Functional Analysis

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Contents

1 Lecture 1

(10 January 2014)

The main text for this course is [KREY89]. An additional good resources (and the traditional textbook for this course at U Calgary) is [CON90].

The final mark will be based on 5 assignments throughout the semester, as well as a final exam.

1.1 Normed Spaces

Definition 1.1. A *normed space* is a pair $(X, \|\cdot\|)$, where X is a vector space and $\|\cdot\| : X \to \mathbb{R}$ satisfying

i) $\|x\| \geq 0$ for all $x \in X$

ii) $\|x\| = 0$ if and only if $x = 0$

iii) $\|\alpha x\| = |\alpha| \, \|x\|$

iv) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

Remark 1.2. The norm must be continuous. We can define a metric by $d(x, y) = \|x - y\|$. We have

$$\big|\|x\| - \|y\|\big| \leq \|x - y\| \tag{1.1}$$

since $\|y\| = \|y - x + x\| \leq \|y - x\| + \|x\|$, which implies $\|y\| - \|x\| \leq \|y - x\|$. Similarly, swapping the roles of x and y, $\|x\| - \|y\| \leq \|x - y\|$. So $\big|\|x\| - \|y\|\big| \leq \|x - y\|$.

Equation (1.1) means that "if x is 'close' to y, then $\|x\|$ is 'close' to $\|y\|$" (namely, this is our concept of continuity).

Definition 1.3. A normed space is *Banach* if it is complete.

In this sense of *completeness* we mean that the space contains all of its limit points. That is, if a sequence $\{x_n\}$ in X is Cauchy, then it converges to something that is in X.

Example 1.4. i) The Euclidean norm on $\mathbb{R}^n$ or $\mathbb{C}^n$ given by

$$x = (\alpha_1, \ldots, \alpha_n) \qquad ||x|| = \sum_{i=1}^{n} |\alpha_i|^2.$$

Proof that this is a norm is easy (left as an exercize). The triangle inequality comes from the Cauchy-Schwartz inequality. This space is also complete.

ii) The space ℓ^p (sequence space) where $p \geq 1$ is a real number. (This is not a norm for $p < 1$). The space is defined by

$$\left\{ x = (\alpha_1, \alpha_2, \ldots) = \{\alpha_n\}_{n \in \mathbb{N}} \;\middle|\; \sum_{j=1}^{\infty} |\alpha_j|^p < \infty \right\}$$

and the norm is defined by $||x||_p = \left(\sum_{j=1}^{\infty} |\alpha_j|^p\right)^{1/p}$.

Proof of the triangle inequality for this norm uses the Hölder inequality and Minkowski inequality (homework).

iii) The space ℓ^∞

$$\left\{ x = (\alpha_1, \alpha_2, \dots) = \{\alpha_n\}_{n\in\mathbb{N}} \;\middle|\; \sup_j |\alpha_j| < \infty \right\}$$

and the norm is $||x|| = \sup_j |\alpha_j|$.

iv) $C[a,b]$ the space of continuous real-valued functions on the interval $[a,b]$. For $x \in C[a,b]$, $x : t \mapsto x(t)$, we can define the norm

$$||x|| = \max_{t\in[a,b]} |x(t)|.$$

Proposition 1.5. *$C[a,b]$ with the norm defined above is Banach.*

Proof. Let x_n be Cauchy, and let $\epsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that for $n, m > N$ we have $||x_n - x_m|| < \epsilon$. That is,

$$\max_{t\in[a,b]} |x_n(t) - x_m(t)| < \epsilon$$

and so $|x_n(t) - x_m(t)| < \epsilon$ for all $t \in [a,b]$. Thus, for a fixed t, the sequence $\{x_n(t)\}$ is Cauchy in $\mathbb{R}$, so the limit exists in $\mathbb{R}$. Define $x(t) = \lim_{n\to\infty} x_n(t)$. Then for $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n > N$

$$\max_{t\in[a,b]} |x_n(t) - x(t)| < \epsilon,$$

so $|x_n(t) - x(t)| < \epsilon$ for all $t \in [a,b]$. Thus $x_n(t)$ converges uniformly on $[a,b]$, so x is continuous. ☐

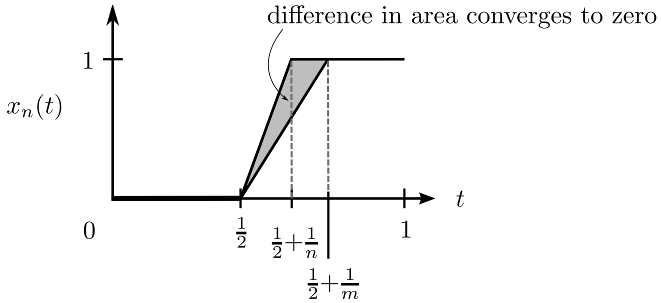
v) Let X be the space of continuous funcitons on $[0,1]$ with norm

$$||x|| = \int_0^1 |x(t)|dt.$$

This space is *not* complete. Indeed, we can define the sequence of continuous functions $x_n : [0,1] \to \mathbb{R}$ by

$$x_n(t) = \begin{cases} 0, & 0 \le t \le \frac{1}{2} \\ n(t - \frac{1}{2}), & \frac{1}{2} < t < \frac{1}{2} + \frac{1}{n} \\ 1, & \frac{1}{2} + \frac{1}{n} \le t \le 1. \end{cases}$$

We can visualize the Cauchy-ness of this sequence in the following diagram:



Indeed, we have $\|x_n - x_m\| \to 0$ as $n, m \to \infty$. But the limit of this sequence (the step function) is clearly not continuous, so it is not in $C[a,b]$.

The completion of this space is $L^1[0,1]$.

2 Lecture 2

(13 January 2014)

2.1 The space $L^p[a,b]$

Let $p \in \mathbb{R}$ $p \geq 1$ and consider the space X of continuous functions on the interval $J = [a,b]$, with the norm $||\cdot||$ defined by

$$||x|| := \left[\int_a^b |x(t)|^p\right]^{1/p}.$$

Along the lines of the example in the previous lecture, this space is not complete.

Definition 2.1. An *isometry* of normed vector spaces is a vector space isomorphism $T : X \to Y$ that preserves the norm, i.e. $||Tx||_Y = ||x||_X$ for all $x \in X$.

Theorem 2.2. *Let $(X, ||\cdot||)$ be a normed space. Then there exists a Banach space $\tilde{X}$ and an isometry $T : X \to W$ with $W \subset \tilde{X}$ such that W is dense in $\tilde{X}$ (i.e. for all $\tilde{x} \in \tilde{X}$ there exists a sequence $\{w_n\}$ in W such that $\lim_{n\to\infty} w_n = x$.)*

Furthermore, $\tilde{X}$ is unique up to isometry.

Proof (sketch). Consider an equivalence relation of Couchy sequences in X given by

$$\{x_n\} \sim \{y_n\} \iff \lim_{n\to\infty} ||x_n - y_n|| = 0.$$

(Exercise: show that this is an equivalence relation.) Let $\hat{X}$ be the set of all Cauchy sequences in X, and define $\tilde{X}$ as the set of equivalence classes in $\hat{X}$ with the equivalence relation given above.

(The rest of the proof may be seen in [KREY89]). □

3 Lecture 3

(15 January 2014)

3.1 The space $L^p[a,b]$ (continued)

1. $L^p[a,b]$ is the completion of the space of all continuous (real-valued) functions on $[a,b]$ with the norm
$$||x||_p = \left[\int_a^b |x(t)|^p\right]^{1/p}.$$
2. Two functions on $[a,b]$ are equivalent in $L^p[a,b]$ if they are the same *almost everywhere* (i.e. $f \sim g$ if $f(x) \neq g(x)$ for only countable many x.)
3. So $L^p[a,b]$ is the space of all equivalence classes of functions on $[a,b]$.

3.2 Properties of normed spaces

Theorem 3.1. *Let X be a Banach space and $Y \subset X$ a subspace. Then Y is complete ifand only if Y is closed in X.*

Proof. Suppose Y is complere. Then consider the closure $\bar{Y}$ of Y. An element $y \in \bar{Y}$. has a sequence $\{y_n\}$ in Y such that $y_n \to y$. Thus $\{y_n\}$ is Couchy. SInce Y is complete, $y \in Y$. Thus $\bar{Y} = Y$.

Now suppose that Y is closed. Let $\{y_n\}$ be a Couchy sequence in Y. Since X is complete, $\{y_n\}$ converges to an element y in X. But $y \in \bar{Y}$. So $s \in Y$ and thus y is complete. □

3.3 Linear Operators

Definition 3.2. A linear operator is a map T such that

i) the domain $\mathcal{D}(T)$ is a vector space

ii) for all $x, y \in \mathcal{D}(T)$ and all scalars $\alpha \in \mathbb{R}(\mathbb{C})$
$$T(x+y) = T(x) + T(y) \qquad T(\alpha x) = \alpha T(x).$$

Example 3.3. The following are some examples of linear operators.

1. Let X be the vector space of all polynomials on $[a,b]$, then the differential operator
$$Tx = x'$$
is a linear operator.
2. On the space $C[a,b]$ of continuous functions, the integration map $T : C[a,b] \to C[a,b]$ given by
$$Tx = \int_a^\tau x(\tau)d\tau$$
is a linear operator.
3. On $C[a,b]$ the operator $Tx(t) = tx(t)$ is a linear operator.

Remark 3.4. Properties of linear operators:

1. The range $\mathcal{R}(T)$ of a linear operator is a vector space.
2. If $\dim \mathcal{D}(T) = n < \infty$, then $\dim \mathcal{R}(T) \leq n$.
3. The null space of T (as a subspace of $\mathcal{D}(T)$) is a vector space

3.3.1 The inverse operator

Let X, Y be vector spaces and let $T : \mathcal{D}(T) \to Y$ be a linear operator, with $\mathcal{D}(T) \subset X$ and $\mathcal{R}(T) \subset Y$. Then

1. The inverse operator $T^{-1} : \mathcal{R}(T) \to \mathcal{D}(T)$ exists if and only if $Tx = 0$ implies $x = 0$.
2. If T^{-1} exists, it is a linear operator.
3. If $\dim \mathcal{D}(T) = n < \infty$ and T^{-1} exists, then $\dim \mathcal{R}(T) = n$.

3.3.2 Bounded linear operators

Definition 3.5. Let X, Y be normed vector spaces and $T : \mathcal{D}(T) \to Y$ a linear operator. Then T is *bounded* if there exists a real number c such that $||Tx|| \leq c||x||$ for all $x \in X$.

Definition 3.6. The *norm* of a bounded linear operator is defined as

$$||T|| := \sup_{x \in X,\, x \neq 0} \frac{||Tx||}{||x||} = \sup_{||x||=1} ||Tx||.$$

Note 3.7. i) $|Tx|| \leq ||T||\,||x||$

ii) $||T_1 T_2|| \leq ||T_1|||, ||T_2||$

Example 3.8. i) For the identity operator $T = I$, $||T|| = 1$. for the zero operator $T = 0$, $||T|| = 0$.

ii) Let X be the space of polynomials on $[0, 1]$ with norm $||x|| = \max_{t \in [0,1]} |x(t)|$. Then the derivative operator $Tx = x'$ is *unbounded*.

Indeed, consider the polynmials $x_n(t) = t^n$, then $||x_n|| = 1$ for all n. But

$$||Tx_n(t)|| = ||T(t^n)|| = ||nt^{n-1}|| = n,$$

so for all $c \in \mathbb{R}$ there is an $n \in \mathbb{N}$ such that $n > c$ such that

$$||Tx_n|| = n > c||x_n|| = c.$$

So T is unbounded.

iii) Consider the space $C[0, 1]$ of continuous functions on $[0, 1]$ with the same norm as above. Consider the linear operator

$$\begin{aligned} T :& C[0,1] \to C[0,1] \\ x \mapsto t & \int_0^1 x(t')dt'. \end{aligned}$$

This has norm

$$||Tx|| = \max_{t \in [0,1]} |Tx(t)| = \max_{t \in [0,1]} \left| t \int_0^1 x(t')dt' \right| = \left| \int_0^1 x(t')dt' \right| \leq \max_{t \in [a,b]} |x(t)| = ||x||,$$

So $||T|| \leq 1$. To show that $||T|| = 1$, take $x(t) = 1$.

4 Lecture 4

(17 January 2014)

Theorem 4.1. *Any linear operator on a finite dimensonal normed space is bounded.*

Lemma 4.2. *Let $\{x_1, \ldots, x_n\}$ be a linearly independent set of vectors in a normed space X. Then there exists a constant $c > 0$ such that $|| \sum_{i=1}^n \alpha_i x_i|| \geq c \sum_{i=1}^n |\alpha_i|$ for all $\alpha_1, \ldots, \alpha_n$ in $\mathbb{C}$.*

Proof. Consider a tuple $(\alpha_1, \ldots, \alpha_n)$ in $\mathbb{C}^n$ and let $\beta_i = \dfrac{\alpha_i}{\sum_{i=1}^n |\alpha_i|}$ (assuming not all α_i are zero). Suppose that there is no such c such that $|| \sum_{i=1}^n \beta_i x_i|| \geq c$ for all x. Then there exists a set $(\beta_i^{(m)}) \subset \mathbb{C}$ such that $\sum_{i=1}^n |\beta_i^{(m)}| = 1$ for all $m \in \mathbb{N}$ and, for $y^{(m)} = \beta_i^{(m)} x_1 + \beta_2^{(m)} x_2 + \cdots + \beta_n^{(m)} x_n$, we have $||y^{(m)}|| \to 0$ as $m \to \infty$.

Consider the sequence $(\beta_1^{(1)}, \beta_1^{(2)}, \ldots)$. This sequence bounded since $|\beta_j^{(m)}| \leq 1$, and thus has a convergent subsequence. Since *each* sequence $(\beta_j^{(m)})$ is bounded, we can pick a subsequence of m's such that each sequence $\beta_j^{(m)} \to \beta_j$ converges for each j. Then the corresponding subsequence of $y^{(m)}$ converges

$$y^{(m)} \to y = \sum_{i=1}^n \beta_i x_i.$$

Thus $||y|| = \lim_{m\to\infty} ||y^{(m)}|| = 0$ and so $y = 0$. This is a contradiction to $y \neq 0$. Indeed, at least one β_j is nonzero, since $|\sum \beta_j| = 1$, and the set $\{x_1, \ldots, x_n\}$ is linearly independent, so we have that $y = \sum_i \beta_i x_i$ cannot be the zero vector . □

Proof of Theorem 4.1. Let $x \in X$, $x = \alpha_1 e_1 + \cdots + \alpha_n e_n$, where $\{e_1, \ldots, e_n\}$ is a basis of X. Let T be a linear operator with domain X. Then

$$\begin{aligned} \|Tx\| = \left\| \sum_{i=1}^n \alpha T e_i \right\| &\leq \sum_{i=1}^n |\alpha_i| \|Te_i\| \\ &\leq \max_{i=1,\ldots,n} \|Te_i\| \sum_{i=1}^n |\alpha_i| \\ &\leq \max_{i=1,\ldots,n} \|Te_i\| \frac{1}{c} \|x\|, \end{aligned}$$

where in the last inequality we make use of the Lemma. Hence we have that

$$\frac{\|Tx\|}{\|x\|} \leq \frac{1}{c} \max_{i=1,\ldots,n} \|Te_i\|.$$

Note that the right-hand-side does not depend on x! So T is bounded. □

Theorem 4.3. *Let $T : \mathcal{D}(T) \to Y$ be a linear operator with $\mathcal{D}(T) \subset X$, where X and Y are normed spaces. Then*

i) T is continuous if and only if T is bounded

ii) If T is continuous at a single point then it is continuous.

Proof. We prove part (i). The proof of part (ii) follows directly.

i) First suppose that T is bounded. Then for any $x, y \in \mathcal{D}(T)$

$$||Tx - Ty|| = ||T(x - y)|| \leq ||T|| \, ||x - y||.$$

So if $||x-y||$ is 'almost zero' then $||Tx-Ty||$ is 'almost zero', i.e. Tx is close to Ty whenever x is close to y. This is the definition of the continuity of T.

Now suppose that T is continuous. Then for all $\epsilon > 0$ there exists a $\delta > 0$ such that for $x, y \in \mathcal{D}(T)$ the condition $||x-y|| < \delta$ implies $||Tx-Ty|| < \epsilon$. Take any $z \neq 0$ in $\mathcal{D}(T)$ and $x = y + \frac{\delta}{2}\frac{z}{||z||}$. Then

$$||x-y|| = ||\tfrac{\delta}{2}\tfrac{z}{||z||}|| = \tfrac{\delta}{2} < \delta$$

implies

$$||Tx-Ty|| = \left|\left|T\left(\frac{\delta z}{2||z||}\right)\right|\right| < \epsilon$$

and thus $\frac{||Tz||}{||z||} < \frac{2\epsilon}{\delta}$ for all nonzero $z \in \mathcal{D}(T)$, so T is bounded.

ii) The proof is simple and follows from linearity of T.

(My proof: Suppose T is continuous at a point $x \in \mathcal{D}(T)$. Then for all $\epsilon > 0$ there exists a $\delta > 0$ such that $||x-y|| < \delta$ implies $||T(x-y)|| < \epsilon$. Consider $z \in \mathcal{D}(T)$ and let $w \in \mathcal{D}(T)$ such that $||z-w|| < \delta$. Define $y = w - z + x$ such that $z - w = x - y$. Then

$$||z-w|| = ||x-y|| < \delta \qquad \text{implies} \qquad ||Tz-Tw|| = ||Tx-Ty|| < \epsilon,$$

so T is continuous at all $z \in \mathcal{D}(T)$.)

□

Theorem 4.4. *Let $T : \mathcal{D}(T) \to Y$ be a bounded linear operator in Banach spaces X and Y. Then T has a* linear extension $\tilde{T} : \overline{\mathcal{D}(T)} \to Y$ *such that* $||\tilde{T}|| = ||T||$.

A linear extension of a linear operator T is a linear operator $\tilde{T}$ such that the restriction $\tilde{T}|_{\mathcal{D}(T)} = T$.

Proof. We first need to define $\tilde{T}$ before we show that it is linear and bounded. Let $x \in \overline{\mathcal{D}(T)}$. Then there exists a sequence (x_n) in $\mathcal{D}(T)$ such that $x_n \to x$. Note that

$$||Tx_n - Tx_m|| \leq ||T||\,||x_n - x_m||$$

ando so the sequence $\{Tx_n\}$ is Cauchy in Y. Since Y is complete, there exists a $y \in Y$ such that $Tx_n \to y \in Y$. Define $\tilde{T}x = y$. We need to show that this is well-defined. Indeed, if there is another sequence $\{z_n\}$ in $\mathcal{D}(T)$ that converges to x, then the sequence $\{Tz_n\}$ also converges in Y, say to some element $y' \in Y$. So

$$||Tx_n - Tz_n|| \leq ||T||\,||x_n - z_n||$$

but $||x_n - z_n|| \to 0$ as $n \to \infty$.

(Linearity is easy to show and is left as an exercise).

For boundedness of $\tilde{T}$, $||\tilde{T}x|| \leq ||T||\,||x||$ since

$$||Tx_n|| \leq ||T||\,||x_n||$$

and

$$||y|| = \lim_{n\to 0} ||Tx_n|| \leq ||T||\,||x||.$$

Furthermore, $||\tilde{T}|| \leq ||T||$. But this implies $||\tilde{T}|| = ||T||$.

To show that $\tilde{T}$ is really an extension, consider an element $x \in \mathcal{D}(T)$ and the constant sequence $\{x\}$. Then clearly $\tilde{T}x = Tx$, since the norm of an operator is the supremum.

□

Denote by $\mathcal{B}(X, Y)$ the set of all bounded linear operators fron X to Y.

Proposition 4.5. *The vector space $\mathcal{B}(X, Y)$ is a normed space with norm defined in the usual way*

$$||T|| = \sup_{||x||=1} ||Tx||.$$

Intreresting fact: if Y is Banach then $B(X, Y)$ is Banach and this does not depend on X! (proof next time)

5 Lecture 5

(20 January 2014)

(These notes copied graciously from German Luna in my absence at the lecture.)

Theorem 5.1. *Let X and Y be normed vector spaces. If Y is Banach, then $\mathcal{B}(X, Y)$ is Banach.*

Proof. Let $\{T_n\} \subseteq \mathcal{B}(X, Y)$ be Cauchy. Then for every $\epsilon > 0$ there exists an N such that for all $m, n \geq N$, we have $||T_n - T_m|| < \epsilon$ and thus

$$||T_n x - T_m x|| < \epsilon ||x||$$

for a fixed $x \in X$. So $\{T_n x\}$ is a Cauchy sequence in Y. Define T as the limit $T = \lim_{n\to\infty} T_n$ given by

$$Tx = \lim_{n\to\infty} T_n x.$$

This operator is clearly linear. We must also show that this operator is bounded. Consider

$$||T_n x - Tx|| = \lim_{m\to\infty} ||T_n x - T_m x|| < \epsilon ||x||$$

(by continuity of the norm) for sufficiently large n. But this implies $||T_n x - Tx|| < \epsilon ||x||$, and thus

$$\frac{||(T_n - T)x||}{||x||} < \epsilon$$

for all $x \neq 0$. Thus $||T_n - T|| < \epsilon$ and therefore $T_n - T$ is bounded. Since T_n and $T_n - T$ are bounded, so is T. □

5.1 (Linear) Functionals

Definition 5.2. A (linear) functional is a (linear) operator $f : X \to K$ (where X is a subset of a normed space and K is the underlying field, i.e. $\mathbb{R}$ or $\mathbb{C}$).

Example 5.3.
1. The norm $|| \cdot || : X \to \mathbb{R}$ on any space X is a nonlinear functional. (It is sublinear, actually, since $||\alpha x + (1-\alpha)y|| \leq |\alpha| ||x|| + |(1-\alpha)| ||y||$.)
2. Given a fixed $y \in X$, the inner product $f_y(x) = \langle x, y \rangle$ is a linear functional with norm $||f_y|| = ||y||$.
3. Definite integrals are linear functionals

$$\int_a^b \cdot \, dt : C[a, b] \to \mathbb{R}$$

with norm $|| \int_a^b \cdot \, dt|| = |b - a|$.

Exercise. *Determine the norms of the operators in the examples above.*

Definition 5.4. Let X be a normed space. The *dual space* of X, denoted by X', consists of the set of all bounded linear functionals on X, with norm given by the standard operator norm. (Note that X^* includes the unbounded operators.)

Proposition 5.5. *X' is Banach for any normed space X.*

Proof. Follows from Theorem 5.1 by taking $Y = \mathbb{R}$ or $\mathbb{C}$. □

Example 5.6. $(\mathbb{R}^n)' = (\mathbb{R}^n)^* \simeq \mathbb{R}^n$. This is because all linear operators on finite dimensional spaces are bounded. We identify linear functionals on $\mathbb{R}^n$ as $f(x) = v \cdot x$ with norm $||f|| = ||v||$.

Example 5.7. $(\ell^1)' \simeq \ell^\infty$. The *Schauder basis* of ℓ^1 consists of the sequences

$$e_k = \{\delta_{k,j}\} \qquad \text{where} \qquad \delta_{k,j} = \begin{cases} 1, & k = j \\ 0, & k \neq j. \end{cases}$$

Then every $x \in \ell^1$ can be written as $x = \sum_{k=1}^{\infty} \alpha_k e_k$.

For every $b \in \ell^\infty$ there exists a $g \in (\ell^1)'$ such that $g(x) = \sum_{k=1}^{\infty} \alpha_k \beta_k$, where $b = \{\beta_k\}$. Then

$$|g(x)| \leq \sum_{k=1}^{\infty} |\alpha_k| \, |\beta_k| \leq \sup_k |\beta_k| \cdot \sum_{k=1}^{\infty} |\alpha_k|.$$

So $|g(x)| \leq ||b|| \, ||x||$. Thus g is a bounded linear functional.

Conversely let $f \in (\ell^1)'$, then for $x = \sum \alpha_k e_k$

$$f(x) = \sum_{k=1}^{\infty} \alpha_k f(e_k).$$

Denote $\beta_k = f(e_k) \in \mathbb{R}$, then

$$|\beta_k| = |f(e_k)| \leq ||f|| \, ||e_k|| = ||f|| < \infty,$$

and $\sup_k |\beta_k| < \infty$. So $b = \{\beta_k\}$ is in ℓ^∞, i.e. $||b||_\infty < \infty$. Note that, by the above reasoning, $||b||_\infty \leq ||f||$ since $||b||_\infty = \sup_k |\beta_k|$. But $||f|| \leq ||b||_\infty$ (why?). Thus $||f|| = ||b||_\infty$. Since it preserves norms, this is an isomorphism of normed spaces.

