Functional Analysis

Review

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1 Zorn's Lemma

Definition 1.1. A ***partially ordered set*** is a set S with a binary operation "$\leq$" satisfying the following:

(i) $a \leq a$ for all $a \in S$;

(ii) $a \leq b$ and $b \leq a$ implies $a = b$;

(iii) $a \leq b$ and $b \leq c$ implies $a \leq c$.

Definition 1.2. A ***chain*** is a subset $M \subset S$ of a partially ordered set S with the property that every two elements in M are comparable.

Definition 1.3. Let $C \subset M$ be a subset of a chain M in S. An ***upper bound*** of C in M is an element $u \in M$ such that $x \leq u$ for all $x \in C$. A ***maximal element*** of M is an element $m \in M$ such that $m \leq x$ implies $x = m$.

Zorn's Lemma 1.4. *Let $S \neq \emptyset$ be a partially ordered set. Suppose that every chain $C \subset S$ has an upper bound in S. Then S has at least one maximal element.*

Definition 1.5. Given a subset $M \neq \emptyset$ of a vector space X, the ***span*** of M is the set $\operatorname{span}(M)$ of all finite linear combinations of vectors in M,

Theorem 1.6. *Every vector space $X \neq \{0\}$ has a (Hamel) basis.*

2 Normed and Banach spaces

Definition 2.1. A ***metric space*** is a pair (X, d) where X is a set and $d\colon X \times X \longrightarrow \mathbb{R}$ is a mapping such that

(i) $d(x, y) \geq 0$ for all $x, y \in X$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, y) = 0$ if and only if $x = y$

(iv) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, , y, z \in X$.

Definition 2.2. A ***normed space*** is a pair $(V, \|\cdot\|)$ consisting of a vector space X and a function $\|\cdot\| : X \longrightarrow \mathbb{R}$ satisfying

i) $\|x\| \geq 0$ for all $x \in X$;

ii) $\|x\| = 0$ if and only if $x = 0$;

iii) $\|\alpha x\| = |\alpha| \, \|x\|$ for all scalars α;

iv) $\|x + y\| \leq \|x\| + \|y\|$.

A normed space is ***Banach*** if it is complete. A normed space is a metric space with $d(x, y) = \|x - y\|$.

Example 2.3. Some examples of normed spaces

(i) The space ℓ^p for $1 \leq p$

$$\left\{ x = (\xi_1, \xi_2, \dots) = (\xi_n)_{n \in \mathbb{N}} \,\middle|\, \sum_{j=1}^{\infty} |\xi_j|^p < \infty \right\}$$

with norm given by $\|x\|_p = \left(\sum_{j=1}^{\infty} |\xi_j|^p \right)^{1/p}$.

(ii) The space ℓ^∞

$$\left\{ x = (\xi_1, \xi_2, \dots) = (\xi_n)_{n\in\mathbb{N}} \;\middle|\; \sup_{n\in\mathbb{N}} |\xi_n| < \infty \right\}$$

with norm given by $\|x\|_\infty = \sup_{n\in\mathbb{N}} |\xi_n|$.

(iii) For an interval $[a, b]$, the space $\mathcal{C}([a, b])$ of continuous real-valued functions on $[a, b]$ with norm

$$\|x\| = \max_{a\le y\le b} |x(t)|$$

and this is complete (since convergence in this norm implies uniform convergence).

(iv) For an interval $[a, b]$, the space $L^p[a, b]$ of equivalence classes of real-valued functions on $[a, b]$ where $[x] = [y]$ if $x(t) = y(t)$ almost everywhere on $[a, b]$. The norm is given by

$$\|x\| = \left(\int_a^b |x(t)|^p \right)^{1/p}.$$

Theorem 2.4. *Let X be a Banach space and $Y \subset X$ a subspace. Then Y is complete if and only if Y is closed in X.*

3 Linear operators

Definition 3.1. A ***linear operator*** is a map $T\colon \mathcal{D}(T) \longrightarrow Y$ such that

1. the ***domain*** $\mathcal{D}(T)$ and the target space Y are vector spaces over the same field,
2. for all $x, y \in \mathcal{D}(T)$ and scalars α,

$$T(x + y) = T(x) + T(y) \qquad \text{and} \qquad T(\alpha x) = \alpha T(x).$$

The set $\mathcal{R}(T) = T\big(\mathcal{D}(T)\big)$ is called the ***range*** of T.

Proposition 3.2. *Let X, Y be vector spaces and let $T\colon \mathcal{D}(T) \longrightarrow Y$ be a linear operator with $\mathcal{D}(T) \subset X$.*

1. *The inverse operator $T^{-1}\colon \mathcal{R}(T) \longrightarrow \mathcal{D}(T)$ exists if and only if $Tx = 0$ implies $x = 0$.*
2. *If T^{-1} exists, then it is a linear operator.*
3. *If $\dim \mathcal{D}(T) = n < \infty$ and T^{-1} exists, then $\dim \mathcal{R}(T) = n$.*

Definition 3.3. Let X, Y be normed spaces and $T\colon \mathcal{D}(T) \longrightarrow Y$ a linear operator. Then T is ***bounded*** if there exists a positive constant $c > 0$ such that $\|Tx\| \le c\,\|x\|$ for all $x \in \mathcal{D}(T)$. The ***norm*** of a bounded operator is defined as

$$\|T\| := \sup_{\substack{x\in\mathcal{D}(T)\\ x\neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\|.$$

The set of all bounded linear operators $T\colon X \longrightarrow Y$ is denoted $\mathcal{B}(X, Y)$

Proposition 3.4. *Let X, Y be normed spaces. The vector space $\mathcal{B}(X, Y)$ is a normed space with the usual operator norm. If Y is Banach, then $\mathcal{B}(X, Y)$ is also Banach.*

