\omega_2$ for some integers $a, b, c, d \in \mathbb{Z}$ such that $ad - bc = 1$. Then the pairs (ω_1, ω_2) and (ω_1', ω_2') define the same lattice. Indeed, we can represent any lattice point $p \in L_{\omega_1',\omega_2'}$ as $p = m\omega_1' + n\omega_2'$ for some $m, n \in \mathbb{Z}$, and thus

$$p = m\omega_1' + n\omega_2' = m(a\omega_1 + b\omega_2) + n(c\omega_1 + d\omega_2) = (ma + nc)\omega_1 + (mb + nd)\omega_2$$

so $p \in L_{\omega_1,\omega_2}$ and thus $L_{\omega_1',\omega_2'} \subseteq L_{\omega_1,\omega_2}$. Similarly, we can write ω_1 and ω_2 as linear combinations of ω_1' and ω_2' by

$$\omega_1 = d\omega_1' - b\omega_2' \qquad \text{and} \qquad \omega_2 = -c\omega_1' + a\omega_2'$$

(using the inverse transform $\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$ of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$). Hence $L_{\omega_1,\omega_2} \subseteq L_{\omega_1',\omega_2'}$ and thus $L_{\omega_1',\omega_2'} = L_{\omega_1,\omega_2}$. Since these two pairs of points define the same lattice, they must also define the same torus, and thus the tori T_{ω_1,ω_2} and $T_{\omega_1',\omega_2'}$ are identical.

From part (a), we see that T_τ and T_{ω_1,ω_2} are conformally equivalent, and similarly that $T_{\tau'}$ and $T_{\omega_1',\omega_2'}$ are conformally equivalent, where

$$\frac{a\tau + b}{c\tau + d} = \frac{a\frac{\omega_1}{\omega_2} + b}{c\frac{\omega_1}{\omega_2} + d} = \frac{a\omega_1 + b\omega_2}{c\omega_1 + d\omega_2} = \frac{\omega_1'}{\omega_2'} = \tau'$$

Since both $T_{\tau'}$ and T_τ are both conformally equivalent to $T = T_{\omega_1,\omega_2} = T_{\omega_1',\omega_2'}$, they are conformally equivalent to each other.

Now suppose there is a conformal equivalence $\phi\colon T_\tau \to T_{\tau'}$. This map can be 'lifted'[1] to a conformal equivalence $\tilde{\phi}\colon \mathbb{C} \to \mathbb{C}$ that satisfies $\tilde{\phi}(0) = 0$ and $\pi' \circ \tilde{\phi} = \phi \circ \pi$, where $\pi\colon \mathbb{C} \to T_\tau$ and $\pi'\colon \mathbb{C} \to T_{\tau'}$ are the canonical projections

$$\pi(\zeta) = \zeta + L_\tau \qquad \text{and} \qquad \pi'(\zeta) = \zeta + L_{\tau'}.$$

The only conformal mappings of the entire complex plane are the linear maps[2] $z \mapsto \alpha z + \beta$ for $\alpha \neq 0$. Since $\tilde{\phi}\colon \mathbb{C} \to \mathbb{C}$ is conformal and $\tilde{\phi}(0) = 0$, it must be a dilation map $\tilde{\phi}(z) = \alpha z$ for some nonzero $\alpha \in \mathbb{C}$. From the relation $\pi' \circ \tilde{\phi} = \phi \circ \pi$, we see that[3] any point on the lattice T_τ gets mapped to point on the lattice $T_{\tau'}$. Hence, we have

$$\tilde{\phi}(\tau) = \alpha\tau = a\tau' + b \qquad \text{and} \qquad \tilde{\phi}(1) = \alpha = c\tau' + d$$

for some $a, b, c, d \in \mathbb{Z}$. Since $\tilde{\phi}$ is invertible, we have $ad - bc = \pm 1$, and τ has the desired form

$$\tau = \frac{\alpha\tau}{\alpha} = \frac{a\tau' + b}{c\tau' + d}.$$

Note that $ad - bc = 1$, since both τ and τ' are both in the upper half-plane. Indeed, since the imaginary part of τ' is positive, the sign of the imaginary part of $\tau' = \frac{a\tau'+b}{c\tau'+d}$ is the same as the sign of $ad - bc$. This concludes the proof.

[1] I didn't really understand this part. I got the idea from *Glimpses of Algebra and Geometry* by Gabor Toth. Springer Science & Business Media, 2002. Pages 187-188. (link: `http://books.google.ca/books?id=U7sSXFIHcl0C&pg=187`)

[2] See Problem IX.2.7 on page 265 of *Complex Analysis* by Gamelin.

[3] Again, I somewhat understand this, but couldn't figure out how to show it.