Exercise. *Show that* $(\ell^p)' \simeq \ell^q$ *for* $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

6 Lecture 6

(22 January 2014)

Since we have shown in the last lecture that $(\ell^p)' \simeq \ell^q$, we know that ℓ^q is complete since the dual space of any normed space is complete. This follows also for ℓ^1 and ℓ^∞. Similarly, this can be shown for the spaces L^p, since $(L^p)' = L^q$ (to show this is an exercise).

6.1 Hilbert spaces

Definition 6.1. An *inner product space* is a vector space X with an *inner product*, i.e. a mapping

$$\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R} \text{ or } \mathbb{C}$$

into the base scalar field with the following properties

1. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
2. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
3. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (which implies $\langle x, \alpha y = \overline{\alpha} \langle x, y \rangle$)
4. $\langle x, x \rangle \geq 0$
5. $\langle x, x \rangle = 0$ if and only if $x = 0$.

Remark 6.2. We have the following hierarchy

$$\text{(inner product spaces)} \subset \text{(normed spaces)} \subset \text{(metric spaces)}.$$

Remark 6.3. Any inner product space can be turned into a normed space by defining the norm

$$||x|| = \sqrt{\langle x, x \rangle}.$$

Definition 6.4. A *Hilbert space* is a complete inner product space. It is also a Banach space with norm given by the previous remark.

Example 6.5. i) The euclidean spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ with the standard inner products are inner product spaces.

ii) The spaces of matrices $n \times m$-matrices $\mathbb{C}^{n \times m}$ is a inner product space with the Hilbert-Schmidt inner product

$$\langle A, B \rangle = \mathrm{Tr}(AB^*)$$

iii) The space $L^2[a, b]$. For $f, g \in L^2$, $\langle f, g \rangle = \int f(t)\overline{g(t)} dt$

iv) The space ℓ^2. For $x = \{x_n\}$, $y = \{y_n\}$, $\langle x, y \rangle = \sum_{n=1}^{\infty} x_n \overline{y_n}$.

v) ℓ^p is *not* a Hilbert space for $p \neq 2$.

vi) $C[a, b]$ (with the max norm) is not a Hilbert space.

How does one prove that a space is *not* a Hilbert space? The parallelogram identity! If a norm $||x||$ comes from an inner product, then it satisties

$$||x + y||^2 + ||y - x||^2 = 2 \left(||x||^2 + ||y||^2 \right). \tag{6.1}$$

Claim 1. ℓ^p is not a Hilbert space for $p \neq 2$.

Proof. Consider the sequences $x = (1, 1, 0, 0, \dots)$ $y = (1, -1, 0, 0, \dots)$. Then

$$x + y = (2, 0, 0, 0, dots) \qquad y - x = (0, -2, 0, 0, 0, \cdots),$$

so $||x + y||_p = 4 = ||y - x||_p$ and the LHS of eq. (6.1) is equal to 8 in this case. But

$$||x||_p^2 = \left[(1^p + 1^p)^{1/p}\right]^2 = 2^{2/p} = ||y||_p,$$

so $2\left(||x||_p^2 + ||y||_p^2\right) = 4 \cdot 2^{2/p} = 8$ is satisfied if and only if $p = 2$.

□

Proposition 6.6. *Properties of inner product spaces:*

1. *The Schwarz inequality holds*
$$|\langle x, y\rangle| \leq ||x||\,||y|| \tag{6.2}$$
with equality if and only if x and y are linearly independent.

2. *The inner product is a continuous function. That is, if $x_n \to x$ and $y_n \to y$, then $\langle x_n, y_n\rangle \to$ $\langle x, yrangle$.*

   Proof.
$$\begin{aligned}|\langle x_n, y_n\rangle| &\leq |\langle x_n, y_n\rangle - \langle x, y_n\rangle| + |\langle x_n, y_n\rangle - \langle x, y\rangle| \\ &= |\langle x_n - x, y_n\rangle| + |\langle x, y_n - y\rangle| \\ &\leq ||x_n - x||\,||y_n|| + ||x||\,||y_n - y|| \to 0\end{aligned}$$
   as $n \to \infty$. □

3. (***Completion***) *As with Banach spaces, there is a unique completion (up to isomporphism), where an* inner product space isomorphism $T : X \to Y$ *is a bijective linear operator that preserves the inner product*
$$\langle Tx, Ty\rangle = \langle x, y\rangle.$$

4. (***Subspaces***) *A subspace $Y \subset H$ of a Hilbert space is complete if and only if it is closed in H.*

Remark 6.7. In Hilbert spaces we have the concept of *orthogonality*, which we don't necessarily have in Banach spaces. That is, $x \perp y$ (x is orthogonal to y) if $\langle x, y\rangle = 0$.

Theorem 6.8 (Minimizing vector theorem)**.** *Let X be an inner product space and $M \neq \emptyset$ a convex subset of X which is complete. (By convexity of M, we mean that for all $x, y \in M$ and $t \in [0, 1]$, the convex combination $tx + (1 - t)y \in M$ is in M.) Then for any $x \in X$ there exists a* unique *$y \in M$ such that*
$$\delta = \inf_{y' \in M} ||x - y|| = ||x - y||.$$

This is only true in Hilbert spaces, not necessarily normed spaces in general.

Proof. First we show existence. By definition of inf, there exists a sequence $\{y_n\}$ in M such that

$$\lim_{n\to\infty} ||x - y_n|| = \delta.$$

For simplicity, we instead work with the sequence $z_n \equiv y_n - x$ such that $\delta_n \equiv ||z_n||$ and $\delta_n \to \delta$. Then we have

$$||y_n - y_m||^2 = ||z_n - z_m||^2 = -||z_n + z_m||^2 + 2\left(||z_n||^2 + ||z_m||^2\right)$$

by the parallelogram identity. This is equal to

$$-||y_n + y_m - 2x||^2 + 2(\delta_n^2 + \delta_m^2) = -4||x - \tfrac{1}{2}(y_n + y_m)||^2 + 2(\delta_n^2 + \delta_m^2).$$

But $\frac{1}{2}(y_n + y_m) \in M$ by convexity of M, so $||x - \frac{1}{2}(y_n + y_m)|| \le \delta$ from the infimum. Thus, we have

$$||y_n - y_m||^2 \le -4\delta^2 + 2(\delta_n^2 + \delta_m^2) \to 0$$

as $m, n \to \infty$.

We need to show that $\delta = ||x - y||$. For $\{y_n\}$ Cauchy, $y_n \to y \in M$, we have $||x - y|| \ge \delta$. But

$$||x - y|| \le ||x - y_n|| + ||y_n - y||$$

for all n by the triangle inequality. This implies $||x - y|| \le \delta + 0$, so $||x - y|| = \delta$.

For uniqueness, suppose there exists a $w \in M$ such that $\delta = ||x - y|| = ||x - w||$. We want to show that $||y - w|| = 0$. Then, by the parallelogram identity, we have

$$\begin{aligned}
||y - w||^2 &= ||(y - x) - (w - x)||^2 \\
&= -||y + w - 2x||^2 + 2\left(||y - x||^2 + ||w - x||^2\right) \\
&= -4||\tfrac{1}{2}(y + w) - x||^2 + 2(\delta^2 + \delta^2) \\
&\le -4\delta^2 + 4\delta^2 \\
&= 0,
\end{aligned}$$

since $\frac{1}{2}(y + w) \in M$, and thus $||\frac{1}{2}(y + w) - x|| \ge \delta$. □

Corollary 6.9. *If M is a complete subspace of X, and y is the unique closest element in M to x from the above theorem, then the vector $z = x - y$ is orthogonal to M (i.e. $\langle z, m\rangle = 0$ for all $m \in M$), denoted $z \perp M$.*

Proof. Suppose $z \not\perp M$. Then there exists a nonzero element $y_0 \in M$ such that $\langle z, y_0\rangle = \beta \ne 0$. Note that $||z - \alpha y_0||^2 \ge \delta^2$ for all $\alpha \in \mathbb{R}(\mathbb{C})$, since $z - \alpha y_0 = x - y - \alpha y_0$ and $y + \alpha y_0$ is in M (it is a subspace). Then, since $||z|| = \delta$,

$$\begin{aligned}
||z - \alpha y_0||^2 &= \langle z - \alpha y_0, z - \alpha y_0\rangle \\
&= ||z||^2 - \alpha\langle y_0, z\rangle - \bar{\alpha}\langle z, y_0\rangle + |\alpha^2|\,||y_0|| \\
&= \delta^2 - \alpha\bar{\beta} - \bar{\alpha}\beta + |\alpha|^2||y_0||^2 \\
&= \delta^2 - \alpha\bar{\beta} - \bar{\alpha}\left(\beta - \alpha||y_0||^2\right) = (*).
\end{aligned}$$

For $\alpha = \frac{\beta}{||y_0||^2}$, we get

$$(*) = \delta^2 - \left(\frac{\beta}{||y_0||^2}\right)^2 < \delta^2$$

since $\beta \ne 0$.

□

7 Lecture 7

(24 January 2014)

7.1 Direct sum

Definition 7.1. A vector space X is saod to be a *direct sum* of two subspaces Y and Z of X, written $X = Y \oplus Z$ if each $x \in X$ has a unique representation $x = y + z$ for $y \in Y$ and $z \in Z$.

Theorem 7.2. *Let Y be a closed supspace of a Hilbert space H. Then $H = Y \oplus Y^\perp$, where*

$$Y^\perp = \{z \in H \mid \langle z, y\rangle = 0 \, \forall y \in Y\}$$

is the orthogonal complement to Y.

Proof. Note that $Y = \bar{Y} \subset H$ implies that Y is complete, so Y itself is a Hilbert space. Take $x \in H$, then there exists a $y \in Y$ such that $z \equiv x - y \in Y^\perp$. Then x decomposes as $x = y + z$. To show that this decomposition is unique, suppose that $x = y + z = y' + z$ for some $y \in Y$ and $z \in Y^\perp$. Then $y - y' = z' - z$, but Y and and Z are subspaces so $y - y' \in Y$ and $z' - z \in Y^\perp$. This implies $y' - y = z' - z = 0$.[1] □

Hilbert spaces have this nice notion of orthogonality that give then many properties familiar to finite dimensional vector spaces.

Definition 7.3. For a closed subspace $Y \subset H$ of a Hilbert space H, the *orthogonal projection onto* Y is the linear operator $P_Y : H \longrightarrow Y$ defined by

$$P_Y x = y,$$

where $y \in Y$ is the unique y from the previous theorem.

Remark 7.4. Properties of the orthogonal projection.

i) The restriction of P_Y to Y is the identity on Y, i.e. $P_Y|_Y = I_Y$.

ii) The nullspace of P_Y is $Y^\perp$, i.e. $\mathcal{N}(P_Y) = Y^\perp$.

Lemma 7.5. *If $Y \subset X$, where X is an inner product space, then*

1. $Y \subset Y^{\perp\perp}$
2. *if $X = H$ is a Hilbert space and $Y = \bar{Y} \subset H$, then $Y = Y^{\perp\perp}$.*

Proof. 1. Take $y \in Y$, then $y \perp Y^\perp$ and so $y \in \left(Y^\perp\right)^\perp = Y^{\perp\perp}$.

2. Let $x \in Y^{\perp\perp} \subset H$. Then $x = y + z$ for some $y \in Y$ and $z \in Y^\perp$ and $z = x - y \in Y^{\perp\perp}$. Since $z \in Y^\perp \cap Y^{\perp\perp}$, $z = 0$ and so $x = y \in Y$. Thus $Y^{\perp\perp} \subset Y$ and therefore $Y = Y^{\perp\perp}$.

□

Note that this implies also that $H = Y \oplus Y^\perp = Y^\perp \oplus Y^{\perp\perp}$.

Definition 7.6. Given a subset $M \neq \emptyset$ of a vector space X, the *span* of M is the set $\operatorname{span}(M)$ of all *finite* linear combinations of vectors in M.

Lemma 7.7. *If $M \neq \emptyset$ is any subset of a Hilbert space H, then $\overline{\operatorname{span} M} = H$ if and only if $M^\perp = \{0\}$.*

[1] Note: $Y \cap Y^\perp = 0$. Indeed, suppose $x \in Y \cap Y^\perp$, then $\langle x, x\rangle = 0$ so $x = 0$.

Proof. Suppose $\overline{\operatorname{span} M} = H$ and let $x \in M^\perp \subset \overline{\operatorname{span} M} = H$. Then $x = \lim x_n$ for some Cauchy sequence $\{x_n\}$ in span(M). But $x \in M^\perp$ and $M^\perp \perp \operatorname{span} M$, so $0 = \langle x, x_n\rangle \longrightarrow \langle x, x\rangle$ implies $x = 0$.

Now suppose $M^\perp = \{0\}$, then $(\operatorname{span} M)^\perp = \{0\}$. Define $Y = \operatorname{span} M$. Then $\overline{Y} \subset H$ is a subspace and $H = \overline{Y} \oplus (\overline{Y})^\perp = Y \oplus \{0\} = \overline{Y}$. Thus $H = \overline{\operatorname{span} M}$ □

Theorem 7.8 (Bessel inequality). *Let $\{e_k\}$ be an orthonormal sequence in X. Then for every $x \in X$*

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \leq ||x||^2.$$

Proof. Fix $x \in X$, and define Y_n as the subspaces $Y_n = \operatorname{span}\{e_1, \dots, e_n\}$. Define

$$y_n = \sum_{k=1}^{n} \langle x, e_k\rangle e_k$$

then $||y_n||^2 = \langle y, y\rangle = \sum_{k=1}^n |\langle x, e_k\rangle|^2$. Set $z_n = x - y_n$.
Claim. $z_n \perp y_n$

Proof of claim. $\langle z_n, y_n\rangle = \langle x, y_n\rangle - \langle y_n, y_n\rangle = 0$, and

$$\langle x, y_n\rangle = \left\langle x, \sum_{k=1}^{n} \langle x, e_k\rangle e_k \right\rangle = \sum_{k=1}^{n} \overline{\langle x, e_k\rangle} \langle x, e_k\rangle = \sum_{k=1}^{n} |\langle x, e_k\rangle|^2 = ||y_n||^2.$$

□

Thus $x = y + z$ and so $||x||^2 = ||y_n||^2 + ||z_n||^2$, since $\langle x, x\rangle = \langle y + z, y + z\rangle$. So for all n

$$||z_n||^2 = ||x||^2 - ||y_n||^2 = ||x||^2 - \sum_{k=1}^{n} |\langle x, e_k\rangle|^2,$$

and thus $\sum_{k=1}^n |\langle x, e_k\rangle|^2 \leq ||x||^2$.

□

Corollary 7.9. *If X is an inner product space, then any $x \in X$ can have at most countable many nonzero* Fourier coefficients *$\langle x, e_k\rangle$ with respect to an orthonormal family $(e_\kappa) \subset X$ indexed by some (not necessarily countable) set $\kappa \in I$.*

Proof. For any integer $m \in \mathbb{N}$, define

$$V_m \equiv \left\{ \kappa \in I \,\middle|\, |\langle x, e_\kappa\rangle| > \frac{1}{m} \right\}.$$

Then $|V_m| < \infty$ (i.e. the cardinality is finite) due to the Bessel inequality. Define $V = \bigcup_{m=1}^{\infty} V_m$ □

8 Lecture 8

(27 January 2014)

8.1 Series of orthonormal sequences

Here, $\{e_k\}$ is always an orthonormal set of vectors in a Hilbert space H.

Definition 8.1. We say that $\sum_{k=1}^{\infty} \alpha_k e_k$ *converges* (or *exists*) if there exists $s \in H$ such that

$$\lim_{n\to\infty} \Big|\Big| \sum_{k=1}^{n} \alpha_k e_k - s \Big|\Big| = 0.$$

We write $s = \sum_{k=1}^{\infty} \alpha_k e_k$.

Theorem 8.2 (Convergence)**.** *Let $\{\alpha_n\}$ be a sequence of scalars.*

1. $\sum_{k=1}^{\infty} \alpha_k e_k$ *converges if and only if* $\sum_{k=1}^{\infty} ||\alpha_k||^2$ *converges.*
2. *If* $\sum_{k=1}^{\infty} \alpha_k e_k$ *converges, say to $x \in H$, then $\alpha_n = \langle x, e_n \rangle$, i.e.* $x = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$
3. *For all $x \in H$,* $\sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$ *converges.*

Proof. Define the partial sums $s_n = \sum_{k=1}^{n} \alpha_k e_k$ and $\sigma_n = \sum_{k=1}^{n} ||\alpha_k||^2$.

1. For $n, m \in \mathbb{N}$ with $n \geq m$, we have

   $$||s_n - s_m||^2 = ||\alpha_{m+1} e_{m+1} + \cdots + \alpha_n e_n|| = |\alpha_{m+1}|^2 + \cdots + |\alpha_n|^2 = |\sigma_n - \sigma_m|,$$

   so $\{s_n\}$ converges if and only if $\{\sigma_n\}$ does.
2. For $n > m$ there is nothing to prove. So suppose $n \leq m$. Then

   $$\langle s_m, e_n \rangle = \alpha_n,$$

   so in the limit of $m \to \infty$,

   $$\langle x, e_n \rangle = \alpha_n.$$
3. We have $\sum_{n=1}^{\infty} |\langle x, e_k \rangle|^2 \leq ||x||^2$, so this series converges. Then part 3 follows from part 1.

□

Definition 8.3. A *total set* in a normed space X is a subset $M \subset X$ whose span is dense in X. A *total orthonormal set* in an inner product space X is an orthonormal set M which is total in X.

Given a total set M in X, we have $\overline{\operatorname{span} M} = X$. In particular, X itself is total in X. But total orthonormal sets will have cardinality limited by the dimension of X. We will see that all Hilbert spaces have a total orthonormal set.

Theorem 8.4 (Totality I). *Let M be a subset of an inner product space X.*

1. *If M is total in X, then $x \perp M$ implies $x = 0$ (i.e. $M^{\perp} = \{0\}$).*
2. *If X is a Hilbert space, then $x \perp M$ implies $x = 0$ if and only if M is total in X.*

Proof. Let H be the completion of X. Hence, $X \subset H$ is a dense subspace of H.

1. Since $\operatorname{span} M$ is dense in X it is also dense in H. Thus, $M^{\perp} = \{0\}$.
2. (We have proved this before in a previous lecture, without using the 'total' terminology. As an exercise, find which previous theorem corresponds to this statement.)

□

The converse of part 1 of Theorem 8.4 is not true. It does hold for most examples that we will deal with, but there are complex examples of inner product spaces for which it does not hold. We will not consider such spaces in this course.

Theorem 8.5. *In every Hilbert space H there exists a total orthonormal set. If H is separable, then there exists a countable set of vectors $C \subset H$ such that $\overline{C} = H$. By the Gram-Schmidt process, we can make C into an orthonormal set.*

Example 8.6. Note that ℓ^p (for $1 \leq p < 0$) are separable, but ℓ^{∞} is *not* separable. The intuition behind this has to do with the supremum norm in ℓ^{∞}. There are 'many more' sequences in ℓ^{∞} than in ℓ^p.

8.2 Zorn's Lemma

In order to prove Theorem 8.5, we need to make use of Zorn's lemma. We need to introduce some basic concepts in set theory.

Definition 8.7. A *partially ordered set* is a set M with a binary operation "$\leq$" satisfying

1. $a \leq a$ for all $a \in M$
2. $a \leq b$ and $b \leq a$ implies $a = b$ (antisymmetry)
3. $a \leq b$ and $b \leq c$ implies $a \leq c$ (transitivity).

Example 8.8.

The power set $\mathcal{P}(X)$ is a partially ordered with inclusion as the partial order (i.e. "$\leq$" ="$\subset$"). This is indeed a *partial* order. Consider $X = \{1, 2\}$, and $\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. We have

$$\{1\} \not\leq \{2\} \quad \text{and} \quad \{2\} \not\leq \{1\},$$

so we say that these elements are *incomparable* under the partial order $\subset$ in $\mathcal{P}(X)$. In fact, this is the only partial order that can be defined on the power set.

If $a \leq b$ or $b \leq a$, then a and b are said to be *comparable*.

Definition 8.9. A *chain* is a subset $M \subset X$ of a partially ordered set with the property that every two elements in M are comparable.

Definition 8.10. Let $C \subset M$ be a subset of a chain M in X. An *upper bound in M of C* is an element element $u \in M$ such that $x \leq u$ for all $x \in C$. A *maximal element* of M is an element $m \in M$ such that $m \leq x$ implies $x = m$ (i.e. there is nothing 'above' m in the set).

Here, we take Zorn's lemma as an axiom. But historically it follows from the Axiom of Choice. (It is in fact equivalent to AoC). We won't worry too much about this statement, except when we use it to prove Theorem 8.5 and the Hahn-Banach theorem later.

Lemma 8.11 (Zorn's Lemma (Zorn's Axiom)). *Let $M \neq \emptyset$ be a partially ordered set. Suppose that every chain $C \subset M$ has an upper bound. Then M has at least one maximal element.*

Note that the chains do not need to be countable, and that the maximal element in M does not have to be unique (since M itself is not a chain).

8.3 Back to the theorem

Proof of Theorem 8.5. If $H = \{0\}$, then this is trivially true. So suppose $H \neq \{0\}$. Let M be the set of all orthonormal subsets of H. Since $H \neq \{0\}$, there is an element $x \in H$ with $x \neq 0$. Thus M is nonempty, since

$$\left\{ \frac{x}{||x||} \right\} \subset H$$

is in M. The partial order on M is defined by inclusion. Every chain $C \subset M$ has an upper bound. Indeed, consider

$$u = \bigcup_{c \in C} c \in M.$$

By Zorn's lemma there exists a maximal element $F \in M$.

Claim 2. F is total in H.

Proof of claim. If F is not total, then there exists a $z \in F^{\perp}$, $z \neq 0$. Then $F \subset F \cup \left\{ \frac{z}{||z||} \right\} \in M$. □

□

9 Lecture 9

(29 January 2014)

Today we will introduce the idea that every vector space has a basis. This is certainly obvious for finite dimensional vector spaces, but we must use Zorn's lemma for the general infinite dimensional case. Recall that a basis of a vector space X is a linearly independent set of vectors in X that span X. A basis that is not constructed but rather shown to exist via Zorn's lemma is called a *Hamel* basis.

Theorem 9.1. *Every vector space $X \neq \{0\}$ has a* (Hamel) *basis.*

Proof. Let M be the set of all linearly independent subsets in X. As before, we can induce a partial order on M by set inclusion. Clearly, $M \neq \emptyset$ since $X \neq \{0\}$. Every chain $C \subset M$ has an upper bound (namely, the union of all elements in C). By Zorn's lemma, there exists a maximal element B in M. We want to show that $\operatorname{span} B = X$.

Let $Y = \operatorname{span} B$. If $Y \neq X$ then there exists a $z \in X$ which is not in Y. This implies that $B \cup \{z\}$ is a linearly independent set, which contradicts the maximality of B. □

Theorem 9.2 (Totality II)**.** *An orthonormal set M in a Hilbert space H is total if and only if for all $x \in H$*

$$||x||^2 = \sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2, \tag{*}$$

where $\{e_k\}_{k=1}^{\infty} = \{e \in M \mid \langle x, e\rangle \neq 0\}$.

Proof. If M is not total in H, then $M^\perp \neq \{0\}$ by the first totality theorem. So there exists a nonzero $z \in M^\perp \subset H$. That is, $\langle z, e_k\rangle = 0$ for all $e_k \in M$. But $||z||^2 \neq 0$. If (*) holds for $x \in H$ then M must be total.

Conversely, if M is total in H, take $x \in H$. Then there is a sequence of vectors in $\operatorname{span} M$ that converges to x. Furthermore

$$y = \sum_{k=1}^{\infty} \langle x, e_k\rangle e_k$$

converges in X. It remains to show that $y = x$. Note that

$$\langle x - y, e_j\rangle = \langle x, e_j\rangle - \underbrace{\langle y, e_j\rangle}_{\langle x, e_k\rangle} = 0$$

since $\langle y, e_j\rangle = \langle x, e_k\rangle$ by definition of y. Take $e \in M$ with $e \notin \{e_k\}_{k=1}^{\infty}$. Then $\langle x, e\rangle = 0$. Furthermore, $\langle y, e\rangle = 0$ since $\langle e, e_j\rangle = 0$ for all $e_j \in \{e_k\}_{k=1}^{\infty}$. Then $\langle x - y, e\rangle = 0$ for all $e \in M$, and thus $x - y \in M^\perp = \{0\}$. So $x = y$. □

9.1 Separable Hilbert spaces

Proposition 9.3. *Let H be a Hilbert space. Then*

1. *if H is separable then every orthonormal set in H is countable.*

2. *If H contains an orthonormal sequence which is total in H then H is separable.*

If a Hilbert space is separable, then all orthonormal sets are countable, so we can take the cardinality of the dimension of the space to be $\aleph_0$ if it is infinite dimensional.