Theorem 3.5. *Let X, Y be normed spaces and $T\colon \mathcal{D}(T) \longrightarrow Y$ be a linear operator with $\mathcal{D}(T) \subset X$. Then T is continuous if and only if it is bounded. If T is continuous at a single point then it is continuous everywhere.*

Theorem 3.6. *Let X, Y be Banach spaces and $T\colon \mathcal{D}(T) \longrightarrow Y$ be a linear operator with $\mathcal{D}(T) \subset X$. Then T has a linear extension $\tilde{T}\colon \overline{\mathcal{D}(T)} \longrightarrow Y$ such that $\left\|\tilde{T}\right\| = \|T\|$.*

3.1 Linear functionals

Definition 3.7. A (***linear***) ***functional*** is a (linear) operator $f\colon X \longrightarrow \mathbb{K}$ where X is a vector space (and $\mathbb{K}$ is the underlying field of X).

Definition 3.8. Let X be a normed space. The *dual space* of X, denoted X', is a normed spaces that consists of the set of all bounded linear functionals on X. The norm is given by the standard operator norm.

Proposition 3.9. *X' is Banach for any normed space X.*

4 Hilbert spaces

Definition 4.1. An ***inner product space*** is a vector space X with an *inner product*, i.e. a mapping

$$\langle \cdot, \cdot \rangle : X \times X \longrightarrow \mathbb{K}$$

into the scalar field, that satisfies (note that this is *backwards* to the usual physicists notation):

i) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

ii) $ang\alpha x, y = \alpha \langle x, y \rangle$

iii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

iv) $\langle x, x \rangle \geq 0$

v) $\langle x, x \rangle = 0$ if and only if $x = 0$. Any inner product space is a normed space with norm $\|x\| = \sqrt{\langle x, x \rangle}$.

Definition 4.2. A ***Hilbert*** space is a complete inner product space.

Proposition 4.3. *Properties of inner product spaces:*

1. *The Schwartz inequality holds:*
$$|\langle x, y \rangle| \leq \|x\| \, \|y\|$$
with equality if and only if x and y are linearly dependent.

2. *The inner product is a continuous function.*

3. *There is a unique completion (up to isomorphism).*

4. *A subspace $Y \subset H$ of a Hilbert space is complete if and only if it is closed in H.*

Proposition 4.4. *If X is a Hilbert space, then the norm satisfies the parallelogram identity*

$$\|x + y\|^2 + \|x - y\|^2 = 2 \left(\|x\|^2 + \|y^2\| \right).$$

As a corollary, ℓ^p is not a Hilbert space for $p \neq 2$.

Definition 4.5. Let H be a Hilbert space, $x, z \in H$ and $M \subset H$. Then we say that x and y are ***orthogonal*** of $\langle x, y \rangle = 0$, which is denoted $x \perp y$. We say $x \perp M$ if $|x, y| = 0$ for all $y \in M$.

Theorem 4.6 (Minimizing vector theorem)**.** *Let X be an inner product space and $M \neq \emptyset$ a complete convex subset of X. Then for any $x \in X$ there exists a unique $y \in M$ such that*

$$\delta = \inf_{y' \in M} \|x - y'\| = \|x - y\|.$$

Corollary 4.7. *If M is a complete subspace of X and y is the unique closest element in M to x from the above theorem, then the vector $z = x - y$ is orthogonal to M.*

Definition 4.8. A vector space X is said to be a ***direct sum*** of two subspaces Y and Z of X if each $x \in X$ has a unique representation $x = y + z$ for $yinY$ ann $z \in Z$. We write $X = Y \oplus Z$.

Theorem 4.9. *Let Y be a closed subspace of a Hilbert space H. Then $H = Y \oplus Y^{\perp}$.*

Definition 4.10. For a closed subspace $Y \subset H$ of a Hilbert space, the ***orthogonal projection*** onto Y is the linear operator $P_Y : H \longrightarrow Y$ defined by $P_Y x = y$, where y is the unique $y \in Y$ from the previous theorem.

Lemma 4.11. *If $Y \subset X$ where X is an inner product space, then $Y \subset Y^{\perp\perp}$. Furthermore, if $X = H$ is a Hilbert space and $Y = \overline{Y}$, then $Y = Y^{\perp\perp}$.*

Lemma 4.12. *Let $M \neq \emptyset$ be a subset of a Hilbert space. Then $\overline{\operatorname{span}(M)} = H$ if and only if $M^{\perp} = \{0\}$.*

Theorem 4.13 (Bessel inequality)**.** *Let X be an inner priduct space and (e_k) be an orthonormal sequence in X. Then for every $x \in X$ the following inequality holds:*

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2 .$$

Corollary 4.14. *If X is an inner product space, then any $x \in X$ can have at most countably many nonzero* Fourier coefficients *$\langle x, e_\kappa \rangle$ with respect to an orthonormal family $(e_\kappa)_{\kappa \in I} \subset X$ indexed by some (not necessarily countable) set I.*

Definition 4.15. Let $\{e_k\}$ be an orthonormal set of vectors in a Hilbert space H and $\{\alpha_k\}$ scalars. We say that $\sum_{k=1}^{\infty} \alpha_x e_k$ ***converges*** (or ***exists***) if there exists an $s \in H$ such that

$$\lim_{n \to \infty} \left\| \sum_{k=1}^{\infty} \alpha_k e_k - s \right\| = 0.$$

Theorem 4.16 (convergence)**.** *Let (α_n) be a sequence of scalars.*

1. *The sum $\sum_{k=1}^{\infty} \alpha_x e_k$ converges if and only if $\sum_{k=1}^{\infty} \|\alpha_k\|^2$ converges.*

2. *If $\sum_{k=1}^{\infty} \alpha_x e_k$ converges to some $x \in H$, then $\alpha_n = \langle x, e_n \rangle$ for all n.*

3. *For all $x \in H$, the sum $\sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$ converges to x.*

Definition 4.17. A subset $M \subset X$ of a normed space is said to be ***total*** if $\operatorname{span}(M)$ is dense in X. A ***total orthonormal set*** in an inner product space X is an orthonormal set M which is total in X.