Proof. 1. Suppose M is an uncountable orthonormal set in H. Take $x, y \in M$, $x \neq y$. Then

$$||x-y||^2 = \langle x-y, x-y\rangle = ||x||^2 + ||y||^2 = 2.$$

Define a neighborhood N_x of x by

$$N_x = \left\{x' \in H \,\middle|\, ||x-x'|| < \tfrac{1}{4}\right\},$$

and similarly for N_y. Then $N_x \cap N_y = \emptyset$. So we have an uncountable family of disjoint open sets in X. Take B a countably dense set in H. For each $x \in M$, the set N_x contains an element $b_x \in B$ since B is dense in H. Since N_x and N_y are disjoint for $x \neq y$, we have $b_x \neq b_y$ for $x \neq y$. This is a contradiction to the countability of B.

2. Let $\{e_k\}$ be a total orthonormal sequence in H. Let

$$A = \operatorname{span}_{\mathbb{Q}}\{e_k\} := \left\{\sum \alpha_k e_k \,\middle|\, \alpha_k = a_k + ib_k,\ a_k, b_k \in \mathbb{Q}\right\},$$

i.e. finite linear combinations of elements in e_k with coefficients in $\mathbb{Q} + i\mathbb{Q}$. Then A is clearly countable. Let $x \in H$ and $\epsilon > 0$. Since $\{e_k\}$ is total, there exists an $n \in \mathbb{N}$ and $\{\alpha_k\}_{k=1}^n$ such that $||x-y|| < \frac{\epsilon}{2}$ where

$$y = \sum_{k=1}^n \langle x, e_k\rangle e_k.$$

Take $a_k, b_k \in \mathbb{Q}$ such that

$$\left|\left|\sum_{k=1}^n \left(\langle x, e_k\rangle - (a_k + ib_k)\right) e_k\right|\right| < \frac{\epsilon}{2}.$$

Denote $v_n = \sum_{k=1}^n (a_k + ib_k)e_k \in A$. Then

$$\begin{aligned} ||x-v|| &\leq ||x-y|| + ||y-v|| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

Theorem 9.4 (Hilbert dimension)**.** *All total orthonormal sets in a given Hilbert space $H \neq \{0\}$ have the same cardinality, called the Hilbert dimension.*

Theorem 9.5. *Two Hilbert spaces H_1 and H_2 are isomorphic if and only if H_1 and H_2 have the same Hilbert dimension.*

Proof. If $H_1 \simeq H_2$, then there exists an inner product preserving isomorphism $T : H_1 \to H_2$. That is

$$\langle Tx, Ty\rangle_2 = \langle x, y\rangle_1$$

for all $x, y \in H_1$. Then every total orthonormal set in H_1 is mapped to an orthonormal set n H_2.

Suppose $\dim H_1 = \dim H_2$. Let $M_1 \subset H_1$ and $M_2 \subset H_2$ be total orthonormal subsets. For every $x \in H_1$, define $\{e_k\}_{k=1}^\infty$ in M_1 with $\langle x, e_k\rangle \neq 0$. Similarly define $\{f_k\}_{k=1}^\infty$ in M_2.

(to be continued...) □

Note 9.6. In particular, every finite dimensional Hilbert space is isomorphic to $\mathbb{R}^n$ or $\mathbb{C}^n$. For countable infinite dimensional spaces, the standard example is $\ell^p \simeq L^p[a,b]$. Indeed, ℓ^p has a countable basis of finite sequences, and L^p has a basis defined by the *sin* and *cos* functions (Fourier basis)[1].

An example of a nonseparable Hilbert space is a function space with inner product given by

$$\lim_{R\to\infty} \frac{1}{R} \int_{-R}^{R} f(x)g(x)dx.$$

Then each function $L^2(-\infty,+\infty)$ is actually the zero element in this space. But *many* more divergent functions are contained.

[1] See section 3.7 in [KREY89]

10 Lecture 10

(30 January 2014)

Theorem 10.1. *Two Hilbert spaces H_1 and H_2 are isomorphic $H_2 \simeq H_2$ if and only if they have the same Hilbert dimension.*

Proof. Suppose $H_1 \neq \{0\} \neq H_2$. Let $M_1 \subset H_2$ and $M_2 \subset H_2$ be total orthonormal sets. For every $x \in H_1$ define a sequence $\{e_k\}_{k=1}^{\infty} = \{e \in M_1 \mid \langle e, x\rangle \neq 0\}$ and analogous for $\{f_k\}_{k=1}^{\infty}$ in M_2. Then $x = \sum_{k=1}^{\infty} \langle x, e_k\rangle e_k$. Define a linear operator $T : H_1 \to H_2$ through its action on the elements e_k by

$$x \longmapsto Tx = \sum_{k=1}^{\infty} \langle x, e_k\rangle f_k \in H_2.$$

From the Bessel inequality $\sum |\langle x, e_k\rangle|^2 \leq ||x||_1^2$, and we have

$$||Tx||_2{}^2 = \sum_k |\langle x, e_k\rangle|^2 = ||x||_1{}^2.$$

So $\langle Tx, Ty\rangle_2 = \langle x, y\rangle_1$. Examing the real and imaginary parts, we have

$$\mathrm{Re}\langle x, y\rangle = \frac{1}{4}\left(||x+y||^2 - ||x-y||^2\right) \quad \text{and} \quad \mathrm{Im}\langle x, y\rangle = \frac{1}{4}\left(||x+iy||^2 - ||x-iy||^2\right).$$

Then

$$\begin{aligned}\langle Tx, Ty\rangle &= \mathrm{Re}\langle Tx, Ty\rangle + i\,\mathrm{Im}\langle Tx, Ty\rangle \\ &= \frac{1}{4}\left(||x+y||^2 - ||x-y||^2\right) + i\frac{1}{4}\left(||x+iy||^2 - ||x-iy||^2\right).\end{aligned}$$

To show that T is injective, note that $||x-y||_1 = ||Tx-Ty||_2$, so if $Tx = Ty$ then $||Tx-Ty||_2 = 0$ and thus $||x-y||_1 = 0$ implies $x = y$.

It remains to show that T is surjective. If $y = \sum_{k=1}^{\infty} \alpha_k f_k$, take $x = \sum_{k=1}^{\infty} \alpha_k e_k$. Note that $\alpha_k = \langle y, e_k\rangle = \langle x, e_k\rangle$ and $\sum |\alpha_k|^2 \leq ||y||^2$. □

10.1 Functionals on Hilbert spaces

Theorem 10.2 (Riesz's Representation Theorem)**.** *For any funcitonal $f \in H'$, there exists a unique element $z \in H$ such that $f(x) = \langle x, z\rangle$ for all $x \in H$ and $||f|| = ||z||$.*

Note 10.3. In the above theorem $z \perp \mathcal{N}(f)$ which implies $z \in \mathcal{N}(f)^{\perp}$. Therefore $\dim \mathcal{N}(f)^{\perp} = 1$.

Proof of Riesz's Representation Theorem. If $f = 0$ then we may take $z = 0$, so we may assume that $f \neq 0$ and we have $\mathcal{N}(f) \neq H$. Since $\mathcal{N}(f)$ is a proper subspace of H we can decompose H as $H = \mathcal{N}(f) \oplus \mathcal{N}(f)^{\perp}$. Note that $\mathcal{N}(f)^{\perp} \neq \{0\}$, since otherwise $\mathcal{N}(f) = H$, which is a contradiction. Then there exists a nonzero element $z_0 \in \mathcal{N}(f)^{\perp}$. Furthermore, for all $x, y \in H$ we have that

$$f\big(f(x)y - f(y)x\big) = f(x), f(y) - f(y)f(x) = 0$$

and thus $f(x)y - f(y)x \in \mathcal{N}(f)$. For each $x \in H$, define the vector $w_x = f(x)z_0 - f(z_0)x$ such that $w_x \in \mathcal{N}(f)$. Then

$$0 = \langle w_x, z_0\rangle = f(x)\underbrace{\langle z_0, z_0\rangle}_{=||z_0||^2} - f(z_0)\langle x, z_0\rangle.$$

Solving for $f(x)$, we find $f(x) = \frac{f(z_0)}{\|z_0\|^2}\langle x, z_0\rangle = \langle x, z\rangle$, where $z = \frac{\overline{f(z_0)}}{\|z_0\|^2} z_0$.

It remains to show that z is unique. Suppose that $z' \in H$ such that $\langle x, z\rangle = \langle x, z'\rangle$ for all $x \in H$. Then $\langle x, z - z'\rangle = 0$ and take in particular $x = z - z'$. Then $\langle z - z', z - z'\rangle = ||z - z'||^2 = 0$ so $z = z'$.

To show that $||f|| = ||z||$, note that

$$|f(x)| = |\langle x, z\rangle| \leq ||x||\,||z||$$

and thus $||f|| \leq ||z||$. Taking $x = z$, we have

$$\frac{|f(z)|}{||z||} = ||z||$$

and thus $||f|| \geq ||z||$ since , $||f||$ is defined through the supremum. Hence $||f|| = ||z||$. □

10.2 Hilbert-Adjoint operator

Definition 10.4. Let $T : H_1 \to H_2$ be a bounded linear operator on Hilbert spaces. The *Hilbert-adjoint operator* is an operator $T^* : H_2 \longrightarrow H_1$ defined by the relation,

$$\langle Tx, y\rangle_2 = \langle x, T^*y\rangle_1$$

for all $x \in H_1$ and $y \in H : 2$. In particular, for an operator $T : H_1 \longrightarrow H_2$ there exists a *unique* operator $H_2 \longrightarrow H_1$ that satisfies the above relation.

We actually need to prove that T^* not only exists, but it is unique and linear. In addition, it is bounded if T is bounded.

Theorem 10.5. *Let T be as in Definition 10.4. Then T^**

1. *exists*
2. *is unique*
3. *and is a bounded linear ,operator if T is with $||T^*|| = ||T||$.*

Proof. 1. For any $y \in H_2$ define the functional $f_y : H_1 \longrightarrow \mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{C}$ or $\mathbb{R}$) by $f_y(x) = \langle Tx, y\rangle$.

Claim 3. $f_y \in H_1'$

Indeed, note that $|f_y| = |\langle Tx, y\rangle \leq ||Tx||\,||y|| \leq ||T||\,||x||\,||y||$. Then by the Reisz representation theorem, there exists a unique $z_y \in H_1$ such that $f_y(x) = \langle x, z_y\rangle = \langle Tx, y\rangle$ for all $x \in H_1$. Then define T^* as the map $y \mapsto z_y$.

2. From the above arguments, we see that the assignment $y \mapsto z_y$ is unique for each y (by Reisz representation), so T^* is unique.

3. To show that T^* is linear, note that

$$\begin{aligned}\langle Tx, \alpha_1 y_1 + \alpha_2 y_2\rangle &= \overline{\alpha}_1\langle Tx, y_1\rangle + \overline{\alpha}_2\langle Tx, y_2\rangle\\ &\overline{\alpha}_1\langle x, z_1\rangle + \overline{\alpha}_2\langle x, z_2\rangle\\ &\langle x, \alpha_1 z_1 + \alpha_2 z_2\rangle.\end{aligned}$$

Then $T^* : \alpha_1 y_1 + \alpha_2 y_2 \longmapsto \alpha_1 z_1 + \alpha_2 z_2 = \alpha_1 T y_1 + \alpha_2 T y_2$. , To show that T^* is bounded, note that.

$$|\langle x, T^*y\rangle| = |\langle Tx, y\rangle| \leq ||Tx||\,||y|| \leq ||T||\,||x||\,||y||.$$

Taking $x = T^*y$, we have

$$||T^*y||^2 \leq ||T||\,||T^*y||\,||y|| \leq ||T||\,||T||\,||y||\,||y|| = ||T||^2||y||^2$$

so $||T^*|| \leq ||T||$. Furthermore, we have

$$|\langle Tx, y\rangle| = |\langle x, T^*y\rangle \leq ||x||\,||T||\,||y||.$$

So taking $y = Tx$, analogously to above we have

$$||Tx||^2 \leq ||T^*||^2\,||x||^2$$

and so $||T|| \leq ||T^*||$. Thus $||T^*|| = ||T||$.

□

11 Lecture 11

(**3 February 2014**)

Proposition 11.1 (Properties of adjoint operators)**.** *Let* $T : H_1 \longrightarrow H_2$*.*

1. $\langle T^*y, x\rangle_1 = \langle y, Tx\rangle_2$
2. $(S+T)^* = S^* + T^*$
3. $(\alpha T)^* = \bar{\alpha}T^*$
4. $(T^*)^* = T$
5. $||T^*T|| = ||TT^*|| = ||T||^2$
6. $T^*T = 0$ *if and only if* $T = 0$*.*
7. $(ST)^* = T^*S^*$ *(for* $H_1 = H_2$*)*

Proof. Left as an exercise. □

Definition 11.2. An operator T is

i) *self-adjoint* or *hermitian* if $T^* = T$.

ii) *unitary* if $T^{-1} = T^*$

iii) *normal* if $TT^* = T^*T$.

Exercise. *If* $Q : X \longrightarrow Y$ *is a bounded linear operator then*

i) $Q = 0$ *if and only if* $\langle Qx, y\rangle = 0$ *for all* $x \in X$ *and* $y \in Y$*.*

ii) if $Q : x \longrightarrow X$ *and* X *is a complex vector space, then* $\langle Qx, x\rangle = 0$ *for all* $x \in X$ *if and only if* $Q = 0$*.*

Note 11.3. Let $T : H \longrightarrow H$ be a bounded linear operator.

1. If $T = T^*$, then $\langle Tx, x\rangle = \langle x, Tx\rangle$ is real for all x.
2. If H is complex and $\langle Tx, x\rangle$ is real for all $x \in H$, then $T = T^*$.
3. If T is unitary, then $||Tx|| = ||x||$ for all $x \in H$.

Theorem 11.4. *Let* $\{T_n\}$ *be a sequence of bounded self-adjoint linear operators with* $T_n : H \longrightarrow H$*. If* $T_n \to T$ *(i.e.* $||T_n - T|| \to 0$*), then* T *is bounded and* $T^* = T$*.*

Proof. We have that $||T_n^* - T^*|| = ||(T_n - T)^*|| = ||T_n - T|| \to 0$. Similarly, we have $||T_n - T^*|| \to 0$. So $T_n \to T^*$ and $T_n \to T$, since $T_n = T_n^*$ for all n. □

11.1 Hahn-Banach theorem

We will discuss three versions of this main theorem.

Definition 11.5. A *sublinear functional* on a vector space X is a real valued function p that satisfies

i) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$

ii) $p(\alpha x) = \alpha p(x)$ for all $\alpha \geq 0$ real and $x \in X$.

Example 11.6. Norms and absolute values of linear functionals satisfy the requirements to be sublinear functionals. Note that for any sublinear functional p, we have $p(0) = 0$. So $p(0) \leq p(x) + p(-x)$ for all $x \in X$, thus $p(-x) \geq -p(x)$.

Theorem 11.7 (Hahn-Banach - version I)**.** *Let X be a real vector space and p a sublinear functional. Ler f be a linear functional which is defined on a subspace of W of X that satisfies $f(x) \leq p(x)$ for all $x \in W$. Then there exists a linear extension $\tilde{f}$ from W to X of f such that*

1. $\tilde{f}(x) \leq p(x)$ for all $x \in X$

2. $\tilde{f}(x) = f(x)$ for all $x \in W$.

Example 11.8. We can make use of the Hahn-Banach theorem by choosing p in the following cases. Choose $p(x) = ||f||\,||x||$, since $f(x) \leq |f(x)| \leq ||f||\,||x|| = p(x)$. If X is an inner product space, then $f(x) = \langle x, z\rangle$ for some $z \in X$, and $|f(x)| = |\langle x, z\rangle| = p(x)$.

Proof. Let E be the set of all linear extensions g of f such that $g(x) \leq p(x)$ for all $x \in \mathcal{D}(g)$ with $\mathcal{D}(f) \subset \mathcal{D}(g)$ (and of course $g(x) = f(x)$ for all $x \in \mathcal{D}(f)$ since g is a proper extension). Then $E \neq \emptyset$ since $f \in E$ (i.e. f is an extension of itself).

We define a partial order on E in the following way. Say that $g \leq h$ if h is an extension of g. That is, $\mathcal{D}(g) \subset \mathcal{D}(h)$ and $g(x) = h(x)$ for all $x \in \mathcal{D}(g)$. For every chain $C \subset E$ define $u(x) = g(x)$ if $x \in \mathcal{D}(g)$ for each $g \in C$. Then the domain of u is $\mathcal{D}(u) = \bigcup_{g\in C} \mathcal{D}(g)$. Note that $g \leq u$ for all $g \in C$, so u is an upper bound of C. By Zorn's Lemma, there exists a maximal element $\tilde{f} \in E$. By definition, this means that $\tilde{f}(x) \leq p(x)$ for all $x \in X$ by definition of E. It remains to show that $\mathcal{D}(\tilde{f}) = X$.

Claim. The domain of $\tilde{f}$ is X.

Proof of claim. Otherwise there exists a $y_0 \in X \setminus \mathcal{D}(\tilde{f})$ and define the subspace $Y = \operatorname{span}\{\mathcal{D}(f), y_0\}$. Then there exists an extension of f whose domain is Y. For every $y \in Y$ there exists unique $z \in \mathcal{D}(\tilde{f})$ and $\alpha \in \mathbb{R}$ such that $y = z + \alpha y_0$. This is indeed unique, since supposing $y = z' + \alpha' y_0$ implies $z - z' = (\alpha' - \alpha)y_0 = 0$. Define a functional h on Y by $h(y) = \tilde{f}(z) + \alpha c$ where $c \in \mathbb{R}$ is some constant. Then this is a linear extension of $\tilde{f}$.

Then we need to show that there exists a $c \in RR$ such that $h(y) \leq p(y)$ for all $y \in Y$. Indeed, $h(y) \leq p(y)$ is equivalent to $\tilde{f}(z) + \alpha c \leq p(z + \alpha y_0)$. This is true if and only if

$$\alpha c \leq p(z + \alpha y_0) - \tilde{f}(z) \qquad \text{for all } y \in Y. \tag{11.1}$$

Case 1: $\alpha > 0$. Then $c \leq p(\frac{1}{\alpha} + y_0) - \tilde{f}(\frac{1}{\alpha}z)$ for all $z \in \mathcal{D}(\tilde{f})$. This is true if and only if $c \leq p(z+y_0) - \tilde{f}(z)$ for all $z \in \mathcal{D}(\tilde{f})$.

Case 2: $\alpha < 0$. Then for all $z \in \mathcal{D}(\tilde{f})$,

$$\begin{aligned} c &\geq \frac{1}{\alpha}\left[p(z + \alpha y_0) - \tilde{f}(z)\right] \\ &= -p(-\frac{1}{\alpha} - y_0) - \tilde{f}(\frac{1}{\alpha}z). \end{aligned}$$

This is true if and only if $c \geq -p(-z' - y_0) - \tilde{f}(z')$ for all $z' \in \mathcal{D}(\tilde{f})$.

Combining Cases 1 and 2, we have that (11.1) holds if and only if

$$-p(-z' - y_0) - \tilde{f}(z') \leq p(z + y_0) - \tilde{f}(z)$$

for all $z, z' \in \mathcal{D}(\tilde{f})$. Then

$$\tilde{f}(z - z') \leq p(z - z') = p(z + y_0 - y_0 - z') \leq p(z + y_0) + p(-y_0 - z'),$$

and since $f(z) - f(z') = \tilde{f}(z - z')$, this proves the claim. □

□

12 Lecture 12

(**5 February 2014**)

Note 12.1. If the dimension of X is finite or countable, then Zorn's Lemma is not needed to prove the Hahn-Banach theorem.

Analogous to last time, if $\mathcal{D}(f) \neq X$ we can define the function

$$g_1 \equiv f(z) + \alpha c_1 \quad \text{for all } z \in Y_1 = \operatorname{span}(\mathcal{D}(f), y_1),$$

where y_1 is some element not in $\mathcal{D}(f)$. If the domain $\mathcal{D}(g_1)$ is stilll not all of X, then we can continue this process and define

$$g_2 \equiv g_1(z) + \alpha c_2 \quad \text{for all } z \in Y_2 = \operatorname{span}(Y_1, y_2).$$

By induction, we can continue this process until $\mathcal{D}(\tilde{f}) = X$.

Theorem 12.2 (Hahn-Banach, version II – generalized to complex vector spaces)**.** *Let X be a real or compled vector space and p a real-valued functional on X which is subadditive (i.e. $p(x+y) \leq p(x)+p(y)$ and $p(\alpha x) = |\alpha| p(x)$ for any scalar α). Let f be a linear functional on a subspace $W \subset X$ which satisfies $|f(x)| \leq p(x)$ for all $x \in W$. Then f has a linear extension $\tilde{f}$ from W to X satisfying $|\tilde{f}(x)| \leq p(x)$ for all $x \in X$.*

Proof. If X is real, then $|f(x)| \leq p(x)$ implies $f(x) \leq p(x)$ for all x. By the first version of the Hahn-Banach theorem, there exists an extension $\tilde{f}$ such that $\tilde{f}(x) \leq p(x)$. Then $-\tilde{f}(x) = \tilde{f}(-x) \leq p(-x) = p(x)$ by the conditions in the statement of the theorem. So $|\tilde{f}(x)| \leq p(x)$ for all $x \in X$.

If X is complex, then $f(x) = r(x) + i\,m(x)$ for some real-valued functionals r and m. Then

$$i(r(x) + i\,m(x)) = i f(x) = f(ix) = r(ix) + i\,m(ix),$$

and thus $i\,r(x) - m(x) = r(ix) + i\,m(x)$. Equating the real parts, we have that $r(ix) = -m(x)$. Hence $f(x) = r(x) + i\,r(x)$ for the real functional r which may be given by

$$r(x) = \frac{f(x) + \overline{f(x)}}{2}.$$

So r is a real-valued linear functional.

Consider the vector space $W_{\mathbb{R}}$, which is W as a real vector space. Then

$$r(x) \leq |f(x)| \leq p(x) \quad \text{for all } x \in W_{\mathbb{R}}$$

implies that there exists an extension $\tilde{r}$ of r such that $\tilde{r}(x) \leq p(x)$ for all $x \in X_{\mathbb{R}}$. Define $\tilde{f}(x) = \tilde{r}(x) - i\tilde{r}(ix)$.

Claim. $\tilde{f}$ is a linear functional on X and $|\tilde{f}(x)| \leq p(x)$ for all $x \in X$.

Note that $\tilde{f}$ is linear by construction. Consider some complex number $a + ib$, then

$$\begin{aligned} \tilde{f}\left((a+ib)x\right) &= \tilde{r}\left((a+ib)x\right) - i\tilde{r}\left((ia-b)x\right) \\ &= a\tilde{r}(x) + b\tilde{r}(ix) - ia\tilde{r}(ix) + ib\tilde{r}(x) \\ &= (a+ib)\tilde{r}(x) - i(a+ib)\tilde{r}(x) \\ &= (a+ib)\tilde{f}(x). \end{aligned}$$

So $\tilde{f}$ is homogeneous with respect to complex scalars.

We can write in polar form $\tilde{f}(x) = |\tilde{f}(x)|e^{i\theta_x}$ for some θ_x, thus

$$
\begin{aligned}
|\tilde{f}(x)| = e^{-i\theta_x}\tilde{f}(x) &= \tilde{f}(e^{-i\theta_x}x) \\
&= \tilde{r}(e^{-i\theta_x}x) \\
&\leq p(e^{-i\theta_x}x) = p(x),
\end{aligned}
$$

where the second equality comes from the fact that the value must be real (imaginary part is zero). □

Theorem 12.3 (Hahn-Banach, version III – Normed spaces)**.** *Let f be a bounded linear functional on a subspace W of a normed space X. Then there exists a a bounded linear functional $\tilde{f}$ on X which is an extension of f such that $||f||_W = ||f||_X$.*

Proof. Take $p(x) = ||f||_W||x||$. This implies $|f(x)| \leq p(x)$. From version II of the Hahn-Banach theorem, there exists an extension $\tilde{f}$ such that $|\tilde{f}(x)| \leq p(x) = ||f||_W||x||$. Hence $\tilde{f}$ is bounded and $||\tilde{f}||_X \leq ||f||_W$. Since $\tilde{f}$ is an extension, we must have $||\tilde{f}||_X \geq ||f||_W$, hence $||\tilde{f}||_X = ||f||_W$. □

Corollary 12.4. *Let X be a normed space and $x_0 \in X$ a nonzero vector. Then there exists a functional $f \in X'$ such that $||f|| = 1$ and $f(x_0) = ||x_0||$.*

Note 12.5. This proposition implies that for all $0 \neq x_0$ there is always a functional f such that

$$|f(x_0)| = ||f||\,||x_0||.$$

Normally, we would just have $|f(x_0)| \leq ||f||\,||x_0||$.

Proof of Corollary 12.4. Let $W = \text{span}\{x_0\}$ and define $g(x) = g(\alpha x_0) \overset{\text{def}}{=} \alpha||x_0||$ for all $x \in W$. Since

$$|g(x)| = |\alpha|\,|||x_0|| = ||\alpha x_0|| = ||x||,$$

we have that $||g|| = 1$. By version III of the Hahn-Banach theorem, there exists an extention f of g from W to X such that $||f|| = 1$, and $f(x_0) = g(x_0) = ||x_0||$. □

13 Lecture 13

(7 February 2014)

We have two important corollaries from last time.

Corollary 13.1 (from last time)**.** *For $0 \neq x_0 \in X$, there exists a functional $f \in X'$ such that $||f|| = 1$ and $f(x_0) = 0$.*

Corollary 13.2. *For any $x \in X$, we have*

$$||x|| = \sup_{0 \neq \tilde{f} \in X'} \frac{|\tilde{f}(x)|}{||\tilde{f}||}.$$

In particular, if there exists an x_0 such that $f(x_0) = 0$ for all $f \in X'$, then $x_0 = 0$.

Proof. For $0 \neq x \in X$, there exists an $f \in X'$ such that $||f|| = 1$ and $f(x) = ||x||$. Since $|\tilde{f}(x)| \leq ||\tilde{f}||\,||x||$, we have

$$\sup_{0 \neq \tilde{f} \in X'} \frac{|\tilde{f}(x)|}{||\tilde{f}||} \geq \frac{f(x)}{||f||} = \frac{||x||}{1}.$$

□

Gilad laments the fact that there is only one course in functional analysis at Univeristy of Calgary. There are many important applications of the above results, but we don't have time to discuss them here.

13.1 The Adjoint Operator

Definition 13.3. Let $T : X \longrightarrow Y$ be a bounded linear operator. Then the ***adjoint operator***, $T^\times : Y' \longrightarrow X'$, is defined by

$$(T^\times g)(x) = g(Tx)$$

for all $g \in Y'$.

Note 13.4. The adjoint is the analogue of the transpose for matrices.

We need to show that the functional defined by $f(x) = (T^\times g)(x)$ is indeed an element of X'. We have

$$|f(x)| = |T^\times g(x)| = |g(Tx)| \leq ||g||\,||T||\,||x||.$$

It remains to show that f is linear and bounded.

Theorem 13.5. *The adjoint operator $T^\times$ is linear and bounded, and $||T^\times|| = ||T||$.*

Proof. To show linearity, we have

$$\begin{aligned} T^\times(\alpha g_1 + \beta g_2)(x) &= (\alpha g_1 + \beta g_2)(Tx) \\ &= \alpha g_1(Tx) + \beta g_2(Tx) \\ &= \alpha T^\times(g_1) + \beta T^\times(g_2). \end{aligned}$$

For boundedness, with f given as above, we have

$$||T^\times g|| = ||f|| \leq ||g||\,||T||$$

so $T^\times$ is bounded and $||T^\times|| \le ||T||$. For all $0 \neq x \in X$ there exists a $g \in Y'$ such that $g(Tx) = ||Tx||$. Then (assuming $||g|| = 1$, which we can do if $Tx \neq 0$)

$$||Tx|| = g(Tx) = f(x) \le ||f||\,||x||\,||T^\times g||\,||x|| \le ||T^\times||\,||g||\,||x|| = ||T^\times||\,||x||,$$

and thus $||T|| \le ||T^\times||$. □

Proposition 13.6. *In finite dimensions, if T is represented by a matrix A then $T^\times$ is represented by the matrix $A^\intercal$ in the dual basis to the basis chosen for A.*

Proof. Let $\{e_k\}_{k=1}^n$ be a basis of X and $\{f_k\}_{k=1}^n$ its dual basis in X'. Then $f_k(e_{k'}) = \delta_{kk'}$ and we can define

$$e_k(f_{k'}) \equiv f_{k'}(e_k) = \delta_{k'k}.$$

Without loss of generality, assume $Y = X$, and let $y = Tx$. Then

$$x = \sum_{k=1}^n \alpha_k e_k \quad \text{and} \quad y = \sum_{k=1}^n \beta_k e_k,$$

and we have $\vec{\beta} = A\vec{\alpha}$, where $A_{kk'} = f_k(Te_{k'})$.

Now let $g, h \in X'$ with $h = T^\times g$. Then

$$h = \sum_{k=1}^n \gamma_k f_k \quad \text{and} \quad g = \sum_{k=1}^n \delta_k f_k,$$

and we have $\vec{\gamma} = B\vec{\delta}$, where $B_{kk'} = e_k(T^\times f_{k'})$. Then

$$B_{kk'} = e_k(T^\times f_{k'}) = \left(T^\times f_{k'}\right)(e_k) = f_{k'}\,(Te_k) = A_{k'k} = A^\intercal_{kk'}.$$

□

Proposition 13.7. *Let $S, T : X \longrightarrow Y$ be bounded linear operators of normed spaces.*

1. $(S+T)^\times = S^\times + T^\times$
2. $(\alpha T)^\times = \alpha T^\times$
3. $(ST)^\times = T^\times S^\times$
4. *if $T \in \mathcal{B}(X,Y)$ and T has an inverse $T^{-1} \in \mathcal{B}(Y,X)$, then $(T^\times)^{-1} = \left(T^{-1}\right)^\times$*

13.2 The relation between $T^\times$ and T^*

We have the correspondence

$$\begin{array}{ccc} H_1 & \overset{T}{\underset{T^*}{\rightleftarrows}} & H_2 \\ {\scriptstyle C_1}\downarrow & & \downarrow{\scriptstyle C_2} \\ H_1' & \overset{T^\times}{\longleftarrow} & H_2' \end{array}$$

where C_1 and C_2 are conjugate-linear isometric bijections. We will now discuss that that means.

Consider functionals $f \in H_1'$ and $g \in H_2'$. By the Riesz representation theorem, there are vectors $z \in H_1$ and $w \in H_2$ such that $f(x) = \langle x, z\rangle$ and $g(x) = \langle x, w\rangle$. So denote $f = f_z$ and $g = g_w$. In fact, this assignment of vectors in H_1 and H_2 to functionals in H_1' and H_2', i.e.

$$\begin{array}{rl} C_1 : H_1 & \longrightarrow H_1' \\ z & \longmapsto f_z \end{array} \quad \text{and} \quad \begin{array}{rl} C_2 : H_2 & \longrightarrow H_2' \\ w & \longmapsto g_w, \end{array}$$

are the isometries we are looking for. By the Riesz representation theorem, C_1 and C_2 are isometric bijections, since $||f_z|| = ||z||$.

Note that, for vectors z, z' in a Hilbert space H and corresponding functionals $f_z, f_{z'}$ in H', we have

$$\langle f_z, f_{z'} \rangle_{H'} = \langle z', z \rangle_H = \overline{\langle z, z' \rangle}_H$$

and

$$f_{\alpha z + \alpha' z'}(x) = \langle x, \alpha z + \alpha' z' \rangle = \overline{\alpha} \langle x, z \rangle + \overline{\alpha'} \langle x, z' \rangle = \overline{\alpha} f_z(x) + \overline{\alpha'} f_{z'},$$

so these isometric bijections are conjugate-linear.

Composition gives us the operator $T^* = C_1^{-1} T^\times C_2$. For a functional $g_z \in H_2'$, we have $T^\times g_z = f_{T^* z} \in H_1'$. Indeed, we have

$$(T^\times g_z)(x) = g_z(T(x)) = \langle Tx, z \rangle = \langle x, T^* z \rangle = f_{T^* z}(x)$$

and for $h_v \in H_1'$, we have

$$\langle T^\times g_z, h_v \rangle_{H_1'} = \langle f_{T^* z}, h_v \rangle_{H_1'} = \langle v, T^* z \rangle = \overline{\langle T^* z, v \rangle}$$

15 Lecture 15

(12 February 2014)

15.1 Uniform Boundedness

There are really four important cornerstones of functional analysis. The first is the Hahn-Banach theorem. Today we will introduce the second one: the uniform boundedness theorem.

Theorem 15.1 (Uniform boundedness)**.** *Let $\{T_n\}$ be a sequence in $\mathcal{B}(X,Y)$, where X is a Banach space and Y is a normed space. If there exists constants $c_x \geq 0$ such that $||T_n x|| \leq c_x$ for all $n \in \mathbb{N}$. Then $\{||T_n||\}$ is bounded. That is, there exists a constant $c > 0$ such that $||T_n|| \leq c$ for all n.*

Thus if X is Banach, pointwise boundedness of a sequence of operators implies uniform boundedness.

Definition 15.2. A subset $M \subset X$ in a metric space is

1. ***rare*** in X if its closure $\overline{M}$ has no interior points (in X)—namely, it doesn't contain any open balls;
2. ***meager*** in X if it is a countable union of rare sets in X;
3. ***non-meager*** if it is not meager.

(Historical notaion called these categories of type I and II).

Theorem 15.3 (Baire's Category Theorem)**.** *If a metric space $X \neq \emptyset$ is complete, then X is non-meager in itself.*

Proof. Suppose that X is meager. Then $X = \bigcup_{k=1}^{\infty} M_k$ where M_k are rare in X. Then $\overline{M_k}^c$ (the complement of $\overline{M_k}^c$) is nonempty for each k. Then there exists a ball of radius $\varepsilon_1 < \frac{1}{2}$ around a point $x_1 \in \overline{M_k}^c$ such that

$$B(x_1, \varepsilon_1) \subset \overline{M_1}^c.$$

Note that $B(x_1, \varepsilon_1) \not\subset \overline{M_2}$ since M_2 is rare, so $\overline{M_2}^c \cap B(x_1, \varepsilon_1) = \emptyset$. Then there exists an $\varepsilon_2 < \frac{1}{4}$ and $x_2 \in \overline{M_2}^c$ such that

$$B(x_2, \varepsilon_2) \subset \overline{M_2}^c \cap B(x_1, \varepsilon_1).$$

Carrying this out, there exists an $\varepsilon_3 < \frac{1}{8}$ and x_3 such that

$$B(x_3, \varepsilon_3) \subset \overline{M_3}^c \cap B(x_2, \varepsilon_2).$$

We can continue this inductively for all k. For conciseness, denote $B_k = B(x_k, \varepsilon_k)$, and we have

$$x_k \in B_k \cap \overline{M_k}^c \qquad \text{and} \qquad B_{k_1} \subset B_k.$$

Hence the sequence $\{x_k\}$ is Cauchy.

It remains to show that $\{x_k\}$ does not converge in X. For $m \geq n$, we have

$$d(x_n, x) \leq d(x_n, x_m) + d(x_m + x) \leq \varepsilon_n + d(x_m, x).$$

Since this is true for all $m \geq n$, and $d(x_m, x) \longrightarrow 0$, we have

$$d(x_n, x) \leq \varepsilon_n \qquad \text{for all } n.$$

Hence $x \in B_n$ and thus $x \in \overline{M_n}^c$ for all $n \in \mathbb{N}$. So x is not in M_n for all $n \in \mathbb{N}$ and thus $x \notin X = \bigcup\limits_{n=1}^{\infty} M_n$. □

Proof (*of Uniform Boundedness Theorem*). For $k \in \mathbb{N}$, denote A_k to be the set of all $x \in X$ such that $||T_n x|| \leq k$. Then $X = \bigcup_{k=1}^{\infty} A_k$. Note that A_k is closed for each k. From Baire's Category Theorem, there exists a $k_0 \in \mathbb{N}$ such that A_{k_0} is not rare in X. That is, there exists an $x_0 \in X$ and $r > 0$ such that $B_0 = B(x_0, r) \subset A_{k_0}$. Then

$$||T_n x|| \leq k_0 \qquad \text{for all } x \in B(x_0, r) = B_0.$$

Take $\varepsilon > 0$ such that $z \equiv x_0 + \varepsilon x \in B_0$ for all $x \in X$ with $||x|| = 1$. Note that if $\varepsilon < r$, then

$$||z - x_0|| = \varepsilon ||x|| = \epsilon < r$$

which implies $z \in B_0$. Then $||T_n z|| \leq k_0$ for all $n \in \mathbb{N}$. Hence

$$\begin{aligned} ||T_n x|| &= \frac{1}{\varepsilon} ||T_n(z - x_0)|| \\ &= \frac{1}{\varepsilon} \left(||T_n z|| + ||T_n x_0|| \right) \\ &= \frac{1}{\varepsilon} (k_0 + k_0) \\ &= \frac{2k_0}{\varepsilon}. \end{aligned}$$

Hence $||T_n|| \leq \frac{2k_0}{\varepsilon}$.

□

16 Lecture 16

(14 February 2014)

Example 16.1. Let X be the normed space of all polynomials (in $\mathbb{R}$ or $\mathbb{C}$) with norm given by

$$||x|| = \max_{0\leq j\leq N_x} |\alpha_j|$$

where $x(t) = \alpha_0 + \alpha_1 x + \cdots + \alpha_{N_x} x^{N_x}$. Show that X is not complete. First, we denote $x(t)$ by

$$x(t) = \sum_{j=0}^{\infty} \alpha_j t^j \qquad \text{where } \alpha_j = 0 \text{ if } j > N_x.$$

Proof. Suppose that X is Banach. For each n, let T_n be the linear funcitonal on X defined by

$$T_n x = \sum_{j=0}^{n} \alpha_j.$$

Then for each x, we have $|T_n x| \leq n \cdot \max_{0\leq j\leq n} |\alpha_j| \leq n||x||$ which implies that T_n is bounded and thus $T_n \in X'$. Furthermore, note that, for each x, we have

$$|T_n x| \leq (N_x + 1)||x||$$

and define the constants $c_x = N_x + 1$. Thus, by the Uniform Boundedness Theorem, there exists a constant $c > 0$ such that $||T_n|| \leq c$ for all n.

However, for each n we can define the polynomial $x_n(t) = 1+t+t^2+\cdots+t^n$ such that $||x_n|| = 1$. But $|T_n x_n| = n = n||x_n||$. Thus $\lim_{n\to\infty} ||T_n|| = \infty$, a contradiction to the above claim. □

For more examples of uses of the Uniform Boundedness Theorem, see [CON90] and [RUD91].

16.1 Weak and Strong Convergence

Definition 16.2. Let X be a normed space and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$

- converges ***strongly*** to x, denoted $x_n \longrightarrow x$, if and only if $||x_n + x|| \longrightarrow 0$.
- converges ***weakly*** to x, denoted $x_n \xrightarrow{w} x$, if and only if $f(x_n) \to f(x)$ for all functionals $f \in X'$.

Note that we only have one type of convergence in $\mathbb{R}$ and $\mathbb{C}$ since they are finite dimensional as vector spaces.

Proposition 16.3. *If $\{x_n\}$ is weakly convergent, then it converges to a unique element in X.*

Proof. If $x_n \xrightarrow{w} x$ and $x_n \xrightarrow{w} y$, then $f(x_n) \to f(x)$ and $f(x_n) \to f(y)$ for all $f \in H'$. Hence $f(x-y) = 0$ for all $f \in H'$ and thus $x - y = 0$, so $x = y$. □

Theorem 16.4. *If $x_n \xrightarrow{w} x$ then $\{||x_n||\}$ is bounded.*

Note that strong convergence of a sequence $\{x_n\}$ clearly implies that $\{||x_n||\}$ is bounded.

Proof. Take $f \in X'$, then the sequence $\{f(x_n)\}$ is bounded. Define the sequence $\{g_n\}$ in X'' by $g_n(f) = f(x_n)$ for each $f \in X'$. Then $|f(x_n)| \leq c_f$, which implies $|g_n(f)| \leq c_f$. By the Uniform Boundedness Theorem, we have that $\{||g_n||\}$ is bounded. Hence $\{||x_n||\}$ is bounded (why?). □

Theorem 16.5. *Let X be a normed space. Then*

1. *strong convergence implies weak convergence, and*
2. *if* $\dim X < \infty$ *then strong and weak convergence are equivalent.*

Proof. The proof of part 1 is trivial (exercise), so we prove part 2. Take a basis $\{e_k\}_{k=1}^n \subset X$ and let $x \in X$ be given by

$$x = \sum_{j=1}^{n} \alpha_j e_j.$$

Let $\{f_k\}_{k=1}^n$ be the dual basis in X'. Since X is finite dimensional, it is complete, so there is a sequence $\{x_n\}$ in X denoted by

$$x_n = \sum_{j=1}^{n} \alpha_j^{(n)} e_j$$

that converges to x_n. If $x_n \xrightarrow{w} x$, then $f_j(x_n) \longrightarrow f(x)$ for all j implies $\alpha_j^{(n)} \longrightarrow \alpha_j$ for all j. Then

$$\begin{aligned} ||x_n - x|| &= \left|\left| \sum_{j=1}^{n} (\alpha_j^{(n)} - \alpha_j) e_j \right|\right| \\ &\leq \sum_{j=1}^{n} \left|\alpha_j^{(n)} - \alpha_j\right| ||e_j|| \overset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

□

Example 16.6. We don't have to look very hard to find counterexamples of this fact for infinite dimensional spaces. Take H to be a Hilbert space and take $\{e_n\}$ an orthonormal set in H. From Bessel's inequality,

$$\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 \leq ||x||^2$$

which implies that $\lim_{n\to\infty} |\langle x, e_n \rangle| = 0$. Since each $f \in H'$ can be written as $f(x) = \langle x, z \rangle$ for some z, we have

$$f(e_n) = \langle e_n, z \rangle \overset{n\to\infty}{\longrightarrow} 0 \qquad \text{for all } f \in X'.$$

Theorem 16.7. *Let X be a normed space and $\{x_n\}$ a sequence in X. Then $x_n \xrightarrow{w} x$ if and only if*

1. *the sequence $\{||x_n||\}$ is bounded, and*
2. *there exists an $M \subset X'$ such that M is total in X' and $f(x_n) \longrightarrow f(x)$ for all $f \in M$.*

Proof. Let $\{x_n\}$ be a sequence in X that satisfies condition 1 of the theorem and let $M \subset X'$ be a subset that satisfies condition 2. Let $f \in X'$. Then there exists a sequence $\{f_k\}$ in the span of M such that $f_k \longrightarrow f$. Then for each n,

$$\begin{aligned} |f(x_n) - f(x)| &\leq |f(x_n) - f_k(x_n)| + |f_k(x_n) - f_k(x)| + |f_k(x) - f(x)| \\ &= |(f - f_k)(x_n)| + |f_k(x_n - x)| + |(f_k - f)(x)| \\ &\leq \underbrace{||f - f_k||}_{\longrightarrow 0} ||x_n|| + ||f_k|| \underbrace{||x_n - x||}_{\longrightarrow 0} + \underbrace{||f_k - f||}_{\longrightarrow 0} ||x||, \end{aligned}$$

hence $f(x_n) \longrightarrow f(x)$ for all functionals $f \in X'$. □

17 Lecture 17

(24 February 2014)

17.1 Sequences of operators

We now cover different types of convergence inhte space of operators. All the theorems and statements from last time regarding different types (strong and weak) of convergence will have analogues here.

Definition 17.1. Let $\{T_n\}$ be a sequence of operators in $\mathcal{B}(X,Y)$ where X and Y are normed spaces.

(a) **Uniform operator convergence**. The sequence $\{T_n\}$ converges ***uniformly*** to an operator T if $||T_n - T|| \longrightarrow 0$ which we denote

$$T_n \xrightarrow{u} T.$$

Then T is called the *operator limit* of $\{T_n\}$.

(b) **Strong operator convergence**. The sequence $\{T_n\}$ converges ***strongly*** to an operator T if $T_n x \longrightarrow Tx$ for each x, and we write

$$T_n \xrightarrow{s} T.$$

That is $T_n x$ converges strongly in Y, and T is called the *strong operator limit* of $\{T_n\}$.

(c) **Weak operator convergence**. The sequence $\{T_n\}$ converges ***weakly*** to an operator T if $f(T_n x) \longrightarrow f(Tx)$ for all functionals $f \in Y'$ and all $x \in X$, and we write

$$T_n \xrightarrow{w} T.$$

That is, $T_n x$ converges weakly in Y, and T is called the *weak operator limit* of $\{T_n\}$.

Example 17.2. Let $X = \ell^2$ and let $x = \{x_j\}$.

i) The *erasure operators* are defined by

$$E_n x = \{\underbrace{0, 0, \dots, 0}_{n \text{ zeros}}, x_{n+1}, \dots, \}.$$

Note that $||E_n|| = 1$, and E_n must somehow converge to 0. But $E_n \not\xrightarrow{u} 0$. However, we have $E_n \xrightarrow{s} 0$.

ii) The (right)-*shift operators* are defined by

$$S_n^R x = \{\underbrace{0, 0, \dots, 0}_{n \text{ zeros}}, x_1, x_2, \dots, \}.$$

We can also define the left-shift operators $S_n^L x = \{x_{n+1}, x_{n+2}, \dots\}$. Note that $S_n^L \circ S_n^R = \mathrm{id}_{\ell^2}$, but $S_n^R \circ S_n^L = E_n$, so these are one-sided inverses of each other. Furthermore, we have that S_n^R and S_n^L are adjoints, since

$$\langle S_n^R x, y \rangle = \sum_{k=1}^{\infty} \alpha_k \overline{\beta}_{k+n} = \langle x, S_n^L y \rangle.$$

Furthermore, note that $\{S_n^R\}$ must somehow converge to 0, but $S_n^R \overset{u}{\not\to} 0$ and $S_n^R \overset{s}{\not\to} 0$, since $||S_n^R x|| = ||x||$ for all $x \in \ell^2$. However, the sequence of shift operators is weakly operator convergent since for all functionals $f \in X'$ we have $f(x) = \langle x, z \rangle$ for some $z \in X$ and

$$\begin{aligned} |f(S_n^R x)| &= \left|\langle S_n^R x, y \rangle\right| \\ &= \left|\langle x, S_n^L y \rangle\right| \\ &= ||x|| \underbrace{\left||S_n^L z\right||}_{\longrightarrow 0}. \end{aligned}$$

Theorem 17.3. *Let X be a Banach space and Y a normed space. Let $\{T_n\}$ be a sequence in $\mathcal{B}(X,Y)$. If $\{T_n\}$ be a strongly operator convergent sequence with limit $T_n \xrightarrow{s} T$. Then T is bounded.*

Proof. Technically, we first need to prove that T is indeed a linear operator (left as an exercise), but we will just prove boundedness.

Since the sequence $\{T_n x\}$ in Y is convergent for all $x \in X$, it is bounded, so there exists a constant c_x such that $||T_n x|| \le c_x$. By the Uniform Boundedness Theorem, the sequence $\{||T_n||\}$ is bounded since X is complete. Then there exists a constant c such that $||T_n|| \le c$ for all n, and we have

$$\begin{aligned} ||T_n x|| &\le ||T_n|| \, ||x|| \\ &\le c||x||. \end{aligned}$$

Hence $||Tx|| \le c||x||$ and thus $T \in \mathcal{B}(X,Y)$. □

Example 17.4. We now consider a counterexample to the above theorem. Let $X \subset \ell^2$ be the subspace of sequences consisting of finitely many non-zero elements. This subspace of finite sequences is not complete in ℓ^2. For each n, define the operators

$$T_n x = \{x_1, 2x_2, \dots, nx_n, x_{n+1}, \dots\}.$$

Clearly, we have $T_n x \longrightarrow Tx$ where

$$Tx = \{x_1, 2x_2, 3x_3, \dots\},$$

but T is not bounded.

Theorem 17.5. *Let X, Y be Banach spaces. A sequence $\{T_n\}$ in $\mathcal{B}(X,Y)$ is strongly operator convergent if and only if the following hold:*

1. *the sequence $\{||T_n||\}$ is bounded,*
2. *and the sequence $\{T_n x\}$ is Cauchy in Y for all $x \in M$ where M is a total subset of X.*

Proof. The forward direction is trivial, so assume (1) and (2). Let $x \in X$ and $y \in \operatorname{span} M$. Then we have

$$\begin{aligned} ||T_n x - T_m x|| &\le ||T_n x - T_n y|| + ||T_n y - T_m y|| + ||T_m y - T_m x|| \\ &= ||T_n|| \underbrace{||x - y||}_{\to 0} + \underbrace{||T_n - T_m||}_{\to 0} \, ||y|| + ||T_m|| \underbrace{||y - x||}_{\to 0} \end{aligned}$$

sicne $\operatorname{span} M$ is dense in X, so we can choose $y \in \operatorname{span} M$ arbitrarily close to x. □

Note 17.6. If $T_n = f_n$ for some f_n a sequence of bounded linear functionals, then $Y = \mathbb{R}$ or $\mathbb{C}$. But weak and strong convergence is the same in these fields. So $f_n(x) \xrightarrow{w} f(x)$ is equivalent to $f_n(x) \longrightarrow f(x)$.

Definition 17.7. Let X be a normed space and $\{f_n\}$ a sequence of functionals in X'.