Theorem 4.18 (Totality I)**.** *Let M be a sbset of an inner product space X.*

1. *If M is total in X then $x \perp M$ implies $x = 0$.*

2. *If $X = H$ is a Hilbert space, then $x_{\perp} M$ implies $x = 0$ if and only of M is total in X.*

Definition 4.19. A normed space X is ***separable*** if there exists a countable set of vectors $C \subset X$ that is dense in X, i.e such that $\overline{C} = X$.

Theorem 4.20. *In every Hilbert space H there exists a total orthonormal set. If H is separable and $C \subset H$ is a countably dense subset of H, then by the Gram-Schmidt process we can make C into an orthonormal set.*

Theorem 4.21 (Totality II)**.** *An orthonormal set M in a Hilbert space H is total if and only if for all $x \in H$ we have*

$$\|x\|^2 = \sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2$$

where $\{e_k\} = \{e_M \mid \langle x, e \rangle \neq 0\}$ is a countable set of vectors.

Proposition 4.22. *Let H be a Hilbert space. If H is separable then every orthonormal set in H is countable. If H contains an orthonormal sequence which is total in H, then H is separable.*

Theorem 4.23 (Hilbert dimension)**.** *All total orthonormal sets in a given Hilbert space $H \neq \{0\}$ have the same cardinality, called the* **Hilbert dimension**.

Definition 4.24. Two Hilbert spaces H_1 and H_2 are said to be ***isomorphic*** if there exists an inner product preserving isomorphism $T : H_1 \longrightarrow H_2$. That is

$$\langle Tx, Ty \rangle_2 = \langle x, y \rangle_1$$

for all $x, y \in H$.

Theorem 4.25. *Two Hilbert spaces H_1 and H_2 are isomorphic if and only if they have the same Hilbert dimension.*

4.1 Functionals on Hilbert spaces

Theorem 4.26 (Riesz's representation theorem)**.** *For any functional $f \in H'$, there exists a unique element $z \in H$ such that $f(x) = \langle x, z \rangle$ for all $x \in H$ and $\|f\| = \|z\|$.*

4.2 Hilbert-adjoint operator

Definition 4.27. Let $T : H_1 \longrightarrow H_2$ be a bounded linear operator on two Hilbert spaces. The ***Hilbert-adjoint operator*** is an operator $T^* : H_1 \longrightarrow H_2$ defined by the relation

$$\langle Tx, y \rangle_2 = \langle x, T^* y \rangle_1$$

for all $x \in H_1$ and $y \in H_2$. Furthermore, for an operator $T : H_2 \longrightarrow H_2$ there exists a *unique* operator T^* that satisfies the above relation.

Theorem 4.28. *Let T be as defined in the definition above. Then*

1. *T^* exists,*
2. *is unique,*
3. *and is a bounded linear operator with $\|T^*\| = \|T\|$.*

Proposition 4.29 (Properties of adjoint operators)**.** *Let $T, S : H_1 \longrightarrow H_2$ be a bounded linear operators on Hilbert spaces. Then for all $x, y \in H$ and scalar α the following hold.*

1. $\langle T^* y, x \rangle_1 = \langle y, Tx \rangle_2$

2. $(S+T)^* = S^* + T^*$
3. $(\alpha T)^* = \overline{\alpha} T^*$
4. $(T^*)^* = T$
5. $\|T^*T\| = \|TT^*\| = \|T\|^2$
6. $T^*T = 0$ *if and only if* $T = 0$.
7. $(ST)^* = T^*S^*$ *(for* $H_1 = H_2$*).*

Definition 4.30. Let $T\colon H \longrightarrow H$ be a bounded linear operator. Then T is

(i) ***self-adjoint*** (or ***hermitian***) if $T^* = T$;

(ii) ***unitary*** if $T^{-1} = T^*$;

(iii) ***normal*** of $TT^* = T^*T$.

Theorem 4.31. *Let H be a Hilbert space and (T_n) be a sequence of bounded self-adjoint linear operators with $T_n\colon H \longrightarrow H$. If $T_n \longrightarrow T$ (i.e. $\|T_n - T\| \longrightarrow 0$) then T is bounded and $T^* = T$.*

5 Hahn-Banach theorem

Definition 5.1. A ***sublinear functional*** on a vector space X is a real valued function $p\colon X \longrightarrow \mathbb{R}$ that satisfies

(i) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$

(ii) $p(\alpha x) = \alpha p(x)$ for all $\alpha \geq 0$ real and $x \in X$.

Theorem 5.2 (Hahn-Banach version I - real vector spaces)**.** *Let X be a real vector space and p a sublinear functional. Let f be a linear funcitonal which is defined on a subspace Y of X that satisfies $f(x) \leq p(x)$ for all $x \in Y$. Then there exists a linear extension $\tilde{f}$ from Y to X of f such that*

1. $\tilde{f}(x) \leq p(x)$ *for all* $x \in X$
2. $\tilde{f}(x) = f(x)$ *for all* $x \in Y$.