(a) **Strong functional convergence**. The sequence $\{f_n\}$ converges *strongly* to a functional $f \in X'$ if $||f_n - f|| \longrightarrow 0$, and this is denoted

$$f_n \longrightarrow f.$$

(b) **Weak* functional convergence**. The sequence $\{f_n\}$ is *weakly** convergent to a functional $f \in X'$ if $f_n(x) \to f(x)$ for all $x \in X$, and this is denoted

$$f_n \xrightarrow{w^*} f.$$

Next time we will discuss the Open Mapping Theorem. For this we need to understand a basic concept of continuous mappings. A map $T : X \to Y$ is continuous if and only if $T^{-1}(U) \subset X$ is open in X for all open sets $U \subset Y$.

18 Lecture 18

(**26 February 2014**)

18.1 Open Mapping Theorem

This is our third main cornerstone of functional analysis. The other two "cornerstones", but not as important as the other three, might be Baire's Category Theorem and the Closed Graph Theorem (which is really a consequence of the Open Mapping Theorem, and we will cover this later). The remainder of this course will be focused on spectral theory and self-adjoint operators, and unbounded operators if we have time.

Recall: A mapping T of a metric space X into a metric space Y is continuous if and only if $T^{-1}(A)$ is open in X for any open subset $A \in Y$. This notation $T^{-1}(A)$ does not necessarily mean that T has an inverse, this is just the inverse image of the mapping.

Definition 18.1. Let X and Y be metric spaces. Then a a mapping $T : X \longrightarrow Y$ is said to be an ***open*** mapping if for every open set $U \subset \mathcal{D}(T)$ the image under T, i.e. $T(U) \subset Y$, is open in Y.

Note 18.2. Note that in our definition of open, the sets must be open in Y, *not* that they must be open in the range $\mathcal{R}(T) \subset Y$, since openess in Y is stronger than openess in a subset of Y under the subset topology.

Question: What conditions do we need for a bounded continuous linear mapping $T : X \longrightarrow Y$ to be open? It is nice to work in spaces that are Banach, and in this case it turns out that the mapping must be surjective.

Theorem 18.3 (Open Mapping Theorem)**.** *Every bounded linear operator from a Banach space X onto a Banach space Y is an open map. Hence if $T : X \longrightarrow Y$ is bijective, then T^{-1} is continuous (which is equivalent to boundedness).*

Proof. Let $A \subset X$ be open. We need to show that $T(A)$ is open. Take $y \in Tx \in T(A)$, and we need to show that there is an open neighborhood of y that is contained in $T(A)$. We know that there exists an $\varepsilon > 0$ such that the open ε-ball around x is contained in A, i.e. $B(x; \varepsilon) \subset A$.

Instead of looking at the ε-ball around an arbitrary x, it is more convenient to translate and expand the problem to look at the 1-ball around 0, i.e. this is equivalent to looking at

$$B(0; 1) \subset \frac{1}{\varepsilon}(A - x)$$

where $A - x$ is the set $\{y - x \in X \mid y \in A\}$, and similarly $cA = \{cy \in X \mid y \in A\}$. We can do this by linearity of the maping. Hence, we have

$$T(B(0; 1)) \subset \frac{1}{\varepsilon}(T(A) - Tx).$$

We need to show that there exists a ball of any radius around $0 \in Y$ that is contained in $T(B(0; 1))$, then we are done.

Denote $B_n \equiv B(0; 2^{-n}) \subset X$. So we have $B_0 = B(0; 1)$ and $B_1 = B(0; \frac{1}{2})$. Note that we can write X as the countable union

$$X = \bigcup_{k=1}^{\infty} B_1,$$

since for each $x \in X$ there is an integer k such that $\frac{k}{2} > \|x\|$ and thus $x \in kB_1$. So we have

$$\begin{aligned} Y = T(X) &= \bigcup_{k=1}^{\infty} T(kB_1) \\ &= \bigcup_{k=1}^{\infty} kT(B_1) \\ &= \bigcup_{k=1}^{\infty} \overline{kT(B_1)}, \end{aligned}$$

where we can take the closure of each set, since the union will still be equal to the closure of Y in Y, which is just Y. Now we can make use of Baire's Category Theorem. Since Y is complete, and we have written it as a countable union of closed sets, at least one of these closed sets must be non-meager. So there exists a $y_0 \in Y$ and $\delta > 0$ such that $B_Y(y_0;\delta) \subset \overline{T(B_1)}$.[1] Hence

$$B_Y(0;\delta) \subset \overline{T(B_1)} - y_0,$$

and we will show that this is contained in $\overline{T(B_0)}$. Indeed, take $y \in B_Y(0;\delta)$, then $y + y_0 \in \overline{T(B_1)}$. In addition, we have $y_0 \in \overline{T(B_1)}$. Hence, $y - y_0$ can be written as a limit of elements in $T(B_1)$ (since it is in the closure), that is

$$y - y_0 = \lim_{n\to\infty} Ta_n$$

for some sequence $a_n \in B_1$. Similarly, $y_0 \in \overline{T(B_1)}$ so y_0 can be written as a limit

$$y_0 = \lim_{n\to\infty} Tb_n$$

for some sequence $b_n \in B_1$. Then

$$y = \lim_{n\to\infty} T(a_n - b_n),$$

and since a_n, b_n we have $\|a_n - b_n\| \le \|a_n\| + \|b_n\| < \frac{1}{2} + \frac{1}{2} = 1$. So $(a_n - b_n) \in B_0$ for each n and thus $y \in \overline{T(B_0)}$.

To summarize, we have that there exists a $\delta > 0$ such that $B_y(0;\delta) \subset \overline{T(B_0)}$ if and only if $B_Y(0;\frac{\delta}{2}) \subset \overline{T(B_n)}$, and thus $B_Y(0;\frac{\delta}{2}) \subset \overline{T(B_1)}$.

Finally, we claim that $B(0;\frac{\delta}{2}) \subset T(B_0)$. Indeed, take $y \in B(0,\frac{\delta}{2})$ and thus there exists an $x_1 \in B_1$ such that $\|y - Tx_1\| < \frac{\delta}{4}$. Hence

$$y - Tx_1 \in B(0,\tfrac{\delta}{4}) \subset \overline{T(B_2)},$$

and thus there exists an $x \in B_2$ such that $\|y - Tx_1 - Tx_2\| < \frac{\delta}{8}$. By induction, we can continue to carry out this process to construct a sequence $\{x_k\}$ with each $x_k \in B_k$ such that the series $\sum_{k=1}^{\infty} x_n$ converges in X. Indeed, for each n we have

$$\left\| y - T\sum_{k=1}^{n} x_k \right\| \le \frac{\delta}{2^{n+1}}.$$

Define the partial sums of the series $s_n \equiv \sum_{k=1}^{n} x_k$, then

$$\|s_n - s_m\| \le \sum_{k=m+1}^{n} \|x_k\| \le \sum_{k=m+1}^{\infty} \|x_k\| \xrightarrow{m\to 0} 0,$$

[1] Note that we have removed the k, since there exists some k (say $k = 17$) such that there is a y_0 and $\delta' > 0$ with$B_Y(y_0;\delta') \subset \overline{kT(B_1)}$ and so choose $\delta = \frac{1}{k}\delta'$.

so $\{s_n\}$ is a Cauchy sequence in X. Since X is Banach, it converges $s_n \longrightarrow x$ to some x in X. Hence $Tx_n \longrightarrow Tx = y$, and thus $y = \lim_{n\to\infty} Ts_n$. But

$$\|x\| \leq \sum_{k=1}^{\infty} \|x_k\| < 1$$

since $x_k \in B_k$ and thus $\|x_k\| < \frac{1}{2^k}$ for each k. Then $\|x\| < 1$ and thus $x \in B(0;1)$, so $y \in T(B(0;1))$ and we are done.

□

19 Lecture 19

(**28 February 2014**)

Theorem 19.1 (Bounded Inverse theorem[1]). *A bijective linear bounded map $T : X \longrightarrow Y$, where X and Y are Banach spaces, has a bounded linear inverse.*

Today we will discuss another of the important 'cornerstones' of functional analysis: the Closed Graph theorem. In addition, we will talk about unbounded operators, since many important linear operators in physics (such as the momentum operator $p = i\frac{d}{dx}$) are unbounded.

Definition 19.2. Let X and Y be normed spaces and $T : \mathcal{D}(T) \longrightarrow Y$ a linear operator with $\mathcal{D}(T) \subset X$. Then T is said to be a *closed linear operator* if its graph is closed in $X \times Y$.

We recall that the *graph* of the mapping T is

$$\mathcal{G}(T) = \{(x, Tx) \,|\, x \in \mathcal{D}(T)\} \subset X \times Y.$$

Note that $X \times Y$ has the structure of a normed space, where

$$\alpha(x, y) = (\alpha x, \alpha y) \quad \text{and} \quad (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and the norm is given by $\|(x, y)\| = \|x\| + \|y\|$. Another norm could also be defined by $\|(x, y)\|' = \sqrt{\|x\|^2 + \|y\|^2}$, but in fact all norms on a normed space define the same topology (homework), so it does not matter.

Proposition 19.3. *If X and Y are Banach spaces, then so is $X \times Y$.*

Proof. Let $z_n = (x_n, y_n)$ be a Cauchy sequence in $X \times Y$. Then

$$\|z_n - z_m\| = \|x_n - x_m\| + \|y_n - y_m\| \longrightarrow 0.$$

□

There are two ways to define the notion of a *closed operator*. We will give the second definition here, then show that the two notions are the same.

Lemma 19.4. *Let $T : \mathcal{D}(T) \longrightarrow Y$ be a linear operator, where $\mathcal{D}(T) \subset X$ and X and Y are normed spaces. Then T is a closed linear operator if and only if the following holds: if $x_n \in \mathcal{D}(T)$ is a sequence in X such that $x_n \longrightarrow x$ and $Tx_n \longrightarrow y \in Y$, then $x \in \mathcal{D}(T)$ and $y = Tx$.*

That is, in any situation in which Tx_n converges to something in Y, then it must converge to Tx where x is the limit of $\{x_n\}$. Thus, this is similar to, but a weaker condition, then continuity of T.

Proof. Note that the graph $\mathcal{G}(T)$ is closed in $X \times Y$ if and only if for every Cauchy sequence of the form $z_n = (x_n, Tx_n)$, we have that $\{z_n\}$ converges to something in $\mathcal{G}(T)$. In particular, $z_n \longrightarrow z = (x, Tx)$ where $x \in \mathcal{D}(T)$. But this is true if and only if for all convergent sequences $x_n \longrightarrow x$ in $\mathcal{D}(T)$ with $Tx_n \longrightarrow y \in Y$ we have $y = Tx$. □

Example 19.5. Let $X \subset C[0, 1]$ be the subspace of all functions with continuous derivative, and $T : \mathcal{D}(T) \longrightarrow X$ with $\mathcal{D}(T)$ as a subspace[2] of X. Then T is an unbounded closed linear operator.

[1] This is an important corollary to (or perhaps an important part of) the Open Mapping theorem.
[2] In particular, it is the subspace of $C[0, 1]$ of functions with continuous first and second derivatives

Proof. Let $x_n \in \mathcal{D}(T)$ be a sequence such that $x_n \longrightarrow x \in \mathcal{D}(T)$ and $Tx_n \longrightarrow y$ where $y \in X$. Then

$$y = \lim_{n\to\infty} x_n',$$

and we may take the integral of both sides to obtain

$$\begin{aligned}\int_0^t y(\tau)\,d\tau &= \int_0^t \lim_{n\to\infty} x_n'(\tau)\,d\tau \\ &= \lim_{n\to\infty}\int_0^t x_n'(\tau)\,d\tau \\ &= \lim_{n\to\infty}(x_n(t) - x_n(0)) \\ &= x(t) - x(0)\end{aligned}$$

where we interchange the limit and the integral since the limit converges uniformly. Then we have

$$x(t) = x(0) + \int_0^t y(\tau)\,d\tau$$

and thus $x'(t) = y(t)$ so $y = Tx$. □

Example 19.6. Let $T : \mathcal{D}(T) \longrightarrow X$ where $\mathcal{D}(T) \subset X$ is a proper dense subset of X and T is the identity operator. Then T is bounded but not closed. It is clearly bounded, since $\|Tx\| = \|x\|$ for each $x \in \mathcal{D}(T)$, so $\|T\| = 1$. But it is not closed, since for an $x \in X \setminus \mathcal{D}(T)$, there is a sequence $\{x_n\}$ in $\mathcal{D}(T)$ that converges to x. Then $\{Tx_n\}$ converges to x, which is not in $\mathcal{D}(T)$, so the graph is not closed in $\mathcal{D}(T) \times X$.

Lemma 19.7. *Let $T : \mathcal{D}(T) \longrightarrow Y$ be a bounded linear operator with $\mathcal{D}(T) \subset X$ where X and Y are normed spaces.*

1. *If $\mathcal{D}(T)$ is closed in X, then T is closed.*
2. *If T is closed and Y is complete, then $\mathcal{D}(T)$ is closed.*

Proof. Next time. (Note: no class next Monday) □

20 Lecture 20

(**5 March 2014**)

Recall that an operator T is closed if and only if the graph $\mathcal{G}(T)$ is closed. As we saw before, this is equivalent to the statement: if $\{x_n\}$ in X such that $x_n \longrightarrow x$ and $Tx_n \longrightarrow y$ for some $y \in Y$, then $x \in \mathcal{D}(T)$ and $y = Tx$.

Lemma 20.1. *Let $T : \mathcal{D}(T) \longrightarrow Y$ be a bounded linear operator between normed spaces X and Y where $\mathcal{D}(T) \subset X$.*

1. *If $\mathcal{D}(T)$ is closed in X, then T is closed.*
2. *If T is closed and Y is Banach, then $\mathcal{D}(T)$ is closed.*

Proof.

1. Let $\{x_n\}$ be a sequence such that $x_n \longrightarrow x$ and $Tx_n \longrightarrow y$ for some $y \in Y$. Since $\mathcal{D}(T)$ is closed, $x \in \mathcal{D}(T)$. Then $Tx_n \longrightarrow Tx$ and thus $y = Tx$.
2. Let $x \in \overline{\mathcal{D}(T)}$. Then there exists a sequence $\{x_n\}$ in $\mathcal{D}(T)$ such that $x_n \longrightarrow x$. But $\|Tx_n - Tx\| \leq \|T\|\|x_n - x\| \longrightarrow 0$, and thus $\underline{Tx_n \text{ is}}$ Cauchy. Since Y is complete, we have $Tx_n \longrightarrow y$ for some $y \in Y$. Hence $x \in \mathcal{D}(T)$, so $\overline{\mathcal{D}(T)} = \mathcal{D}(T)$.

□

Theorem 20.2 (Closed Graph Theorem). *Let X, Y be Banach spaces, and let $T : \mathcal{D}(T) \longrightarrow Y$ be a closed lienar operator such that $\mathcal{D}(T) \subset X$. If $\mathcal{D}(T)$ is closed then T is bounded.*

Proof. By assumption, $\mathcal{G}(T)$ is closed in $X \times Y$. Since $X \times Y$ is Banach, we have that $\mathcal{G}(T)$ is Banach. Similarly, $\mathcal{D}(T)$ is Banach. Define the mapping[1] $P : \mathcal{G}(T) \longrightarrow \mathcal{D}(T)$

$$(x, Tx) \longmapsto x.$$

This is clearly linear (exercise), so we show that P is bounded. Note that

$$\|P(x, Tx)\| = \|x\| \leq \|x\| + \|Tx\| = \|(x, Tx)\|$$

and thus $\|P\| \leq 1$. Note that P is bijective, since $P(x, Tx) = P(x', Tx')$ if and only if $x = x'$, and P is also open by the Open Mapping Theorem. Hence P^{-1} is bounded by the Bounded Inverse Theorem. Then

$$\|Tx\| \leq \|x\| + \|Tx\| = \|(x, Tx)\| = \left\|P^{-1}(x)\right\| \leq \left\|P^{-1}\right\|\|x\|$$

and thus T is bounded. □

Hence the Closed Graph Theorem is really a consequence of the Open Mapping Theorem. These theorems have very important applications in spectral theory, which is what we will be covering pretty much for the rest of the semester.

20.1 Spectral theory in normed spaces

Spectral theory itself has many applications in this such as solving differential equations, since this often boils down to finding eigenvalues of linear operators. However, spectral theory is still interesting in its own right, not just applications. Our focus here will mostly be on the following (in the given order):

1. bounded self-adjoint operators,
2. compact operators,
3. unbounded operators.

[1] This is kind of like a projection mapping, so we use the letter P.

20.1.1 Eigenvalues and eigenvectors

Given an operator T on a vector space[2], a nonzero vector x is an *eigenvector* of T if $Tx = \lambda x$ for some scalar λ, and λ is said to be an *eigenvalue* of T. If $Tx = \lambda x$, then we have

$$Tx - \lambda x = 0.$$

For each λ, we can define the operator $T_\lambda \equiv T - \lambda I$, where I is the identity mapping. Then λ is an eigenvalue of T if and only if T_λ is not invertible. We define

$$R_\lambda \equiv T_\lambda^{-1} = (T - \lambda I)^{-1},$$

if it exists. This is known as the *resolvant*. In many cases, the resolvant may be unbounded, which, in terms of spectrum, is a similar condition it not existing at all.

Definition 20.3. Let $T : \mathcal{D}(T) \longrightarrow X$ be a linear operator with $\mathcal{D}(T) \subset X$.

1. The ***point spectrum*** or ***discrete spectrum*** of T is the set

$$\sigma_p(T) \equiv \{\lambda \in \mathbb{C} \,|\, R_\lambda \text{ does not exist}\}.$$

2. The ***continuous spectrum*** of T is the set

$$\sigma_c(T) \equiv \left\{\lambda \in \mathbb{C} \,\middle|\, R_\lambda \text{ exists but is not bounded, and } \overline{\mathcal{D}(R_\lambda)} = X\right\}.$$

3. The ***residual spectrum*** of T is the set

$$\sigma_r(T) \equiv \left\{\lambda \in \mathbb{C} \,\middle|\, R_\lambda \text{ exists and } \overline{\mathcal{D}(T)} \neq X\right\}.$$

4. The ***spectrum*** of T is the set

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T).$$

5. The ***resolvant set*** of T is $\rho(T) \equiv \mathbb{C} \setminus \sigma(T)$, and $\lambda \in \rho(T)$ is called a ***regular value*** of T.

Hence $\lambda \in \rho(T)$ if and only if all of the following are satisfied:

i) R_λ exists,

ii) R_λ is bounded,

iii) and $\mathcal{D}(R_\lambda)$ is dense in X (i.e. $\overline{\mathcal{D}(T_\lambda)} = X$).

Note that we can partition the complex plane into four regions:

$$\mathbb{C} = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T) \cup \rho(T).$$

Example 20.4. In finite dimensions, the continuous and residual spectra are empty.

Example 20.5. Consider the infinite-dimensional Hilbert space ℓ^2 and define the *shift* operator $S : \ell^2 \longrightarrow \ell^2$ defined by

$$S : (\alpha_1, \alpha_2, \dots) \longmapsto (0, \alpha_1, \alpha_2, \dots).$$

Take $\lambda = 0$. Note that $R_0 = T^{-1}$ exists on the range of T, since $T^{-1} : T(\ell^2) \longrightarrow \ell^2$ is defined by

$$(0, \alpha_1, \alpha_2, \dots) \longmapsto (\alpha_1, \alpha_2, \dots).$$

But $\overline{\mathcal{D}(R_0)} = \mathcal{R}(T) = \left\{(0, \alpha_1, \alpha_2, \dots) \,\middle|\, (\alpha_1, \alpha_2, \dots) \in \ell^2\right\}$ which is not dense in ℓ^2. So $\lambda = 0$ is in the residual spectrum of S.

[2] We are not necessarily working in normed spaces or any particular kind of space yet.

21 Lecture 21

(**7 March 2014**)

Proposition 21.1. *Let $T : X \longrightarrow X$ be a linear operator on a Banach space X.*

1. *If T is bounded and for some $\lambda \in \mathbb{C}$ such that $R_\lambda(T)$ exists with $\mathcal{D}(R_\lambda(T)) = X$, then R_λ is bounded.*
2. *If $\lambda \in \rho(T)$ and T is either closed or bounded, then $\mathcal{D}(R_\lambda) = X$.*
3. *$R_\mu - R_\nu = (\mu - \nu)R_\mu R_\nu$, where $\mu, \nu \in \mathbb{C}$.*
4. *If $[S, T] = ST - TS = 0$, then $[S, R_\mu(T)]$ for all μ such that R_μ exists.*
5. *$[R_\mu, R_\nu] = 0$ for all $\mu, \nu \in \mathbb{C}$.*

Proof. .

1. This is an immediate consequence of the Bounded Inverse Theorem.
2. If T is closed then T_λ is closed. Then $R_\lambda = T_\lambda^{-1}$ is closed, since

$$\mathcal{G}(R_\lambda) = \mathcal{G}(T_\lambda^{-1}) = \{(T_\lambda x, x) \mid x \in X\}$$

   which is isomorphic to the closed set $\mathcal{G}(T_\lambda) = \{(x, T_\lambda x) \mid x \in X\}$. By assumption, $\lambda \in \rho(T)$ and thus R_λ is also bounded. Since R_λ is both closed an bounded, $\mathcal{D}(R_\lambda) = \overline{\mathcal{D}(R_\lambda)}$, but $\mathcal{D}(R_\lambda)$ is dense in X and thus $\mathcal{D}(R_\lambda) = X$.

   If T is bounded and $\mathcal{D}(T)$ is closed in X, then T is closed. But in our case here, $\mathcal{D}(T) = X$ and thus T is closed.
3. Note that $R_\nu = T_\nu^{-1}$ and so $T_\nu R_\nu = I$, so we have

$$\begin{aligned} R_\mu - R_\nu &= R_\mu \underbrace{T_\nu R_\nu}_{I} - \underbrace{R_\mu T_\mu}_{I} R_\nu \\ &= R_\mu (T_\nu - T_\mu) R_\nu \\ &= R_\mu (T - \nu I - (T - \mu I)) R_\nu \\ &= (\mu - \nu) R_\mu R_\nu. \end{aligned}$$

4. Note that $R_\mu S = R_\mu S T_\mu R_\mu$, but S commutes with T and thus $[S, T_\lambda] = 0$. Hence

$$R_\mu S = R_\mu S \underbrace{T_\mu R_\mu}_{I} = \underbrace{R_\mu T_\mu}_{I} S R_\mu = S R_\mu$$

   and thus $[S, R_\mu] = 0$.
5. This follows from (4). Indeed, R_μ commutes with T, so chose $S = R_\nu$.

□

21.1 Spectral properties of bounded linear operators

Lemma 21.2. *Let $T \in \mathcal{B}(X, X)$ where X is a Banach space. If $\|T\| < 1$ then $(I - T)^{-1}$ exists and*

$$(I - T)^{-1} = I + T + T^2 + \cdots = \sum_{k=0}^{\infty} T^k.$$

Proof. Note that $\|T^n\| \leq \|T\|^n$, and follow the proof of convergence for a geometric series in $\mathbb{R}$ (left as an exercise).[1] □

Theorem 21.3 (Closed Speectrum Theorem)**.** *Let $T : X \longrightarrow X$ be a bounded linear operator on a Banach space X. The resolvent set $\rho(T)$ is open in $\mathbb{C}$, hence $\sigma(T)$ is closed in $\mathbb{C}$.*

Proof. Take a fixed $\lambda_0 \in \rho(T)$. We will show that there exists a neighborhood of λ_0 that is in $\rho(T)$. Take some $\lambda \in \mathbb{C}$ and consider T_λ and note that

$$\begin{aligned} T_\lambda &= T - \lambda I \\ &= T - \lambda_0 I + \lambda_0 I - \lambda I \\ &= (T - \lambda_0 I)\left[I - (T - \lambda_0 I)^{-1}(\lambda_0 - \lambda)I\right] \\ &= T_{\lambda_0}\left[I - (\lambda_0 - \lambda)R_{\lambda_0}\right]. \end{aligned}$$

If $\|(\lambda_0 - \lambda)R_{\lambda_0}\| < 1$ then $\lambda \in \rho(T)$, and thus if $|\lambda - \lambda_0| < \frac{1}{R_{\lambda_0}}$ then $\lambda \in \rho(T)$. □

From above, we have that $T_\lambda = T_{\lambda_0}\left[I - (\lambda - \lambda_0)R_{\lambda_0}\right]$, and thus we have the useful formulae

$$R_\lambda = \left[I - (\lambda - \lambda_0)R_{\lambda_0}\right]^{-1} \qquad \text{and thus} \qquad R_\lambda = \sum_{k=0}^{\infty}(\lambda - \lambda_0)^k R_{\lambda_0}^{k+1}.$$

Theorem 21.4. *Let X be a Banach space and $T \in \mathcal{B}(X, X)$. Then $\sigma(T)$ is compact and $|\lambda| \leq \|T\|$ for all $\lambda \in \sigma(T)$.*

So the spectrum is always contained inside a disk of radius $\|T\|$ about the origin.

Proof. Suppose that $|\lambda| > \|T\|$. Then $R_\lambda = (T - \lambda I)^{-1}$ exists if $\left\|\frac{1}{|\lambda|}T\right\| < 1$, since this is equivalent to $|\lambda| > \|T\|$. Hence $\lambda \in \rho(T)$ and thus $\lambda \notin \sigma(T)$. □

Definition 21.5. The ***spectral radius*** $r_\sigma(T)$ is defined as

$$r_\sigma(T) := \sup_{\lambda \in \sigma(T)} |\lambda|.$$

Theorem 21.6. [2]*Let X be a Banach space and $T \in \mathcal{B}(X, X)$. Then*

$$r_\sigma(T) = \lim_{n\to\infty} \sqrt[n]{\|T^n\|} \leq \|T\|.$$

Proof. We will show that

$$r_\sigma(T) = \sqrt[n]{r_\sigma(T^n)} \leq \sqrt[n]{\|T^n\|}$$

(to be continued next time). □

[1] Gilad says this might be a good question for the final.
[2] This is related to the Hadamard theorem in complex analysis.

22 Lecture 22

(10 March 2014)

From last time: we looked at expanding $(I-T)^{-1}$ if $\|T\| < 1$. In particular, this means that $1 \in \rho(T)$, since in this case $(I-T)^{-1}$ is bounded and defined on all X.

Theorem 22.1 (The Spectral Mapping Theorem for polynomials)**.** *Let X be a complex Banach space and $T \in \mathcal{B}(X,X)$. Let $p(\lambda) = \alpha_0 + \alpha_1\lambda + \cdots + \alpha_n\lambda^n$ be a degree n polynomial (with $\alpha_n \neq 0$) then we can define the operator $p(T)$. Then $\sigma(p(T)) = p(\sigma(T))$.*

That is, the spectrum of $p(T)$ is the image of the spectrum of T under the polynomial p. We are familiar with this fact for matrices. For example, if λ is an eigenvalue of the matrix A, then λ^2 is an eigenvalue of A^2. We must be careful with our intuition, however, in the infinite dimensional case.

Proof. We first prove that $\sigma(p(T)) \subset p(\sigma(T))$. Fix $\mu \in \sigma(p(T))$, then we need to show that $\mu = p(\lambda)$ for some $\lambda \in \sigma(T)$. Since μ is just some constant, we can define a new polynomial $q_\mu(\lambda) = p(\lambda) - \mu$. Since this is a polynomial over $\mathbb{C}$, we can factor it as

$$q_\mu(\lambda) = \alpha_n(\lambda - a_1)(\lambda - a_2)\cdots(\lambda - a_n)$$

for some nonzero[1] complex numbers a_k. This gives us the operator

$$Q_\mu = \alpha_n(T - a_1I)(T - a_2I)\cdots(T - a_nI) = P(T) - \mu I.$$

Suppose that $(T - a_kI)^{-1}$ exists and $a_k \in \rho(T)$ for all $k = 1, \dots n$. Then

$$(P(T) - \mu I)^{-1} = Q_\mu^{-1} = \frac{1}{\alpha_n}(T - a_nI)^{-1}\cdots(T - a_2I)^{-1}(T - a_1I)^{-1}$$

and thus $(P(T)-\mu I)^{-1}$ exists and is bounded, hence $\mu \in \rho(P(T))$, a contradiction to the assumption that $\mu \in \sigma(P(T))$. Thus for at least one a_k we must have that $(T-a_kI)^{-1}$ does not exist and therefore $a_k \in \sigma(T)$. So we have $\mu = p(a_k)$.

Now we show that $\sigma(p(T)) \supset p(\sigma(T))$. Take $\nu \in p(\sigma(T))$, then $\nu = p(\lambda_0)$ for some $\lambda_0 \in \sigma(T)$. We need to show that $\nu \in \sigma(P(T))$. Define a polynomial $q_\nu(\lambda) = p(\lambda) - \nu$, then we can factor this as

$$q_\nu(\lambda) = (\lambda - \lambda_0)g(\lambda)$$

for some polynomial $g(\lambda)$. We have the operator

$$Q_\nu = P(T) - \nu I = (T - \lambda_0 I)g(\lambda).$$

If Q_ν^{-1} does not exist, then $\nu \in \sigma(P(T))$. We still need to consider the case when Q_ν^{-1} does exist.

- Case I: $R_{\lambda_0} = (T - \lambda_0 I)^{-1} = T_{\lambda_0}^{-1}$ does not exist. Then

$$I = Q_\nu^{-1}\underbrace{(T - \lambda_0 I)g(T)}_{Q_\nu} = \underbrace{(T - \lambda_0 I)g(T)}_{Q_\nu}Q_\nu^{-1}.$$

  Then (?????)

- Case II: $R_{\lambda_0} = (T-\lambda_0 I)^{-1} = T_{\lambda_0}^{-1}$[2] exists. Then $\mathcal{D}(R_{\lambda_0}) \neq X$. Indeed, otherwise $T_{\lambda_0} : X \longrightarrow X$ is onto and thus $\mathcal{R}(T_{\lambda_0}) = X$. By the Bounded Inverse Theorem, we have that R_{λ_0} is bounded.

  Last time we proved that if $T : X \longrightarrow X$ is bounded or closed and $\lambda \in \rho(T)$ then $\mathcal{D}(R_\lambda(T)) = X$. If $\nu \in \rho(P(T))$ then $\mathcal{D}(P(T)) = X$.

□

[1] indeed, otherwise $\alpha_n = 0$

[2] Gilad later erased this and wrote Q_ν^{-1}

23 Lecture 23

(12 March 2014)

We go back to correct the proof of the theorem from last time.

Proposition 23.1. $p(\sigma(T)) \subset \sigma(p(T))$

Proof. Let $\nu \in p(\sigma(T))$ then $\nu = p(\lambda_0)$ for some $\lambda_0 \in \sigma(T)$. Define the new polynomial

$$q_\nu(\lambda) = p(\lambda) - \nu$$

which factors as $q_\nu(\lambda) = (\lambda - \lambda_0)g(\lambda)$ for some polynomial $g(\lambda)$. Then we have the operator

$$Q_\nu = p(T) - \nu I = (T - \lambda_0 I)g(T) = g(T)(T - \lambda_0 I).$$

- If Q_ν^{-1} does not exist, then $\nu \in \sigma(p(T))$.
- If Q_ν^{-1} exists, then $(T - \lambda_0)^{-1} = R_{\lambda_0}$ exists. Then $\mathcal{R}(T_{\lambda_0}) \neq X$ and thus $\mathcal{R}(Q_\nu) \neq X$ since $\mathcal{R}(Q_\nu) \subset \mathcal{R}(T_{\lambda_0})$.

  We have shown previously that, if $T : X \longrightarrow X$ is bounded or closed and $\lambda \in \rho(T)$, then $\mathcal{D}(R_\lambda) = \mathcal{R}(T_\lambda) = X$. So suppose that $\nu \in \rho(p(T))$, that is $\nu \notin \sigma(p(T))$. Then $Q_\nu^{-1} = R_\nu(p(T))$ and thus $\mathcal{R}(Q_\nu) = X$, a contradiction.

Hence Q_ν^{-1} does not exist and thus $\nu \in \sigma(p(T))$. □

23.1 Banach Algebras

Definition 23.2. An ***algebra*** $\mathcal{A}$ is a vector space with a binary operation that defines a product $x \cdot y \in \mathcal{A}$ that is associative. That is

$$x \cdot (y + z) = x \cdot y + y \cdot z$$

for all $x, y, z \in \mathcal{A}$. The algebra has an identity if there exists an element $e \in \mathcal{A}$ such that $e \cdot x = x \cdot e$ for all $x \in \mathcal{A}$, and the algebra is said to be abelian if the product is commutative. A ***normed algebra*** is an algebra $\mathcal{A}$ that is normed as a vector space and satisfies

$$\|x \cdot y\| \leq \|x\| \|y\|$$

for all $x, y \in \mathcal{A}$, and if $\mathcal{A}$ has a multplicative identity e then[1] $\|e\| = 1$. A ***Banach algebra*** is a normed algebra whose underlying normed space is Banach.

Note 23.3. The product in a normed algebra is continuous. Indeed, we have

$$\begin{aligned} \|xy - x_0y_0\| = \|xy - xy_0 + xy_0 - x_0y_0\| &\leq \|x(y - y_0)\| + \|(x - x_0)y_0\| \\ &\leq \|x\|\|y - y_0\| + \|y\|\|x - x_0\|, \end{aligned}$$

so $\|x_0y_0\|$ is close to $\|x_0y_0\|$ if x_0 and y_0 are close to x and y.

Example 23.4. Some examples of normed and Banach algebras are

1. $\mathbb{R}$ and $\mathbb{C}$ with standard product and norm;
2. $C[a, b]$ is an abelian Banach algebra with identity where product is given by $(x \cdot y)(t) := x(t)y(t)$;
3. the space of $n \times n$ matrices $\mathcal{M}_n(\mathbb{C})$ (this is abelian if and only if $n = 1$)

[1] To figure out (exercise): is this requirement necessary, or does it follow from $\|x \cdot y\| \leq \|x\| \|y\|$?

4. the space $\mathcal{B}(X, X)$ for any normed space X (this space has an identity).

Definition 23.5. Let $\mathcal{A}$ be a complex Banach algebra with identity.

- The ***resolvent set*** $\rho'(x)$ for $x \in \mathcal{A}$ is the set of all $\lambda \in \mathbb{C}$ such that $x - \lambda e$ is invertible.
- The ***spectrum*** is $\sigma'(x) := \mathbb{C} \setminus \rho'(x)$ and any $\lambda \in \sigma'(x)$ is called a ***spectral value*** of x.

Theorem 23.6. *If $\mathcal{A} = \mathcal{B}(X, X)$, then $\rho'(T) = \rho(T)$. That is, our notion of the resolvent set of an operator $T \in \mathcal{B}(X, X)$ in the algebra-sense coincides with our definition of the resolvent set of T as an operator in Definition 20.3.*

Proof. If $\lambda \in \rho'(T)$, then $(T - \lambda e)^{-1}$ exists and belongs to $\mathcal{B}(X, X)$, hence $\lambda \in \rho(T)$. If $\lambda \in \rho(T)$ then $(T - \lambda e)^{-1}$ exists[2] with domain dense in X. By the previous theorem, we must have $\mathcal{D}(R_\lambda) = X$. Hence $(T - \lambda e)^{-1}$ exists in $\mathcal{A} = \mathcal{B}(X, X)$ and thus $\lambda \in \rho'(T)$. □

Theorem 23.7. *For a complex Banach algebra $\mathcal{A}$ with identity, if $\|x\| < 1$ then $e - x$ is invertible with*

$$(e - x)^{-1} = e + \sum_{k=1}^{\infty} x^k$$

and this converges.

Proof. Define $s_n = \sum_{k=1}^{n} x^k$ and $s = \sum_{k=1}^{\infty} x^k$. For all $n \in \mathbb{N}$, note that

$$\left\| \sum_{k=1}^{n} x^k \right\| \leq \sum_{k=1}^{n} \left\| x^k \right\| \sum_{k=1}^{n} \|x\|^k \underset{n\to\infty}{<} \infty$$

and thus $\sum_{k=1}^{\infty} \left\| x^k \right\|$ converges absolutely. Since $\mathcal{A}$ is Banach, this means that $\sum x^k$ converges. It remains to show that $e - s$ is the inverse of $e - x$. Indeed,

$$\begin{aligned}(e - x)(e + s_n) &= (e - x)(e + x + x^2 + \cdots + x^n) = (e + x + x^2 + \cdots + x^n)(e - x) \\ &= e - x^{n+1} \underset{n\to\infty}{\longrightarrow} e.\end{aligned}$$

□

Theorem 23.8. *Let $\mathcal{A}$ be a complex Banach algebra with identity. The group $G \subset \mathcal{A}$ of all invertible elements is open in $\mathcal{A}$. (Hence, $\mathcal{A} \setminus G$ is closed in $\mathcal{A}$.)*

Proof. Note that G is nonempty since $e \in G$. Take $x_0 \in G$ and $x \in \mathcal{A}$. Then

$$\begin{aligned}x = x_0 - (x_0 + x) &= x_0 - x_0(e + x_0^{-1}x) \\ &= x_0(e - x_0^{-1}(x_0 + x).\end{aligned}$$

Then x had an inverse if $\left\| x_0^{-1}(x_0 - x) \right\| < 1$.
$\left\| x_0^{-1} \right\| \|x_0 - x\| < 1$... Suppose $\|x_0 - x\| < 1$. ??? □

[2] "But where does it exist? On the moon? It exists somewhere, but it important to know where it exists." – Gilad

24 Lecture 24

(14 March 2014)

Theorem 24.1 (Spectral radius in a Banach algebra)**.** *For a complex Banach algebra with identity, the following hold:*

1. $r_\sigma(x) := \sup_{\lambda\in\sigma(x)} |\lambda| \leq \|x\|$
2. *The spectrum $\sigma(x)$ is compact.*

Proof. 1. Take $\lambda \in \mathbb{C}$. Then

$$x - \lambda e = -\lambda(e - \tfrac{1}{\lambda}x)$$

and this is invertible if $\left\|\frac{1}{\lambda}\right\| < 1$, that is, if $|\lambda| > \|x\|$. Hence, if $\lambda \in \sigma(x)$ then $|\lambda| < \|x\|$.

2. First note that $\rho(x)$ is nonempty, since part (1) implies that $\lambda_0 \in \rho(x)$ whenever $|\lambda_0| > \|x\|$. Take $\lambda_0 \in \rho(x)$. Then $x - \lambda_0 e \in G$, where G is the group of invertible elements in the algebra. Since G is open (which we proved last time), there exists a $\delta > 0$ such that $x - \lambda e \in G$ for all λ such that $|\lambda - \lambda_0| < \delta$. Hence $\lambda \in \rho(x)$ if $|\lambda - \lambda_0| < \delta$. Therfore $\rho(x)$ is open, and thus $\sigma(x)$ is closed and bounded. So it is compact.

□

Theorem 24.2. *The spectrum of every element is nonempty. That is,*

$$\sigma(x) \neq \emptyset$$

for all $x \in \mathcal{A}$, where $\mathcal{A}$ is a complex Banach space.

Proof. Let $f \in \mathcal{A}'$ be a bounded linear functional and define a mapping

$$\begin{aligned} h : \rho(x) &\longrightarrow \mathbb{C} \\ \mu &\longmapsto f\left([x - \mu e]^{-1}\right). \end{aligned}$$

We will show that h is analytic. If $\rho(x) = \mathbb{C}$, then f is a bounded entire function, and therefore must be constant. We will show that this constant must be zero.

To show that the derivative of h exists everywhere, note that

$$\frac{h(\mu) - h(\nu)}{\mu - \nu} = f\left(\frac{(x - \mu e)^{-1} - (x - \nu e)^{-1}}{\mu - \nu}\right) \tag{24.1}$$

since f is linear. Examing the numerator of the fraction above, we have

$$\begin{aligned} (x - \mu e)^{-1} - (x - \nu e)^{-1} &= (x - \nu e)^{-1}\left[(x - \nu)(x - \mu)^{-1} - e\right] \\ &= (x - \nu e)^{-1}\left[(x - \mu e + (\mu - \nu)e)(x - \mu e)^{-1}\right] \\ &= (\mu - \nu)(x - \nu e)^{-1}(x - \mu e)^{-1} \end{aligned}$$

and thus eq. (24.1) becomes

$$\frac{h(\mu) - h(\nu)}{\mu - \nu} = f\left((x - \nu e)^{-1}(x - \mu e)^{-1}\right).$$

Hence, h is analytic on its domain. In particular, the derivative is equal to

$$h'(\mu) = f\left((x - \mu e)^{-2}\right).$$

Suppose that $\rho(x) = \mathbb{C}$, then h is an entire function. We have

$$\begin{aligned} |h(\mu)| &= \left|f\left((x-\mu e)^{-1}\right)\right| \\ &\leq \|f\| \left\|(x-\mu e)^{-1}\right\| \\ &= \|f\| \left\| \mu^{-1}\left(\tfrac{1}{\mu}x - e\right)^{-1} \right\| \\ &= \frac{1}{\|\mu\|}\|f\| \left\| \left(\tfrac{1}{\mu}x - e\right)^{-1} \right\| \xrightarrow{|\mu|\to\infty} 0. \end{aligned}$$

So h is bounded. By Liouville, h is constant Hence $h(\mu)$ for all $\mu \in \mathbb{C}$. .... (something else here, he erased before I wrote it down) ☐

24.1 Spectral Theory of Self-Adjoint Operators

Recall that an operator $T : H \longrightarrow H$ on a Hilbert space is *self-adjoint* if

$$\langle Tx, y\rangle = \langle x, Ty\rangle$$

for all $x, y \in H$.

Theorem 24.3. *If T is a self-adjoint operator, then T is bounded.*

Proof. Note that T is closed. Indeed, given a sequence $x_n \longrightarrow x$ in H such that $Tx_n \longrightarrow y$ for some $y \in Y$, then

$$\langle Tx_n, z\rangle = \langle x_n, Tz\rangle \xrightarrow{n\to\infty} \langle x, Tz\rangle = \langle Tx, z\rangle.$$

Hence $Tx_n \xrightarrow{w} Tx$, but the weak limit is unique. So $Tx = y$, and thus T is closed. Since $\mathcal{D}(T) = H$ is closed, we have that T is bounded by the Closed Graph Theorem[1]. ☐

For Hilbert spaces, most of our useful theorems for finite dimensional self-adjoint matrices holds.

Proposition 24.4. *All eigenvalues of self-adjoint operators are real. Eigenvectors of self-adjoint linear operators corresponding to different eigenvalues are orthogonal.*

However, in infintite dimensions, we have other notions of spectrum that we have to be careful about, although most of our intuition carries over

Theorem 24.5. *The spectrum $\sigma(T)$ of a self-adjoint linear operator $T : H \longrightarrow H$ on a complex Hilbert space is real.*

Lemma 24.6. *For a self-adjoint linear operator $T : H \longrightarrow H$, we have $\lambda \in \rho(T)$ if and only if there exists a constant $c > 0$ such that $\|T_\lambda x\| > c\|x\|$ for all $x \in H$.*

Proof (of the theorem). First note that, for all $\lambda \in \mathbb{C}$, we have

$$\langle T_\lambda x, x\rangle = \langle Tx, x\rangle - \lambda\langle x, x\rangle \qquad \text{and} \qquad \overline{\langle T_\lambda x, x\rangle} = \langle Tx, x\rangle - \overline{\lambda}\langle x, x\rangle$$

since $\langle x, x\rangle = \|x\|^2$ and $\langle Tx, x\rangle$ are real. Looking at the imaginary part, we have

$$\operatorname{Im}\langle T_\lambda x, x\rangle = -\operatorname{Im}\lambda\|x\|^2,$$

and taking the absolute value of this gives

$$\|T_\lambda x\|\|x\| \geq |\langle T_\lambda x, x\rangle| \geq |\operatorname{Im}\lambda|\|x\|^2$$

which implies $\|T_\lambda x\| \geq |\operatorname{Im}\lambda|\|x\|$. By the lemma, this imples that $\operatorname{Im}\lambda = 0$ unless $\lambda \in \rho(T)$. Hence, the spectrum of T is real. ☐

[1] As an assignment, we will prove this theorem using a different method

Proof (*of the lemma*). If $\lambda \in \rho(T)$ then $R_\lambda = T_\lambda^{-1}$ exists and thus $\|R_\lambda\| = a > 0$ for some consant a. Then

$$\|x\| = \|R_\lambda T_\lambda x\| \leq \|R_\lambda\|\|T_\lambda x\| = a\|T_\lambda y\|,$$

which implies that $\|T_\lambda x\| \geq \frac{1}{a}\|x\|$, where $\frac{1}{a} = c$ is the constant we use.

Now suppuse that there exists a constant $c > 0$ such that $\|T_\lambda x\| \geq c\|x\|$. We need to show that the inverse exists, for which we first need to show that T_λ is injective. Indeed,

$$a\|T_\lambda x_1 - T_\lambda x_2\| = \|T_\lambda(x_1 - x_2)\| \geq c\|x_1 - x_2\|$$

and htus $x_1 = x_2$ if $T_\lambda x_1 = T_\lambda x_2$.

Furthermore, we claim that the range $T_\lambda(H)$ is dense in H. Indeed, suppose $x_0 \perp T_\lambda(H)$ for some $x_0 \in H$. For all $x \in H$, we have

$$0 = \langle T_\lambda x, x_0\rangle = \langle x, T_\lambda x_0\rangle$$

and thus $T_\lambda(x_0) = 0$. Since T_λ is injective, $x_0 = 0$, so $T_\lambda(H)$ is dense in H. □

25 Lecture 25

(17 March 2014)

Extra lecture: scheduled for next Monday (March 24) 10-11am (room to be determined).

Return to proofof the lemma from last time.

Lemma 25.1. *For a self-adjoint operator $T : H \longrightarrow H$, we have $\lambda \in \rho(T)$ if and only if there exists a constant c such that $\|T_\lambda x\| \geq c\|x\|$ for all $x \in X$.*

Proof. We have already proved one direction, so it remains to prove the following claim.

Claim: If there exists a $c > 0$ such that $\|T_\lambda x\| \geq c\|x\|$ for all $x \in X$, then $\lambda \in \rho(T)$.

We first need to show that R_λ is defined on the whole space H. We show that $T_\lambda(H)$ is closed and dense in H. Let $x_0 \perp T_\lambda(H)$, then we show that $x_0 = 0$. Indeed, for all $x \in H$ we have

$$\begin{aligned} 0 &= \langle x_0, T_\lambda x\rangle \\ &= \left\langle T_{\overline{\lambda}} x_0, x\right\rangle \end{aligned}$$

which implies $T_{\overline{\lambda}} x_0 = 0$ and thus $T x_0 = \overline{\lambda} x_0$. Note that one solution is $x_0 = 0$. Since by the previous theorem, we must have that λ is real, so $\overline{\lambda} = \lambda$. Then $(T - \lambda I)x_0 = T_\lambda x_0 = 0$. But this is impossible, due to the assumption that $\|T_\lambda x\| \geq c\|x\| > 0$ if $x_0 \neq 0$. So we have that $\overline{T_\lambda(H)} = H$.

Now let $y \in \overline{T_\lambda(H)} = H$. Then there exists a sequence $y_n \in T_\lambda(H)$ sch that $y_n \longrightarrow y$. Since each y_n is in the image of T, for each n there is an $x_n \in H$ such that $T x_n = y_n$. Then $x_n \longrightarrow x$ for some $x \in H$. Indeed,

$$\|x_n - x_m\| \leq \frac{1}{c}\|T_\lambda x_n - T_\lambda x_m\| = \frac{1}{c}\|y_n - y_m\|$$

which goes to zero. □

Proposition 25.2 (Properties of the spectrum of self-adjoint linear operators)**.** *Let $T : H \longrightarrow H$ be a bounded self-adjoint linear operator.*

1. *$\sigma(T) \subset [m, M]$, where*

$$m := \inf_{\substack{x \in H \\ \|x\|=1}} \langle Tx, x\rangle \qquad \text{and} \qquad M := \sup_{\substack{x \in H \\ \|x\|=1}} \langle Tx, x\rangle.$$

2. *$\|T\| = \max(|m|, |M|) = \sup_{\substack{x \in H \\ \|x\|=1}} |\langle Tx, x\rangle|$.*

3. *$m, M \in \sigma(T)$.*

4. *$\sigma_r(T) =$ (a self-adjoint linear operator has an empty residual spectrum).*

Proof. .

1. Let $c > 0$ be some positive constant and let $\lambda = m - c$. Let $x \in H$ with $\|x\| = 1$. By the Schwartz inequality, we have $\langle T_\lambda x, x\rangle \leq \|T_\lambda x\|\|x\|$ and thus

$$\|T_\lambda x\| \geq \langle T_\lambda x, x\rangle = \langle Tx, x\rangle - \lambda \geq m - \lambda = c.$$

Hence, from the lemma, we have $\lambda \in \rho(T)$. A similar argument follows if we take $\lambda = M + c$. So $\lambda \notin [m, M]$ implies $\lambda \in \rho(T)$.