Lemma. *Let g be a linear functional on a strict subspace $Y \subsetneq X$ of a real vector space X and p a sublinear functional such that $g(x) \leq p(x)$ for all $x \in Y$. Then there exists a linear extension h of g such that h is defined on a subspace $Z \subset X$ with $Y \subsetneq Z$ such that $h|_Y = g$ and $h(x) \leq p(x)$ for all $x \in Z$.*

Proof. There is an element $z_0 \in X \setminus Y$ and consider the subspace Z of X defined by $Z = \operatorname{span}\{Y, z_0\}$. Note that $z_0 \neq 0$. Thus any $x \in Z$ can be written uniquely as

$$x = y + tz_0 \tag{$*$}$$

for some $y \in Y$ and $t \in \mathbb{R}$. Indeed, if $y + tz_0 = y' + t'z_0$, then $y - y' = (t' - t)z_0$. But $y - y' \in Y$ wheras $z_0 \notin Y$. Hence $y - y' = 0$ and $t = t'$ so the representation in $(*)$ is unique.

Using this unique representation, we can define a linear extension h of g to Z by

$$h(y + tz_0) = g(y) + ct \qquad \text{for all } y \in Y \text{ and } t \in \mathbb{R}$$

for some $c \in \mathbb{R}$. It remains to show that we can pick $c \in \mathbb{R}$ such that $h(x) \leq p(x)$ for all $x \in Z$. Hence we need to show that

$$g(y) + ct \leq p(y + tz_0) \qquad \text{for all } y \in Y \text{ and } t \in \mathbb{R}. \tag{$**$}$$

If $t = 0$, there is nothing to show. For $t > 0$, the requirement in $(**)$ is equivalent to the statement tnat

$$\begin{aligned} c &\le \tfrac{1}{t}p(y + tz_0) - \tfrac{1}{t}g(y) \\ &= p\left(\tfrac{1}{t}y + z_0\right) - g\left(\tfrac{1}{t}y\right), \end{aligned}$$

for all $y \in Y$ and $t > 0$. So we may define $w = \frac{1}{t}y \in Y$ and this is equivalent to the statement that

$$c \le p(w + z_0) + g(w) \text{ for all } w \in Y. \tag{$\dagger$}$$

Similarly, for $t < 0$ we may write $s = -t$ such that $s > 0$ we require that $g(y) - cs \le p(y - sz_0)$ for all $y \in Y$ and $s > 0$. This is equivalent to the requirement that

$$\begin{aligned} c &\ge \tfrac{1}{s}g(y) - \tfrac{1}{s}p(y - sz_0) \\ &= g\left(\tfrac{1}{s}y\right) - p\left(\tfrac{1}{s}y - z_0\right) \end{aligned}$$

for all $s > 0$ and $y \in Y$. Setting $v = \frac{1}{s}y \in Y$ for each y, we see that this is equivalent to the statement that

$$c \ge g(v) - p(v - z_0) \text{ for all } v \in Y. \tag{$\dagger\dagger$}$$

Putting together $(\dagger)$ and $(\dagger\dagger)$, the requirement in $(**)$ is equivalent to the statement that

$$g(v) - p(v - z_0) \le c \le p(w + z_0) + g(w)$$

for all $v, w \in Y$. This is true if there is a c number such that

$$\sup_{v \in Y}\big(g(v) - p(v - z_0)\big) \le c \le \inf_{w \in Y}\big(p(w + z_0) + g(w)\big),$$

but this is equivalent to having

$$g(v) - p(v - z_0) \le p(w + z_0) - g(w)$$

for all $v, w \in Y$. But this is indeed true, since

$$g(v) + g(w) = g(v + w) \le p(v + w) = p((v - z_0) + (w + z_0)) \le p(w + z_0) + p(v - z_0)$$

by the asusmption that g is dominated by p and p is sublinear. □

Proof. (of Hahn-Banach I) Define E as the set of all linear extensions of f that are dominated by p. That is

$$E := \{g \in V' \,|\, W \subset V \subset X \text{ a subspace, } g|_W = f, \text{ and } g(x) \le p(x)\ \forall x \in V\}.$$

Note that E is nonempty since $f \in E$. Define a partial order on E by

$$g \le h \qquad \text{if } h \text{ is an extension of } g.$$

That is, $\mathcal{D}(g) \subset \mathcal{D}(h)$ and $h|_{\mathcal{D}(g)} = g$.

- **Claim**: *There exists a maximal element $\tilde{f}$ of E.*
  Indeed, let $\mathcal{C}$ be a chain of elements in E, and define $g_{\mathcal{C}} \in E$ by

  $$g_{\mathcal{C}}(x) = g(x) \qquad \text{for all } g \in \mathcal{C} \text{ and } x \in \mathcal{D}(g)$$

  with domain given by

  $$\mathcal{D}(g_{\mathcal{C}}) = \bigcup_{g \in \mathcal{C}} \mathcal{D}(g).$$

  Hence $g_{\mathcal{C}} \in E$ is an upper bound of the chain $\mathcal{C}$. By Zorn's lemma, there exists an element $\tilde{f}$ that is maximal in E.

- **Claim**: *The domain $\mathcal{D}(\tilde{f})$ is all of X.*

  Suppose otherwise, then $\mathcal{D}(\tilde{f}) \subsetneq X$. But by the above lemma, we can define a proper linear extension of $\tilde{f}$ that is dominated by p, a contradiction to the maximality of $\tilde{f}$.

□

Definition 5.3. Let X be a vector space. A sublinear functional $p\colon X \longrightarrow \mathbb{R}$ is ***subadditive*** if $p(x+y) \leq p(x) + p(y)$ and $p(\alpha x) = |\alpha|\, p(x)$ for any scalar α.