2. First note that, by the Schwartz inequality, we have that

$$t = \sup_{\|x\|=1} |\langle Tx, x\rangle| \leq \sup_{\|x\|=1} \|Tx\|\|x\| = \|T\|,$$

and thus $t \leq \|T\|$. If $Tx = 0$ for all $x \in H$, then $T = 0$. If $T \neq 0$, let $x \in H$ such that $\|x\| = 1$ and $Tx \neq 0$. For an arbitrary linear combination $y = \alpha x + \beta Tx$, we have

$$\langle Ty, y\rangle = |\alpha|^2 \langle Tx, x\rangle + |\beta|^2 \left\langle T^2 x, Tx\right\rangle + \alpha\bar{\beta}\|Tx\|^2 + \bar{\alpha}\beta\|Tx\|^2.$$

Take two linear combinations $y_1 = \alpha_1 x + \beta_1 Tx$ and $y_2 = \alpha_2 x + \beta_2 Tx$ with $\alpha_1 = \alpha_2$ real and $\beta_1 = -\beta_2$ real. Then we have

$$\langle Ty_1, y_1\rangle - \langle Ty_2, y_2\rangle = 4\alpha\beta\|Tx\|^2.$$

Furthermore, we have that $|\langle Ty_1, y_1\rangle - \langle Ty_2, y_2\rangle| \leq |\langle Ty_1, y_1\rangle| + |\langle Ty_2, y_2\rangle|$. Putting this together with the equation above, we obtain

$$|4\alpha\beta|\|Tx\|^2 \leq |\langle Ty_1, y_1\rangle| + |\langle Ty_2, y_2\rangle| \leq t\|y_1\|^2 + t\|y_2\|^2$$

where t is defined above. Then we have

$$\begin{aligned} t &\geq |4ab| \frac{\|Tx\|^2}{\|y_1\|^2 + \|y_2\|^2} \\ &= \frac{2|\alpha\beta|\|Tx\|^2}{|\alpha|^2\|x\|^2 + |\beta|^2\|Tx\|^2} \end{aligned}$$

□

26 Lecture 26

(19 March 2014)

26.1 Spectral theory (continued)

Proof (continued from last time). .

3. Since $\sigma(T + aI) = \sigma(T) + a$ by the spectral mapping theorem, we may assume without loss of generality that $0 \leq m \leq M$ by adding some constant of I to T. From part (2) of the proposition, we have that

$$M = \sup_{\|x\|=1} |\langle Tx, x\rangle| = \sup_{\|x\|=1} \langle Tx, x\rangle = \|T\|.$$

So there exists a sequence $\{x_n\}$ in H with $\|x_n\| = 1$ such that $\langle Tx_n, x_n\rangle \longrightarrow M$. With $T_M = T - MI$, we have

$$\begin{aligned} \|T_M x_n\|^2 &= \langle T_M x_n, T_M x_n\rangle \\ &= \langle (T - MI)x_n, (T - MI)x_n\rangle \\ &= \underbrace{\langle Tx_n, Tx_n\rangle}_{\leq M^2} - M\langle x_n, Tx_n\rangle - M\langle Tx_n, x_n\rangle + M^2 \underbrace{\langle x_n, x_n\rangle}_{=\|x_n\|=1} \\ &\leq 2M^2 - 2M \underbrace{\langle Tx_n, x_n\rangle}_{\to M} \longrightarrow 0. \end{aligned}$$

Hence $M, m \in \sigma(T)$.

4. Suppose the residual spectrum of T is not empty. Then $\sigma_r(T) \neq \emptyset$ and take $\lambda \in \sigma_r(T)$. This implies that T_λ^{-1} exists but $\overline{\mathcal{D}(T_\lambda^{-1})} \neq H$. So we can take a nonzero $y \in H \setminus \overline{\mathcal{D}(T_\lambda^{-1})} \neq H$ such that $y \perp \mathcal{D}(T_\lambda^{-1})$. Note also that $\mathcal{R}(T_\lambda) = \mathcal{D}(T_\lambda^{-1})$. So for all $x \in H$ we have that

$$\langle T_\lambda x, y\rangle = 0 \quad \text{and thus} \langle x, T_\lambda y\rangle = 0$$

and thus $T_\lambda y = 0$. Hence $Ty = \lambda y$, that is $\lambda \in \sigma_p(T)$, a contradiction since $\sigma_p(T) \cap \sigma_r(T) = \emptyset$.

□

26.1.1 Partial order on the set of self-adjoint operators

Let H be a Hilbert space and consider the set of all self-adjoint operators on H. Then we can define a partial order on this set in the following manner:

$$T_1 \leq T_2 \quad \text{if and only if} \quad \langle T_1 x, x\rangle \leq \langle T_2 x, x\rangle \quad \text{for all } x \in H.$$

This is equivalent to the statement that $0 \leq T_2 - T_1$ since

$$0 \leq T_2 - T_1 \quad \text{if and only if} \quad \langle (T_2 - T_1)x, x\rangle \geq 0.$$

In finite dimensions, we have $T \geq 0$ if and only if all of the eigenvalues in the point spectrum are non-negative.

We now state a few useful lemmas without proof.

Lemma 26.1. *If both $T_1 \geq 0$ and $T_2 \geq 0$ and $[T_1, T_2] = 0$ then $T_1 T_2 \geq 0$.*

Lemma 26.2. *If $T \geq 0$ then there exists a unique operator $A \geq 0$ such that $T = A^2$.*

26.1.2 Projection operators

In finite dimensions, we can decompose any self-adjoint matrix A as

$$A = \sum_{k=1}^{n} \lambda_k v_k v_k^*$$

where $Av_k = \lambda v_k$ and v_k^* is the conjugate transpose. Then $P_k = v_k v_k^*$ is a projection operator.

Recall that, if $Y \subset H$ is a closed subspace, then $H = Y \oplus Y^\perp$ and any element $x \in H$ can be written as

$$x = y + z \qquad \text{with } y \in Y \text{ and } z \in Y^\perp.$$

We can define the *projection* onto Y as the operator

$$\begin{aligned} P\colon H &\longrightarrow Y \\ y + z &\longmapsto y. \end{aligned}$$

Note that $x = Px + (I-P)x$ and $I - P$ is a projection from H onto $Y^\perp$.

Theorem 26.3. *A bounded linear operator $P\colon H \longrightarrow H$ is a projection if and only if $P = P^*$ (self-adjoint) and $P^2 = P$ (idempotent).*

Proof. If P is a projection operator then

$$P^2 x = P(Px) = Py = y = Px$$

for all $x \in H$, so $P^2 = P$. now let $x_1 = y_1 + z_1$ and $x_2 = y_2 + z_2$, with $y_1, y_2 \in Y$ and $z_1, z_2 \in Y^\perp$. Then

$$\langle Px_1, x_2\rangle = \langle y_1, y_2 + z_2\rangle = \langle y_1, y_2\rangle = \langle y_1, Py_2\rangle = \langle y_1 + z_1, Px_2\rangle = \langle x_1, Px_2\rangle,$$

so $\langle Px_1, x_2\rangle = \langle x_1, Px_2\rangle$ for all $x_1, x_2 \in H$, and thus P is self-adjoint.

Now suppose that $P = P^*$ and $P^2 = P$. Define $Y = P(H)$.

Claim: For all $x \in H$, we have $(I-P)x \in Y^\perp$.

Indeed, take $Px' \in Y$ for some $x' \in H$. Then

$$\langle (I-P)x, Px'\rangle = \langle P(I-P)x, x'\rangle = \langle \underbrace{(P-P^2)}_{0} x, x'\rangle = 0$$

since $P = P^2$.

Claim: $Y = P(H)$ is closed.

Indeed, Y is the nullspace $\mathcal{N}(I-P)$, and nullspaces of bounded operators are closed (since it is the preimage of the closed set $\{0\}$).

Claim: $P|_Y = \mathrm{id}_Y$.

Indeed, for $y \in Y = P(H)$, there is an $x \in H$ such that $Px = y$, and

$$Py = P(Px) = P^2 x = Px = y.$$

□

27 Lecture 27

(21 March 2014)

Proposition 27.1 (Properties of projections). *Let P, P_1, P_2 be projections on a Hilbert space H.*

1. $\langle Px, x\rangle = \|Px\|^2$, *hence* $P \geq 0$;
2. $\|P\| = 1$ *if* $P(H) \neq \{0\}$ (*i.e.* $P \neq 0$);
3. $Q = P_1P_2$ *is a projection if and only if* $[P_1, P_2] = 0$; *if* P_1 *and* P_2 *commute we have* $Q(H) = P_1(H) \cap P_2(H)$;
4. *given vectors* $w, v \in H$, *we have* $w \perp v$ *if and only if* $P_wP_v = 0$;
5. $Q = P_1 + P_2$ *is a projection if and only if* $P_1(H) \perp P_2(H)$.

Note that $P_1(H) \perp P_2(H)$ and $Q = P_1 + P_2$ implies $Q(H) = P_1(H) \oplus P_2(H)$.

Theorem 27.2 (Partial Order Theorem for Projections). *Let P_1 and P_2 be projections on H. Deonote $Y_1 = P_1(H)$ and $Y_2 = P_2(H)$, and let $\mathcal{N}(P_1)$ and $\mathcal{N}(P_2)$ be the nullspaces. The following are equivalent:*

1. $P_2P_1 = P_2P_2 = P_1$;
2. $\|P_1x\| \leq \|P_2x\|$ *for all* $x \in H$;
3. $P_1 \leq P_2$;
4. $\mathcal{N}(P_2) \subset \mathcal{N}(P_1)$;
5. $Y_1 \subset Y_2$.

Proof. .

(1) ⇒ (2): Since $\|P_2\| \leq 1$ (it is either 1 or 0), we have

$$\|P_1x\| = \|P_1P_2x\| \leq \|P_1\|\|P_2x\|\|P_2x\|.$$

(2) ⇒ (3): For all $x \in X$, we have that

$$\langle P_1x, x\rangle = \|P_1x\|^2 \leq \|P_2x\|^2 = \langle P_2x, x\rangle,$$

and thus $\langle P_1x, x\rangle \leq \langle P_2x, x\rangle$ for all $x \in X$, which is equivalent to $P_1 \leq P_2$.

(3) ⇒ (4): Take $x \in \mathcal{N}(P_2)$, and thus $P_2x = 0$. But $P_1 \leq P_2$ and the fact that P_1 and P_2 are both positive, we have $P_1x \leq P_2x = 0$, and thus $P_1x = 0$.

(4) ⇒ (5): Note that $\mathcal{N}(P_1) = Y_1^\perp$ and $\mathcal{N}(P_2) = Y_2^\perp$. But we have $\mathcal{N}(P_2) \subset \mathcal{N}(P_1)$ and thus $Y_2^\perp \subset Y_1^\perp$. Hence $Y_1 \subset Y_2$.

(5) ⇒ (1): Let $x \in H$, then $P_1x \in Y_1 \subset Y_2$. But this implies $P_2P_1x = P_1x$ for all $x \in X$ and thus $P_2P_1 = P_1$. By self-adjointness of projections, we also have $P_1P_2 = P_1$.

□

Proposition 27.3 (Difference of projections). *Let P_1 and P_2 be projections on a Hilbert space H. Let Y_1 and Y_2 be as defined above. Then $P = P2 - P1$ is a projection if and only if $Y_1 \subset Y_2$. Furthermore, if P is a projection, then $P(H) = Y_1^\perp \cap Y_2$.*

Proof. If $Y_1 \subset Y_2$, then $P_2P_1 = P_1P_2 = P_1$. Since $P_1^2 = P_1$ and $P_2^2 = P_2$, we have

$$\begin{aligned} P^2 &= (P_2 - P_1)^2 \\ &= P_2^2 - P_1P_2 - P_2P_1 + P_1^2 \\ &= P_2 - P_1 - P_1 + P_1 \\ &= P_2 - P_1 = P, \end{aligned}$$

and thus $P^2 = P$. Conversely, $P^2 = P$ implies that

$$P_1P_2 + P_2P_1 = 2P_1. \tag{27.1}$$

Multiplying both sides of (27.1) by P_2 yields

$$P_2P_1P_2 + P_2^2P_1 = 2P_2P_1$$

and thus $P_2P_1P_2 = P_2P_1$, since $P_2^2 = P_2$. But the operator $P_2P_1P_2$ is self-adjoint, and thus

$$P_2P_1 = (P_2P_1)^* = P_1^*P_2^* = P_1P_2.$$

Hence $P_1P_2 = P_2P_1 = P_1$. □

27.1 The Spectral Family

Consider a finite dimensional Hilbert space $H = \mathbb{C}^n$, and a self-adjoint linear operator $T : H \longrightarrow H$. Then we may decompose T as

$$T = \sum_{i=1}^{n} a_i P_i \qquad \text{for eigenvalues } \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$$

with rank-one projections $P_i = x_i x_i^*$ given by the complete set of eigenvectors x_i. Then any vector $x \in H$ may be decomposed uniquely as

$$x = \sum_{i=1}^{n} \beta_i x_i$$

for some scalars β_i. Then for all x, we have $P_i x = \beta_i x_i$ for each i, and thus

$$\sum_{i=1}^{n} P_i = \mathbb{1}.$$

Unfortunately, this simple realization of finite-dimensional Hilbert spaces does not extend to infinite dimensions. We need to use different concepts. Instead, for all $\lambda \in \mathbb{R}$, denote

$$E_\lambda := \sum_{\substack{j \\ a_j \leq \lambda}} P_j,$$

where the values a_j comprise the point spectrum of T. This definition only works if the point spectrum is discrete (i.e. countable). Hence, we would have

$$E_{a_1} = P_1 \qquad \text{and} \qquad E_{a_2} = P_1 + P_2$$

and $E_\lambda = P_1$ for all $a_1 \leq \lambda < a_2$.

If the point spectrum is not discrete, we instead define

28 Lecture 28

(24 March 2014)

Definition 28.1. A ***real spectral family*** is a one parameter family $\mathcal{F} = (E_\lambda)_{\lambda\in\mathbb{R}}$ of projections E_λ on a Hilbert space H which satisfies the following properties:

(i) $E_\lambda \le E_\mu$ for $\lambda \le \mu$;

(ii) $\lim_{\lambda\to-\infty} E_\lambda x = 0$ and $\lim_{\lambda\to+\infty} E_\lambda x = x$ for all $x \in H$;

(iii) continuity from the right, that is

$$E_{\lambda+0}x \equiv \lim_{\mu\to\lambda^+} E_\mu x = E_\lambda x$$

for all x.

A ***spectral family*** on an interval $[a, b] \subset \mathbb{R}$ satisfies the properties (i) and (iii) above and the modified property

(ii*) $E_\lambda = 0$ for $\lambda < a$ and $E_\lambda = I$ for $\lambda \ge b$.

Definition 28.2. Let H be a Hilbert space and $T : H \longrightarrow H$ a self-adjoint linear operator. The ***spectral family associated with*** T is the family $\mathcal{F} = (E_\lambda)_{\lambda\in\mathbb{R}}$ of projections that are defined as follows:

$$E_\lambda := H \xrightarrow{\text{onto}} \mathcal{N}(T_\lambda^+)$$

where T_λ^+ is the operator

$$T_\lambda^+ = \frac{1}{2}\left(|T_\lambda| + T_\lambda\right).$$

Here, recall that $T_\lambda = T - \lambda I$ and $|T_\lambda| = \sqrt{T_\lambda T_\lambda^*}$. Note that each nullspace $\mathcal{N}(T_\lambda^+)$ is closed and thus has a projection operator defined onto it.

In finite dimensions, a self-adjoint operator is an hermitian matrix

$$\begin{aligned} A = A^* &= \sum_{k=1}^{n} \lambda_k V_k V_k^* \\ &= \sum_{\substack{k \\ \lambda_k \ge 0}} \lambda_k V_k V_k^* - \sum_{\substack{k \\ \lambda_k < 0}} |\lambda_k| V_k V_k^* \\ &= A^+ - A^- \end{aligned}$$

such that $A^\pm \ge 0$. The absolute value of the matrix is

$$|A| = A^+ + A^- \qquad \text{such that} \qquad A^\pm = \frac{1}{2}(|A| \pm A)$$

Note that A, A^+ and A^- all commute with each other.

We can now consider the infinite-dimensional case.

Proposition 28.3 (Properties of self-adjoint operators)**.** *Let $T : H \longrightarrow H$ be a self-adjoint hermitian operator. Denote $E : H \xrightarrow{onto} \mathcal{N}(T^+)$ the projection onto the the nullspace of T^+.*

1. $[T, |T|] = [T, T^\pm] = 0$;

2. $T^+T^- = 0$;
3. $[E, T] = [E, |T|] = 0$;
4. $T^+E = ET^+ = 0$ *and* $T^-E = ET^- = T^-$;
5. $TE = -T^-$ *and* $T(I-E) = T^+$;
6. $T^\pm \geq 0$.

Proof. Parts (1) and (2) are trivial (exercise left to reader).

3. Take $x \in H$ and $y = Ex \in \mathcal{N}(T^+)$. Then $T^+y = 0$ and $TT^+y = T^+Ty = 0$. Hence $Ty \in \mathcal{N}(T^+)$, and thus
$$T^+TEx = TT^+Ex = TT^+y = 0.$$
So $TEx \in \mathcal{N}(T^+) = Y$ and $ETEx = TEx$ since E projects H onto Y. Since this is true for all $x \in H$, we have $TE = ETE$. Taking the adjoint, we obtain
$$TE = ETE = (ETE)^\times = (TE)^\times = ET$$
since all operators involved are self-adjoint, and thus $ET = TE$. The argument follows similarly for $|T|$.
4. For all $x \in H$ we have $Ex \in \mathcal{N}(T^+)$. Hence $T^+Ex = 0$ and thus $T^+E = 0$. Taking the adjoint yields $ET^+ = 0$ since both T^+ and E are self-adjoint.
Similarly, $T^+T^-x = 0$ so $T^-x \in \mathcal{N}(T^+)$ for all $x \in H$. Hence $ET^-x = T^-x$ for all x and thus $ET^- = T^-$. Since all operators here are self-adjoint, we also have $T^-E = T^-$.
5. Note that $TE = (T^+ - T^-)E = -T^-$.
6. From (4), we have that $T^- = ET^- + ET^+ = E|T| \geq 0$, since $[E, |T|]$ commute and $E, |T| \geq 0$ (exercise – part of the next assignment[1]). Similarly, we have $T^+ = |T| - T^- = |T|(I-E) \geq 0$.

□

Lemma 28.4. *Let $T : H \longrightarrow H$ be a self-adjoit operator, then for $\mu > \lambda$ and the operators T^+_λ and T^+_μ defined above, we have $T^+_\mu T^+_\lambda \geq (T^+_\mu)^2$.*

Proof. Note that $T^+_\mu T^+_\lambda \geq (T^+_\mu)^2$ is true if and only if $T^+_\mu(T^+_\lambda - T^+_\mu) \geq 0$. This is equivalent to
$$T^+_\mu(T^+_\lambda \underbrace{-T^+_\mu + T^-_\mu}_{-T_\mu}) \geq 0.$$
and thus equivalent to
$$T^+_\mu(T^+_\lambda - T_\mu) \geq 0.$$
But $T^+_\lambda = T_\lambda + T^-_\lambda$ which implies $T^+_\lambda \geq T_\lambda$.

Indeed we have
$$T^+_\lambda - T_\mu \geq T_\lambda - T_\mu = (\mu - \lambda)I \geq 0$$
and
$$\langle T^+_\lambda x, x \rangle = \langle T_\lambda x, x \rangle + \langle T^-_\lambda x, x \rangle.$$

□

Proposition 28.5. *Let $T : H \longrightarrow H$ be a self-adjoint operator of a Hilbert space H. Then the spectral family associated with T is indeed a spectral family.*

[1] If $A, B \geq 0$ are self-adjoint operators and $[A, B] = 0$, then $AB \geq 0$.

Proof. We need to show that it satisfies each of the properties (i), (ii*) and (iii).

(i) Let $\lambda < \mu$, then $E_\lambda \leq E_\mu$ is true if and only if $\mathcal{N}(T_\lambda^+) \subseteq \mathcal{N}(T_\mu^+)$.

Take $x \in \mathcal{N}(T_\lambda^+)$, then $T_\lambda^+ x = 0$. By the above lemma, we have

$$\begin{aligned} 0 = \langle T_\mu^+ T_\lambda^+ x, x \rangle &\geq \langle (T_\mu^+)^2 x, x \rangle \\ &= \left\| T_\mu^+ x \right\|^2 \end{aligned}$$

and thus $T_\mu^+ x = 0$ for all $x \in \mathcal{N}(T_\lambda^+)$. Hence $\mathcal{N}(T_\lambda^+) \supseteq \mathcal{N}(T_\mu^+)$ and thus $E_\lambda \leq E_\mu$ as desired.

(Rest for next time...) □

31 Lecture 31

(31 March 2014)

31.1 Spectral representation of self-adjoint linear operators

Each self-adjoint linear operator $T : H \longrightarrow H$ on a Hilbert space H has the spectral representation

$$T = \int_{m-0}^{M} \lambda dE_\lambda$$

and for all $x, y \in H$ we have the representation

$$\langle Tx, y\rangle = \int_{m-0}^{M} \lambda dw(\lambda)$$

where $w(\lambda) = \langle E_\lambda x, y\rangle$.

Note 31.1. We have some useful generalizations of the spectral representation:

1. Given a polynomial p of degree n, i.e. $p(t) = a_0 + a_1 t + \cdots a_n t^n$, we have a representation of $p(T) = a_0 I + a_1 T + \cdots + a_n T^n$ given by

$$p(T) = \int_{m-0}^{M} p(\lambda) dE_\lambda.$$

2. Since any continuous real-valued function f may be approximated by polynomials, we also have

$$f(T) = \int_{m-0}^{M} f(\lambda) dE_\lambda.$$

   (There are lots of details that we are skipping here. For example, what does $f(T)$ even mean for an arbitrary function f and operator T? For details, see the book.)

31.1.1 Properties of $(E_\lambda)_{\lambda\in\mathbb{R}}$

Theorem 31.2 (Property 1)**.** *Let $T : H \longrightarrow H$ be a self-adjoint linear operator on a Hilbert space H.*

Then $E_{\lambda_0} \neq E_{\lambda_0 - 0}$ if and only if $\lambda_0 \in \sigma_p(T)$, that is, λ_0 is in the point spectrum of T if and only if the family E_λ is continuous at λ_0. Furthermore, $\mathcal{N}(T_{\lambda_0}) = F_0(H)$ where we denote $F_0 = E_{\lambda_0} - E_{\lambda_0 - 0}$.

Proof. We prove that $\mathcal{N}(T_{\lambda_0}) = F_0(H)$, and the first conclusion follows. Let $\lambda_0 \in \sigma_p(T)$, then λ_0 is an eigenvalue and $\mathcal{N}(T_{\lambda_0}) \neq \{0\}$. Note that F_0 is a projection operator since E_{λ_0} and E_{λ_0-0} are both projections and $E_{\lambda_0-0} \leq E_{\lambda_0}$.

First suppose that $x \in F_0(H)$. We will make use of the inequality

$$\left(\lambda_0 - \tfrac{1}{n}\right) \Delta E \leq T\Delta E \leq \lambda_0 \Delta E$$

where $\lambda_0 - \frac{1}{n} \leq \lambda_0$ and $\Delta E = E_{\lambda_0} - E_{\lambda_0 - \frac{1}{n}}$. Note that $E_{\lambda_0 - 0} = \lim_{n\to\infty} E_{\lambda_0} - E_{\lambda_0 - \frac{1}{n}}$, and this implies

$$\lambda_0 F_0 \leq T F_0 \leq \lambda_0 F_0.$$

So $TF_0 = \lambda_0 F_0$ and thus $T_{\lambda_0} F_0 = 0$. Hence $T_{\lambda_0} F_0 x = 0$ for all $x \in H$ and thus $F_0(H) \subset \mathcal{N}(T_{\lambda_0})$.

Now suppose that $\lambda_0 \notin \sigma_p(T)$. If $\lambda_0 \notin [m, M]$, then $\lambda_0 \in \rho(T)$. Hence we have $\mathcal{N}(T_{\lambda_0}) = \{0\} \subset F_0(H)$, and thus without loss of generality we may assume that $\lambda_0 \in [m, M]$. Take $x \in \mathcal{N}(T_{\lambda_0})$, then clearly $T_{\lambda_0}^2 x = 0$. Noting that $(T_{\lambda_0})^2 = (T - \lambda_0 I)(T - \lambda_0 I)$, we have

$$\begin{aligned} 0 &= \langle T_{\lambda_0}^2 x, x\rangle \\ &= \int_{m-0}^{M} (\lambda - \lambda_0)^2 dw(\lambda). \end{aligned}$$

But $w(\lambda) = \langle E_\lambda x, x\rangle$ is an increasing function of λ, hence $dw(\lambda) \geq 0$. So for any $a < \lambda_0$ and $\varepsilon > 0$, we have

$$\begin{aligned} 0 &= \int_{a}^{\lambda_0 - \varepsilon} (\lambda - \lambda_0)^2 dw(\lambda) \\ &\geq \varepsilon^2 \int_{a}^{\lambda_0 - \varepsilon} dw(\lambda) \\ &= \varepsilon^2 w(\lambda_0 - \varepsilon) \\ &= \varepsilon^2 \langle E_{\lambda_0 - \varepsilon}, x\rangle \end{aligned}$$

and similarly

$$\begin{aligned} 0 &= \int_{\lambda_0+\varepsilon}^{M} (\lambda - \lambda_0)^2 dw(\lambda) \\ &\geq \varepsilon^2 \int_{\lambda_0+\varepsilon}^{M} dw(\lambda) \\ &= \varepsilon^2 \left(w(M) - w(\lambda_0 + \varepsilon)\right) \\ &= \varepsilon^2 \left(\langle x, x\rangle - \langle E_{\lambda_0+\varepsilon}, x\rangle\right). \end{aligned}$$

Putting these together, we have $\langle E_{\lambda_0-\varepsilon} x, x\rangle = 0$ and $\langle E_{\lambda_0+\varepsilon} x, x\rangle = \langle x, x\rangle$. Taking the limit $\varepsilon \to 0$, we have

$$\langle F_0 x, x\rangle = \langle x, x\rangle,$$

and thus $\left\|(I - F_0)^2 x\right\|^2 = \langle (I - F_0)x, x\rangle = 0$, so $F_0 x = x$ for all $x \in \mathcal{N}(T_{\lambda_0})$. Hence $F_0(H) \subset \mathcal{N}(T_{\lambda_0})$. ☐

Theorem 31.3 (Property 2)**.** *Let T and $(E_\lambda)_{\lambda\in\mathbb{R}}$ be as above. Then $\lambda_0 \in \rho(T)$ if and only if there exists a constant $c > 0$ such that E_λ is constant on $[\lambda_0 - c, \lambda_0 + c] = J$.*

32 Lecture 32

(2 April 2014)

Theorem 32.1. *Let T and $(E_\lambda)_{\lambda\in\mathbb{R}}$ be as before. Then $\lambda_0 \in \rho(T)$ if and only if there exists a $c > 0$ such that E_λ is a constant on $J = [\lambda_0 - c, \lambda + c]$.*

Proof. Suppose that E_λ is constant on J. Recall that $\lambda \in \rho(T)$ if and only if there exists a $c > 0$ such that $\|T_\lambda x\| > c\|x\|$. For $\lambda \in J$, we have $dw(\lambda) = d\langle E_\lambda x, x\rangle = 0$ since E_λ is constant on this interval. But for $\lambda \notin J$, we have $(\lambda - \lambda_0)^2 \geq c^2$. Then

$$\begin{aligned}
\|T_{\lambda_0}x\|^2 = \langle T_{\lambda_0}x, T_{\lambda_0}\rangle = \left\langle T_{\lambda_0}^2 x, x\right\rangle
&= \int_{m-0}^{M} (\lambda - \lambda_0)^2 d\langle E_\lambda x, x\rangle \\
&\geq c^2 \int_{m-0}^{M} d\langle E_\lambda x, x\rangle \\
&= c^2 \langle E_M x, x\rangle \\
&= c^2 \|x\|^2,
\end{aligned}$$

and thus $\|T_\lambda x\| \geq c\|x\|$, hence $\lambda \in \rho(T)$.

Conversely, suppose that $\lambda_0 \in \rho(T)$, which is true if and only if there exists a $c > 0$ such that $\|T_{\lambda_0}x\| \geq c\|x\|$, and denote $J = [\lambda_0 - c, \lambda + c]$. Using the same arguments as above, we have that this is equivalent to

$$\int_{m-0}^{M} (\lambda - \lambda_0)^2 d\langle E_\lambda x, x\rangle \geq c^2 \int_{m-0}^{M} d\langle E_\lambda x, x\rangle$$

for all $x \in H$. Suppose that E_λ is not constant on J. Then there exists an $\eta > 0$ with $\eta < c$ such that $E_{\lambda_0-\eta} \neq E_{\lambda_0+\eta}$, and thus there is a nonzero $y \in H$ such that $x = (E_{\lambda_0+\eta} - E_{\lambda_0-\eta})y \neq 0$. Next, for an arbitrary λ we calculate

$$E_\lambda x = \begin{cases} 0, & \lambda < \lambda_0 - \eta \\ x, & \lambda > \lambda_0 - \eta \\ (E_\lambda - E_{\lambda_0-\eta})y, & \text{otherwise} \end{cases}$$

since $E_\mu E_\nu = E_\nu$ if $\mu < \nu$ and $E_\mu E_\nu = E_\mu$ if $\mu > \nu$. We can split the integral into three parts

$$\begin{aligned}
&\int_{m-0}^{M} (\lambda - \lambda_0)^2 dw(\lambda) = \\
&\qquad = \int_{m-0}^{\lambda_0-\eta} (\lambda - \lambda_0)^2 dw(\lambda) + \int_{\lambda_0-\eta}^{\lambda_0+\eta} (\lambda - \lambda_0)^2 dw(\lambda) + \int_{\lambda_0+\eta}^{M} (\lambda - \lambda_0)^2 dw(\lambda)
\end{aligned}$$

where $dw(\lambda) = d\langle E_\lambda x, x\rangle$. However, from above, note that E_λ only depends on lambda if $\lambda \in [\lambda_0 - \eta, \lambda_0 + \eta]$, and thus $dw(\lambda) = 0$ outside of this range. Hence we have

$$\int_{\lambda_0-\eta}^{\lambda_0+\eta} (\lambda - \lambda_0)^2 d\langle E_\lambda x, x\rangle \geq c^2 \int_{\lambda_0-\eta}^{\lambda_0+\eta} d\langle E_\lambda x, x\rangle$$

which is false, since $(\lambda - \lambda_0)^2 < \eta^2 < c^2$ for all λ in this range. So we have a contradiction. □

Note that we can choose $\eta < c$. Why not $\eta = c$? If $\eta = c$ were the only positive constant less than or equal to c such that this were true, then E_λ would be constant on the open interval of J.

33 Lecture 33

(4 April 2014)

33.1 Compactness

We have used notions of *compactness* so far in the homework problems in the course, but we have yet to rigorously define this notion.

Definition 33.1. A metric space X is ***compact*** if every sequence has a convergent subsequence. A subset $M \subset X$ is compact if every sequence in M has a convergent subsequence in M.

Proposition 33.2 (Properties of compactness)**.** .

1. *If $M \subset X$ is a compact set, then it is closed and bounded.*
2. *There are sets that are closed and bounded but not compact.*
3. *In finite dimensions, a subset $M \subset X$ is comapct if and only if it is closed and bounded.*
4. *Let X be a normed space. If $B(0;1)$ is compact, then $\dim X < \infty$.*

Lemma 33.3 (Riesz's Lemma)**.** *Let X be a normed space. Let $Z \subset X$ be a subspace and $Y \subsetneq Z$ be a proper closed subspace. Then for any $0 < t < 1$, there exists $z \in Z$ such that $\|z\| = 1$ and $\|z - y\| \geq t$ for all $y \in Y$.*

Proof. Take $x \in Z \setminus Y$ and define $d = \inf_{y \in Y} \|x - y\|$. Note that $d > 0$ since otherwise $x \in Y$ since Y is closed. Then there exists a y_0 such that $d \leq \|x - y_0\| \leq \frac{d}{t}$. Setting $z = \frac{x - y_0}{\|x - y_0\|}$ we have

$$
\begin{aligned}
\|z - y\| &= \left\| \frac{z - y_0}{\|z - y_0\|} - y \right\| \\
&= \left\| \frac{x - y_0 - y\|x - y_0\|}{\|x - y_0\|} \right\| \\
&= \|x - y_0\|^{-1} \|x - y'\| \\
&\geq \frac{t}{d} \|x - y'\| \\
&\geq t
\end{aligned}
$$

where $y' = y_0 - y\|x - y_0\| \in Y$. □

Proof of Proposition 33.2 part (4). Suppose that $B(0;1)$ is compact and $\dim X = \infty$. Take $x_1 \in X$ such that $\|x_1\| = 1$ and define $X_1 = \operatorname{span}\{x_1\}$. Since $X_1 \subset X$ is a closed proper subset of X, by Reisz's Lemma there exists a $x_2 \in X \setminus X_1$ such that $\|x_2\| = 1$ and $\|x_2 - x_1\| \geq \frac{1}{2}$ then define $X_2 = \operatorname{span}\{x_1, x_2\}$. Continuing this process by induction, we contstuct a sequence $\{x_n\}$ such that $\|x_n\| = 1$ and $\|x_n - x_m\| \geq \frac{1}{2}$ for all $m \neq n$. So this sequence does not have a convergent subsequence. □

Definition 33.4. Let X be a normed space and $M \subset X$. Then M is ***relatively compact*** in X if its closure in X is compact.

Definition 33.5. Let X and Y be normed spaces. A linear operator $T \colon X \longrightarrow Y$ is called ***compact*** if for every $M \subset X$ bounded subset of X, $T(M)$ is relatively compact in Y.

Historically, the notion of compact operators were originally called *completely continuous*, since (as we will see) any compact operator is bounded and thus continuous.

Lemma 33.6. *Let X,Y and T be as above. Then*

1. *T is bounded.*
2. *If $\dim X = \infty$ then the identity operator $I\colon X \longrightarrow X$ is not compact.*

Proof. .

1. Take $S = \{x \in X \mid \|x\| = 1\}$, which is bounded. Then $\overline{T(S)}$ is compact and thus
$$\sup_{x\in S} \|T(x)\| < \infty$$
so T is bounded.
2. Consider the unit ball at the origin $B(0;1) \subset X$. If $\dim X = \infty$ then by the previous proposition $I\big(B(0;1)\big) = B(0;1) = \overline{B(0;1)}$ is not compact. So I is not compact.

□

34 Lecture 34

(7 April 2014)

Theorem 34.1. *A linear operator T is compact if and only if every bounded sequence (x_n) in X is mapped to a sequence (Tx_n) that has a convergent subsequence.*

Proof. Suppose T is compact, then $\overline{T\{x_n\}}$ is compact and thus (Tx_n) has a convergent subsequence.

On the other hand, let $M \subset X$ be a bounded set and let $(y_n) \subset T(M)$ be a sequence. Then $y_n = Tx_n$ for some sequence (x_n) in M. Then (x_n) is bounded and $(y_n) = (Tx_n)$ has a convergent subsequence by assumption. □

Theorem 34.2. *Let X, Y be normed spaces and $T\colon X \longrightarrow Y$ a linear operator.*

i) If T is bounded and $\dim T(X) < \infty$, then T is compact.

ii) If $\dim(X) < \infty$ then T is compact.

Proof. .

i) Take a bounded sequence (x_n) in X, then (Tx_n) is bounded in Y. Then

$$\|Tx_n\| \le \|T\|\|x_n\|$$

hence $\overline{\{Tx_n\}}$ is a closed and bounded set in a finite dimensional space.

ii) Note that $\dim T(X) \le \dim(X) < \infty$. Then T is bounded since $\dim(X) < \infty$. From part (i), T is compact.

□

Theorem 34.3. *Let X be a normed space and Y a Banach space and let (T_n) be a sequence of compact linear operators $T_n\colon X \longrightarrow Y$. If $\|T_n - T\| \longrightarrow 0$ as $n \longrightarrow \infty$, then T is compact.*

Proof. Take a bounded sequence (x_m), then $(T_1 x_m)$ has a convergent subsequence which we denote by $(T_1 x_{m_{k_1}})$. But T_2 is compact, so $(T_2 x_{m_{k_1}})$ has a convergent subsequence which we denote by $(T_2 x_{m_{k_2}})$. By induction, we construct a sequence of subsequences

$$\cdots \subset (x_{m_{k_2}}) \subset (x_{m_{k_1}}) \subset (x_m)$$

such that $(T_n x_{m_{k_n}})$ is convergent for all n. Using the Cantor diagonalization argument, we can construct a sequence (y_m) that is a subsequence of $(x_{m_{k_n}})$ for all n and thus $(T_n y_m)_{m\in\mathbb{N}}$ is convergent for all n.

We claim that (Ty_m) is also Cauchy. Indeed, we have

$$\begin{aligned}\|Ty_k - Ty_j\| &\le \|Ty_k - T_n y_k\| + \|T_n y_k - T_n y_j\| + \|T_n y_j - Ty_j\| \\ &\le \underbrace{\|T - T_n\|}_{\to 0}\|y_k\| + \|T_n\|\underbrace{\|y_k - y_j\|}_{\to 0} + \underbrace{\|T_n - T\|}_{\to 0}\|y_j\| \\ &\longrightarrow 0.\end{aligned}$$

□

Question: Can we replace uniform convergence in Theorem 34.3 with strong convergence? No! We make use of the fact that the identity operator I in an infinite-dimensional space is not compact.

Example 34.4. Take $X = \ell^2$ and $T_n \colon X \longrightarrow X$ a sequence of operators given by $y_n = T_n x$ with

$$y_n = (\xi_1, \xi_2, \dots, \xi_n, 0, \dots)$$

where $x = (x_i)$. Then $\|T_n x - x\| \longrightarrow 0$ for all $x \in \ell^2$. Hence we have strong convergence $T_n \xrightarrow{s} I$. Each T_n is a compact operator, but I is not compact.

Example 34.5. Show that $T \colon \ell^2 \longrightarrow \ell^2$ is compact, where given by $y = Tx$ with

$$y = (\xi_1, \tfrac{1}{2}\xi_2, \tfrac{1}{3}\xi_3, \dots)$$

and $x = (\xi_i)$.

Proof. Define the sequence of operators (T_n) given by $y_n = T_n x$

$$y_n = (\xi_1, \tfrac{1}{2}\xi_2, \dots, \tfrac{1}{n}\xi_n, 0 \dots).$$

Then each T_n is compact. We have

$$\begin{aligned} \|(T - T_n)x\|^2 = \sum_{k=n+1}^{\infty} \frac{1}{k^2} |\xi_k|^2 &\leq \frac{1}{(n+1)^2} \sum_{k=n+1}^{\infty} |\xi_k|^2 \\ &\leq \frac{\|x\|^2}{(n+1)^2} \end{aligned}$$

which implies $\|T - T_n\| \leq \frac{1}{n+1}$ and thus $\|T - T_n\| \longrightarrow 0$. □

Theorem 34.6. *Let X, Y be normed spaces and $T \colon X \longrightarrow Y$ be a compact linear operator. If $x_n \xrightarrow{w} x$ in X then $Tx_n \longrightarrow Tx$ in Y.*

Proof. First note that $Tx_n \xrightarrow{w} Tx$ since, for all $g \in Y'$ we have the linear functional $f \in X'$ defined by

$$f(x) = g(Tx).$$

Hence $g(Tx_n) = f(x_n) \longrightarrow f(x) = g(Tx_n)$ by weak convergence of $x_n \xrightarrow{w} x$.

We still need to show string convergence. Suppose that $Tx_n \not\to Tx$. Then there exists a constant $c > 0$ and a subsequence (Tx_{n_k}) such that

$$\|Tx_{n_k} - Tx\| \geq c.$$

But $\{x_{n_k}\}$ is bounded, and thus by compactness of T we must have a subsequence of this subsequence $\left(Tx_{n_{k_j}}\right)$ that converges to $\longrightarrow y = Tx$. □

35 Lecture 35

(**9 April 2014**)

There was an important statement that we used last time but did not prove.

Proposition 35.1. *Let X be a metric space. A subset $B \subset X$ is relatively compact in X if and only if for every sequence (x_n) in B there is a convergent subsequence in X.*

Proof. One direction is clear. For the other direction, let (y_n) be a sequence in $\overline{B}$. Then there exists a sequence (x_n) in B such that $d(x_n, y_n) < \frac{1}{n}$ for all n. By assumption, there is a subsequence (x_{n_k}) that converges to some $x \in X$. Then (y_{n_k}) also converges to x, but then x must be in B since B is closed. □

Definition 35.2 (ε-net, total boundedness)**.** Let $\varepsilon > 0$ and B a subset of a metric space X. A set $M_\varepsilon \subset X$ is called an ***ε-net*** for B if for every point $z \in B$ there is a point of M_ε that is a distance less than ε away from z. The set B is called ***totally bounded*** if for every $\varepsilon > 0$ there is a finite ε-net $M_\varepsilon \subset X$ for B.

Proposition 35.3 (Properties of ε-net and total boundedness)**.** *Let X be a metric space and let $B \subset X$ be a subset.*

1. *If B is relatively compact then B is totally bounded.*
2. *If B is totally bounded and X Banach then B is relatively compact.*
3. *If B is totally bounded then for all $\varepsilon > 0$ there exists a finite ε-net M_ε of B such that $M_\varepsilon \subset B$.*
4. *If B is totally bounded then B is separable.*
5. *Total boundedness implies boundedness, but bounded does not necessarily imply totally bounded.*

Proof. .

1. Suppose that $B \subset X$ is relatively compact and let $\varepsilon > 0$. If $B \neq \emptyset$ is not totally bounded, take $x_1 \in B$. If $d(x_1, x) < \varepsilon$ for all $x \in X$ then $\{x_1\}$ is an ε-net for B and we are done. Otherwise, there exists a $x_2 \in B$ such that $d(x_1, x_2) \geq \varepsilon$. If $d(x_1, x) < \varepsilon$ or $d(x_2, x) < \varepsilon$ for all $x \in X$ then we are done and $\{x_1, x_2\}$ is an ε-net for B. Continuing this process inductively, we construct a sequence (x_n) in B such that $d(x_n, x_{n+1}) \geq \varepsilon$ for all n which does not have a convergent subsequence, a contradiction to the assumption that B is relatively compact.

2. Let $B \subset X$ be a totally bounded subset of a Banach space X. Take a sequence (x_n) in B and $\varepsilon = 1$. Since B is totally bounded, there exists a finite ε-net $M_{\varepsilon=1} \subset X$. For each k define the ball $B(x_k; 1)$ centered at x_k with radius 1. Since
$$B \subset \bigcup_{k=1}^{n} B_k$$
there exists a $k \in \mathbb{N}$ such that $(x_{n_{k_1}})$ is a subsequence of (x_n) in B_k. Similarly, for $\varepsilon = \frac{1}{2}$ there exists a finite ε-net $M_{\varepsilon=\frac{1}{2}} \subset X$ for B. So there exists a subsequence $(x_{n_{k_2}})$ of $(x_{n_{k_1}})$ such that $(x_{n_{k_2}}) \subset B(z_k; \frac{1}{2})$.

   For any $m \in NN$ there exists a subsequence $(x_{m,n})$(??need better notation) such that $(x_{m,n}) \subset$ Ball of radius $\frac{1}{m}$. Using Cantor diagonalization, define the sequence $(y_m) \subset (x_n)$ with $y_m = x_{m,m}$. Then (y_m) is Cauchy, since $\|y_m - y_{m'}\| \leq \frac{1}{m}$ for all $m' < m$.

3. Take $\varepsilon > 0$ and $\varepsilon_1 = \frac{1}{2}\varepsilon$. Take $z_k \in B \cap B(x_k; \varepsilon_1)$ and define the set $M_\varepsilon = \{z_1, \ldots, z_n\}$.

   We claim that $M_\varepsilon \subset B$ is an ε-net for B. Indeed, take $z \in B$. Then there exists a k such that $z \in B(x_k; \varepsilon)$ and thus $d(z, z_k) \leq d(z, x_k) + d(x_k, z_k) < \varepsilon_1 + \varepsilon_1 = \varepsilon$.

4.

5. To show that total boundedness implies boundedness is trivial (left as an exercise). On the otherhand, consider the unit ball $B(0;1) \subset \ell^2$. Then $\overline{B(0;1)}$ is not compact since $\dim \ell^2 = \infty$. From (2), this implies that $B(0;1)$ is not totally bounded.

$\square$

36 Lecture 36

(11 April 2014)

Theorem 36.1. *Let X and Y be normed spaces and $T\colon X \longrightarrow Y$ be a compact linear operator. Then the range of T is separable.*

Proof. We can write $X = \bigcup_{n=1}^{\infty} B(0;n)$. Then we have that

$$T(X) = \bigcup_{n=1}^{\infty} T\big(B(0;n)\big)$$

but $B(0;n)$ is bounded, so $T\big(B(0;n)\big)$ is relatively compact for each n and hence $T\big(B(0;n)\big)$ is totally bounded so it is separable. Then $T(X)$ is separable since it is a countable union of separable sets. □

Theorem 36.2. *If $T\colon X \longrightarrow Y$ is compact, then $T^{\times}\colon Y' \longrightarrow X'$ is also compact.*

Proof. Let $B \subset Y'$ be a bounded subset. Consider $T^{\times}(B) \subset X'$. Since X' is Banach, we need to show that $T^{\times}(B) \subset X'$ is totally bounded. Let $U = \{x \in X \mid \|x\| \leq 1\}$, which is bounded. So $T(U) \subset Y$ is relatively compact by compactness of T, and thus $T(U)$ is totally bounded. For some $\varepsilon > 0$ there exists an ε0-net M_{ε} for $T^{\times}(B)$.

Since $T(U)$ is totally bounded, there exist elements $x_1, \ldots, x_n \in U$ such that for all $x \in U$ there exists a $j \in \{1, \ldots, n\}$ with $\|Tx - Tx_j\| < \varepsilon_1$. Define $A\colon Y' \longrightarrow \mathbb{R}^n$ by

$$\begin{aligned} g \longmapsto & (T^{\times}g(x_1), \ldots, T^{\times}g(x_n)) \\ = & (g(Tx_1), \ldots, g(Tx_n)). \end{aligned}$$

Since A is bounded, $\dim A(Y') \leq n < \infty$, we have that A is compact. Then $A(B)$ is relatively compact, so $A(B)$ is totally bounded. This implies that there exist $g_1, \ldots, g_m \in B$ such that for all $g \in B$ there exists a $k \in \{1, \ldots, m\}$ such that $\|Ag - Ag_k\| < \varepsilon_2$.

Claim: The set $\{T^{\times}g_1, \ldots, T^{\times}g_m\}$ is an ε-net for $T^{\times}(B)$.

Note that we should choose $\varepsilon_1 = \frac{\varepsilon}{3c}$ where c is a constant such that $\|g\| < c$ for all $g \in B$, and $\varepsilon_2 = \frac{\varepsilon}{4}$ □

36.1 Solutions to difficult homework problems

Problem 1 (Problem 7 of assignment 5 (Exercise 7.5.10, p. 394 in Kreyszig))**.** Show that the existence of the limit of $\sqrt[n]{\|T^n\|}$ as $n \longrightarrow \infty$ follows from the fact that $\|T^{n+m}\| \leq \|T^n\|\|T^m\|$.

Solution. Define the sequences $a_n = \|T^n\|$ and $b_n = \ln a_n = \ln \|T^n\|$. By assumption we have that $\|T^{n+m}\| \leq \|T^n\|\|T^m\|$ for all $n, m \in NN$. This implies that that $b_{n+m} \leq b_n + b_m$. Let $n, m \in \mathbb{N}$ and suppose wlog that $n > m$, then $n = mq + r$ for some $q \geq 0$ and $0 \leq r < m$. Then we have

$$\frac{b_n}{n} = \frac{b_{mq+r}}{mq+r} \leq \frac{b_{mq} + b_r}{mq+r} \leq \frac{qb_m + b_r}{mq+r} \overset{q\to\infty}{\longrightarrow} \frac{b_m}{m}.$$

(That is, we take $m, r \in \mathbb{N}$ fixed and let $n = qm + r$ for all $q \in \mathbb{N}$.) Let $\varepsilon > 0$ then there exists an $N \in \mathbb{N}$ such that

$$\frac{b_n}{n} \leq \frac{b_m}{m} + \frac{\varepsilon}{2}$$

for all $n > N$. Take m such that

$$\left| \frac{b_m}{m} - \inf_{m'} \frac{b_{m'}}{m'} \right| < \frac{\varepsilon}{2}.$$

Define $\alpha = \inf_{m'} \frac{b_{m'}}{m'}$, then $\alpha \leq \frac{b_n}{n} \leq \alpha + \varepsilon$.

Problem 2 (Problem 8 of assignment 5 (Exercise 7.6.8, p. 403 in Kreyszig))**.** Let $\mathcal{A}$ be a Banach algebra and let $G \subset \mathcal{A}$ be the subset of all invertible elements. Then the inverse map $T\colon G \longrightarrow G$ given by $Tx = x^{-1}$ is continuous.

Solution. Let $x_0 \in G$ and $\varepsilon > 0$. We need to show that there exists $\delta > 0$ such that $\|x^{-1} - {x_0}^{-1}\| < \varepsilon$ for all x such that $\|x - x_0\| < \delta$.

Note that $\|x^{-1} - {x_0}^{-1}\| = \left\|{x_0}^{-1}(x_0 - x)x^{-1}\right\| \le \|{x_0}^{-1}\|\|x^{-1}\|\|x - x_0\|$. Then

$$\|x^{-1}\| - \|{x_0}^{-1}\| \le \|x^{-1} - {x_0}^{-1}\| < \|{x_0}^{-1}\|\|x^{-1}\|\delta,$$

and from this we can find δ from ε.

36.2 Things to know for final

- Learn the first version of the Hahn-Banach theorem.
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References

[KREY89] Erwin Kreyszig. *Introductory Functional Analysis with Applications.* Wiley, 1989.

[CON90] John B. Conway. *A Course in Functional Analysis.* Graduate Texts in Mathematics, vol. 96. Springer, 1990.

[RUD91] Walter Rudin. *Functional analysis.* McGraw-Hill, 1991.