Theorem 5.4 (Hahn-Banach version II - complex vector spaces)**.** *Let X be a real or complex vector space and p a real-valued sublinear functional on X. Let f be a linear functional which is defined on a subspace W of X that satisfies $|f(x)| \leq p(x)$ for all $x \in W$. Then there exists a linear extension $\tilde{f}$ from W to X of f such that $|\tilde{f}| \leq p(x)$ for all $x \in X$.*

Theorem 5.5 (Hahn-Banach version III - normed spaces)**.** *Let X be a normed space and f a bounded linear functional on a subspace $W \subset X$. Then there exists a bounded linear functional $\tilde{f}$ on X which is an extension of f such that $\|f\|_W = \|\tilde{f}\|_X$.*

Corollary 5.6. *Let X be a normed space and $x_0 \in X$ a nonzero vector. Then there exists a functional $f \in X'$ such that $\|f\| = 1$ and $f(x_0) = \|x_0\|$.*

Corollary 5.7. *Let X be a normed space. For any $x \in X$ we have*

$$\|x\| = \sup_{0 \neq \tilde{f} \in X'} \frac{|\tilde{f}(x)|}{\|\tilde{f}\|}.$$

In particular, if $f(x_0) = 0$ for all $f \in X'$ then $x_0 = 0$.

6 Adjoint operator

Definition 6.1. Let X and Y be normed spaces and $T\colon X \longrightarrow Y$ be a bounded linear operator. Then the ***adjoint operator*** $T^\times : Y' \longrightarrow X'$ is defined by

$$(T^\times g)(x) = g(Tx)$$

for all $g \in Y'$.

Theorem 6.2. *The adjoint operator $T^\times$ is linear and bounded, and $\|T^\times\| = \|T\|$.*

Proposition 6.3. *In finite dimensions, if T is represented by a matrix A, we have that $T^\times$ is represented by the matrix $A^\intercal$ in the basis dual to the basis chosen for A.*

Proposition 6.4. *Let $S, T\colon X \longrightarrow Y$ be bounded linear operators of normed spaces. Then the following hold.*

1. $(S+T)^\times = S^\times + T^\times$
2. $(\alpha T)^\times = \alpha T^\times$
3. $(ST)^\times T^\times S^\times$
4. *if $T \in \mathcal{B}(X,Y)$ and T has an inverse $T^{-1} \in \mathcal{B}(Y,X)$ then $(T^\times)^{-1} = (T^{-1})^\times$.*

7 Uniform boundedness

Definition 7.1. A subset $M \subset X$ of a metric space is called

(i) ***rare*** in X if its closure $\overline{M}$ has no interior points in X;

(ii) ***meager*** in X if if is a countable union of rare sets in X;

(iii) ***non-meager*** if it is not meager.

Theorem 7.2 (Baire's category theorem)**.** *If a metric space $X \neq \emptyset$ is complete, then X is non-meager in itself.*

Proof. Let X be a metric space and suppose that X is meager. Then X may be decomposed as a countable union of rare sets

$$X = \bigcup_{k=1} M_k,$$

where the $\overline{M_k}$ are all rare, i.e. do not contain any open balls. Since $\overline{M_1}$ is closed, there is an element in the complement $x_1 \in \overline{M_k}^c$ and a constant $0 < \varepsilon_1 < \frac{1}{2}$ such that $B(x_1; \varepsilon_1) \subset \overline{M_1}^c$. Since M_2 is rare, we have $B(x_1; \varepsilon_1) \not\subset \overline{M_2}$ and thus there is an element $x_2 \in B(x_1; \varepsilon_1) \cap \overline{M_2}^c$ and a constant $0 < \varepsilon_2 < \frac{1}{4}$ such that $B(x_2; \varepsilon_2) \subset B(x_1; \varepsilon_1) \cap \overline{M_2}^c$. Continuing this process inductively, we contruct sequences (x_n) and (ε_n) such that $0 < \varepsilon_n < \frac{1}{2^n}$ and $x_n \in B(x_{n-1}; \varepsilon_{n-1}) \cap \overline{M_n}^c$. For all $n \in \mathbb{N}$, note that $x_m \in B(x_n; \varepsilon)$ and thus $\|x_n - x_m\| < \epsilon_n \leq \frac{1}{2^n}$ for all $m \geq n$. Hence, the sequence (x_n) is Cauchy. Suppose that this sequences converges to some $x \in X$. But this implies that $x \in B(x_n; \varepsilon_n)$ for all $n \in N$. But $B(x_n; \varepsilon_n) \subset \overline{M_n}^c$ and thus $x \in \overline{M_n}^c$ for all n, which would mean that $x \notin X$. So X would not be complete. □

Theorem 7.3 (Uniform boundedness theorem)**.** *Let X be a Banach space and Y a normed space and (T_n) be a sequence in $\mathcal{B}(X, Y)$. If the sequence $(\|T_n x\|)$ is bounded for all $x \in X$, then $(\|T_n\|)$ is bounded.*

Proof. The proof of the uniform boundedness theorem follows from Baire's category theorem. Define $A_k = \{x \in X \mid \|T_n x\| \leq k \text{ for all } n\}$ such that $X = \cup_{k=1}^{\infty} A_k$. Since X is Banach, it is non-meager in itself, so at least one A_{k_0} is not rare and thus contains an open ball $B(x_0; \varepsilon) \subset A_{k_0}$ for some $x_0 \in A_{k_0}$ and $\varepsilon > 0$. Then for all $x \in X$ with $\|x\| = 1$ we have

$$\begin{aligned}
\|T_n x\| &= \frac{2}{\varepsilon} \left\|T_n(\tfrac{\varepsilon}{2}x - x_0 + x_0)\right\| \\
&\leq \frac{2}{\varepsilon} \left(\left\|T_n(\tfrac{\varepsilon}{2}x - x_0)\right\| + \|T_n x_0\|\right) \\
&= \frac{2}{\varepsilon}(k_0 + k_0) \\
&= \frac{4k_0}{\varepsilon}
\end{aligned}$$

where we note that $\left\|x_0 - (\frac{\varepsilon}{2}x - x_0)\right\| \leq \frac{\varepsilon}{2}$ and thus $\frac{\varepsilon}{2}x - x_0 \in B(x_0; \varepsilon) \subset A_{k_0}$. Hence $\|T_n\| \leq \frac{4k_0}{\varepsilon}$ for all n and thus $(\|T_n\|)$ is bounded. □

8 Weak and strong convergence

Definition 8.1. Let X be a normed space and let (x_n) be a sequence in X. We say that

(i) converges ***strongly*** to x (denoted $x_n \longrightarrow x$) if and only if $\|x_n + x\| \longrightarrow 0$;

(ii) converges ***weakly*** to x (denoted $x_n \xrightarrow{w} x$) if and only if $f(x_n) \longrightarrow f(x)$ for all functionals $f \in X'$.

Proposition 8.2. *Let X be a normed space. If a sequence (x_n) is weakly convergent, then it converges to a unique element in X.*

Theorem 8.3. *Let X be a normed space. If (x_n) is a sequence in X such that $x_n \xrightarrow{w} x$, then $\|x_n\|$ is bounded.*

Theorem 8.4. *Let X be a normed space. Then strong convergence inplies weak convergence. Furthermore, if X is finite dimensional, then weak convergence and strong convergence are equivalent.*

Theorem 8.5. *Let X be a normed space and (x_n) be a sequence in X. Then $x_n \xrightarrow{w} x$ if and only if the sequence $(\|x_n\|)$ is bounded and there exists an $M \subset X'$ such that M is total in X' and $f(x_n) \longrightarrow f(x)$ for all $f \in M$.*

(For weak convergence, we only need to check functionals $f \in M$ in some total subset $M \subset X'$, not all $f \in X'$.)

9 Sequences of operators

Definition 9.1. Let X and Y be normed spaces and (T_n) be a sequence of operators in $\mathcal{B}(X, Y)$.

(i) The sequence (T_n) converges ***uniformly*** to an operator T if $\|T_n - T\| \longrightarrow 0$, and this is denoted $T_n \xrightarrow{u} T$.

(ii) The sequence (T_n) converges ***strongly*** to an operator T if $T_n x \longrightarrow Tx$ for all $x \in X$, and this is denoted $T_n \xrightarrow{s} T$.

(iii) The sequence (T_n) converges ***weakly*** to an operator T if $T_n x \xrightarrow{w} Tx$ for all $x \in X$, and this is denoted $T_n \xrightarrow{w} T$. (That is, $f(T_n x) \longrightarrow f(Tx)$ for all $x \in X$ and $f \in Y'$).

Theorem 9.2. *Let X be a Banach space, Y a normed space and (T_n) a sequence in $\mathcal{B}(X, Y)$. If (T_n) is strongly operator convergent with $T_n \xrightarrow{s} T$, then T is bounded.*

Proof. By the uniform boundedness principle, $(\|T_n\|)$ is bounded. Since (T_n) is strongly convergent, $\|Tx\| - \|T_n x\| \leq \|(T - T_n)x\| \longrightarrow 0$ for all x, so T is bounded. □

Theorem 9.3. *Let X, Y be Banach spaces. A sequence (T_n) of operators in $\mathcal{B}(X, Y)$ is strongly operator convergent if and only if the following hold:*

1. *the sequence $(\|T_n\|)$ is bounded,*
2. *and the sequence $(T_n x)$ is Cauchy in Y for all $x \in M$ where $M \subset X$ is total.*

Proof. One direction is trivial, so we may assume that (T_n) is strongly operator convergent. Since M is dense in X, for each $x \in X$ we may choose $y \in M$ that is arbitrarily close to x. Then $\|T_n x - T_m x\| \leq \|T_n\| \|x - y\| + \|T_n - T_m\| \|y\| + \|T_m\| \|x - y\| \longrightarrow 0$. □

Definition 9.4. Let X be a normed space and (f_n) a sequence of functionals in X'.

(i) The sequence (f_n) converges ***strongly*** to $f \in X'$ if $\|f_n - f\| \longrightarrow 0$, and this is denoted $f_n \longrightarrow f$.

(ii) The sequence (f_n) is ***weak****** convergent to $f \in X'$ if $f_n(x) \longrightarrow f(x)$ for all $x \in X$, and this is denoted $f_n \xrightarrow{w^*} f$.

Theorem 9.5. *Let X be a separable normed space. Every bounded sequence of functionals in X' has a subsequence that is weak* convergent to some element of X'.*

Proof. Let (f_n) be a bounded sequence of functionals and (x_n) be a sequence that is dense in X. Since (f_n) is bounded, there is a constant $c > 0$ such that $\|f_n\| < c$ for all n. Noting that $|f_n(x_1)| \leq \|f_n\| \|x_1\| < c\|x_1\|$, we have that the sequence $\big(f_n(x_1)\big)$ is bounded. So there is a subsequence $(f_n^{(1)})$ of (f_n) such that $\big(f_n^{(1)}(x_1)\big)$ is Cauchy. Similarly, the sequence $\big(f_n^{(1)}(x_2)\big)$ is bounded, so there is a subsequence $(f_n^{(2)})$ of $(f_n^{(1)})$ such that $\big(f_n^{(2)}(x_2)\big)$ is Cauchy. Continuing this process inductively, we can construct a series of subsequences

$$\cdots \subseteq (f_n^{(3)}) \subseteq (f_n^{(2)}) \subseteq (f_n^{(1)}) \subseteq (f_n)$$

such that $\big(f_n^{(k)}(x_k)\big)$ is Cauchy for all k. We may construct another subsequence (g_n) of (f_n) by Cantor diagonalization where we take $g_n = f_n^{(n)}$ for all n. Clearly, the sequence $(g_n(x_k))$ is Cauchy for all k. Note also that the sequence $\big(g_n(x)\big)$ is Cauchy for all $x \in X$. Indeed, for all $\varepsilon > 0$ there is an element $x_k \in (x_n)$ such that $\|x - x_k\| < \frac{1}{3c}\varepsilon$. Furthermore, there is an $N \in \mathbb{N}$ large enough such that $\|g_n(x_k) - g_m(x_k)\| < \frac{1}{3}\varepsilon$ for all $n, m > N$. Then, for all $n, m > N$ we have

$$\begin{aligned}
\|g_n x - g_m x\| &= \|g_n(x) - g_n(x_k) + g_n(x_k) - g_m(x_k) + g_m(x_k) - g_m(x)\| \\
&\leq \|g_n(x) - g_n(x_k)\| + \|g_n(x_k) - g_m(x_k)\| + \|g_m(x_k) - g_m(x)\| \\
&\leq \underbrace{\|g_n\|}_{<c} \underbrace{\|x - x_k\|}_{<\frac{1}{3c}\varepsilon} + \underbrace{\|g_n(x_k) - g_m(x_k)\|}_{<\frac{1}{3}\varepsilon} + \underbrace{\|g_m\|}_{<c} \underbrace{\|x_k - x\|}_{<\frac{1}{3c}\varepsilon} \\
&< c\frac{1}{3c}\varepsilon + \frac{1}{3}\varepsilon + c\frac{1}{3c}\varepsilon \\
&= \varepsilon,
\end{aligned}$$

thus $\big(g_n(x)\big)$ is Cauchy for all $x \in X$. Hence, we may define another functional g on X by

$$g(x) := \lim_{n\to\infty} g_n(x).$$

This is clearly linear, since $g(\alpha x + \beta y) = \lim_{n\to\infty}[\alpha g_n(x) + \beta g_n(y)] = \alpha g(x) + \beta g(y)$. It is also bounded, since $|g(x)| = \left|\lim\limits_{n\to\infty} g_n(x)\right| \leq \limsup\limits_{n\in\mathbb{N}} |g_n(x)| \leq c\|x\|$. So we have that $g_n(x) \longrightarrow g(x)$ for all $x \in X$, where (g_n) is a subsequence of (f_n) and g is a bounded linear functional, and $g_n \xrightarrow{w^*} g$ as desired. □

10 Open mapping theorem

Recall: If X and Y are metric spaces, a mapping $T \colon X \longrightarrow Y$ is continuous if and only if the pre-images $T^{-1}(U)$ are open in X for all open sets $U \subset Y$.

Definition 10.1. Let X and Y be metric spaces. A mapping $T \colon \mathcal{D}(T) \longrightarrow Y$ with $\mathcal{D}(T) \subset X$ is said to be ***open*** if the image $T(U)$ is open in Y for every open set $U \subset \mathcal{D}(T)$.

Theorem 10.2 (Open mapping theorem)**.** *Let X and Y be Banach spaces. Every surjective bounded linear operator from X onto Y is an open map.*

Claim 1. *Let $T \colon X \longrightarrow Y$ be a linear map of Banach spaces. If there exists an $r > 0$ such that*

$$B_Y(0; r) \subset T\big(B_X(0; 1)\big),$$

then T is an open mapping.

Proof. Let $A \subset X$ open and $y = Tx \in T\big(B_X(0;1)\big)$. Then there is an $\varepsilon > 0$ such that $B_X(0;\varepsilon) \subset A$. By linearity of T, we have

$$B_Y(0;r) \subset T\big(B_X(0;1)\big) \iff \underbrace{B_Y(0;r) + \underbrace{Tx}_{y}}_{=B_Y(y,r)} \subset T\big(\underbrace{B_X(0;1) + x}_{B_X(x;1)}\big) \iff B_Y(y;\varepsilon r) \subset T\big(B_X(x;\varepsilon)\big).$$

So there is an open ball of radius εr contained in $T(A)$, since $T\big(B_X(x;\varepsilon)\big) \subset T(A)$. ☐

Claim 2. *If the interior of $\overline{T\big(B_X(0;1)\big)}$ is nonempty, then it contains a ball around the origin of Y. That is, there exists an $r > 0$ such that $B_Y(0;r) \subset \overline{T\big(B_X(0;1)\big)}$.*

Proof. By assumption, there exists a $y \in Y$ and $\varepsilon > 0$ such that $B_Y(y;\varepsilon) \subset \overline{T\big(B_X(0;1)\big)}$. Let $z \in Y$ with $\|z\| < 1$ such that both y and $y + \varepsilon z$ are in $B_Y(y;\varepsilon)$. Since they are both in the closure of $T\big(B_X(0;1)\big)$, there exists sequence (x_n) and (x'_n) in $B_X(0;1)$ such that

$$Tx_n \longrightarrow y \quad \text{and} \quad Tx'_n \longrightarrow y + \varepsilon z.$$

Define the sequence (x''_n) as $x''_n = \frac{1}{\varepsilon}(x_n - x'_n)$ and note that $x''_n \in B_X(0;\frac{2}{\varepsilon})$. Then $Tx''_n \longrightarrow z$, since

$$\begin{aligned}\|Tx''_n - z\| &= \tfrac{1}{\varepsilon}\,\|Tx_n - Tx'_n - \varepsilon z - y + y\| \\ &\le \tfrac{1}{\varepsilon}\big(\underbrace{\|Tx_n - y\|}_{\to 0} + \underbrace{\|Tx'_n - (y + \varepsilon z)\|}_{\to 0}\big) \longrightarrow 0.\end{aligned}$$

Hence $z \in \overline{T\big(B_X(0;\frac{2}{\varepsilon})\big)}$, but $z \in B_Y(0;1)$ was arbitrary. So $B_Y(0;1) \subset \overline{T\big(B_X(0;\frac{2}{\varepsilon})\big)}$. By linearity of T, we see that $B_Y(0;r) \subset \overline{T\big(B_X(0;1)\big)}$, where $r = \frac{\varepsilon}{2}$. ☐

Proof. (of open mapping theorem) Since T is onto, we have

$$Y = T(X) = \bigcup_{k=1}^{\infty}\big(B_X(0;k)\big).$$

But Y is complete, so by Baire's category theorem we know that there is at least one $k_0 \in \mathbb{N}$ such that $\overline{T\big(B_X(0;k_0)\big)}$ has nonempty interior. By linearity of T we see that $k_0\overline{T\big(B_X(0;1)\big)}$ and thus $\overline{T\big(B_X(0;1)\big)}$ have nonempty interior. By Claim 2, there is an $\varepsilon > 0$ such that $B_Y(0;\varepsilon) \subset \overline{T\big(B_X(0;1)\big)}$. We now show that $\overline{T\big(B_X(0;1)\big)} \subset T\big(B_X(0;2)\big)$.

Let $y \in \overline{T\big(B_X(0;1)\big)}$, then there exists an element $x_1 \in B_X(0;1)$ such that

$$\|y - Tx_1\| < \frac{1}{2}\varepsilon.$$

Hence $y - Tx_1 \in B_Y(0;\frac{1}{2}\varepsilon)$. By linearity, $B_Y(0;\frac{1}{2}\varepsilon) \subset \overline{T\big(B_X(0;\frac{1}{2})\big)}$, so there exists an element $x_2 \in B_X(0;\frac{1}{2})$ such that

$$\|(y - Tx_1) - Tx_2\| < \frac{1}{4}\varepsilon.$$

Continuing this process inductively, we find a sequence (x_n) in X such that $x_n \in B_X(0;\frac{1}{2^{n-1}})$ and

$$\left\| y - T\sum_{k=1}^{n} x_n \right\| < \frac{1}{2^n}\varepsilon.$$

The sequence of partial sums $s_n := \sum_{k=1}^{\infty} x_k$ is Cauchy since for $m < n$ we have

$$\|s_n - s_m\| \le \sum_{k=n+1}^{m} \|x_k\| < \sum_{k=n+1}^{m} \frac{1}{2^{k-1}} \longrightarrow 0.$$

So $s_n \longrightarrow x$ for some $x \in X$ since X is Banach, and

$$\|x\| = \left\| \sum_{k=1}^{\infty} x_k \right\| \leq \sum_{k=1}^{\infty} \|x_k\| < \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = 2,$$

and thus $x \in B_X(0;2)$. Since T is bounded, we have $Tx_n \longrightarrow y$ and $x_n \longrightarrow x$ implies $y = Tx$. Hence $y \in T\big(B_X(0;2)\big)$.

So we have the inclusions $B_Y(0;\varepsilon) \subset \overline{T\big(B_X(0;1)\big)} \subset T\big(B_X(0;2)\big)$. By linearity of T we have $B_Y(0;\frac{\varepsilon}{2}) \subset T\big(B_X(0;1)\big)$, and by Claim 1 we have that T is an open mapping. □

Corollary 10.3 (Bounded inverse theorem)**.** *Let X and Y be Banach space. Every bijective bounded linear map $T\colon X \longrightarrow Y$ has a bounded linear inverse.*

Proof. Since T is bijective, its inverse T^{-1} exists. From the open mapping theorem, T is open. But the preimage of every open set in X under T^{-1} is open in Y, since $(T^{-1})^{-1}(U) = T(U)$, so T^{-1} is continuous and thus bounded. □

10.1 Closed graph theorem

Definition 10.4. Let X and Y be vector spaces and $T\colon \mathcal{D}(T) \longrightarrow Y$ a linear operator with $\mathcal{D}(T) \subset X$. The ***graph*** of T is the set

$$\mathcal{G}(T) = \{(x, Tx) \mid x \in \mathcal{D}(T)\}$$

as a subset of $X \times Y$.

Definition 10.5. Let X and Y be normed spaces and $T\colon \mathcal{D}(T) \longrightarrow Y$ a linear operator with $\mathcal{D}(T) \subset X$. Then T is said to be ***closed*** if its graph $\mathcal{G}(T)$ is closed in $X \times Y$.

Proposition 10.6. *Let X and Y be Banach spaces. Then $X \times Y$ is Banach.*

Lemma 10.7. *Let X and Y be normed spaces and $T\colon \mathcal{D}(T) \longrightarrow Y$ be a linear operator with $\mathcal{D}(T) \subset X$. Then T is a closed linear operator if and only if the following holds:*

For all sequences (x_n) in $\mathcal{D}(T)$ such that

$x_n \longrightarrow x$ and $Tx_n \longrightarrow y$ for some $x \in X$ and $y \in Y$,
we have $x \in \mathcal{D}(T)$ and $y = Tx$.

Lemma 10.8. *Let X and Y be normed spaces and $T\colon \mathcal{D}(T) \longrightarrow Y$ be a bounded linear operator with $\mathcal{D}(T) \subset X$.*

1. *If $\mathcal{D}(T)$ is closed in X then T is closed.*
2. *If T is closed and Y is Banach, then $\mathcal{D}(T)$ is closed.*

Theorem 10.9 (Closed graph theorem)**.** *Let X and Y be Banach spaces and let $T\colon \mathcal{D}(T) \longrightarrow Y$ be a closed linear operator with $\mathcal{D}(T) \subset X$. If $\mathcal{D}(T)$ is closed then T is bounded.*

Proof. Note that $\mathcal{G}(T)$ is itself a Banach space, since it is a closed vector space in the Banach space $X \times Y$. Similarly $\mathcal{D}(T)$ is Banach since it is closed in the Banach space X. Define the mapping $P\colon \mathcal{G}(T) \longrightarrow \mathcal{D}(T)$

$$P\colon (x, Tx) \longmapsto x.$$

This is linear, so we need to show that it is bounded. Indeed,

$$\|P(x, Tx)\| = \|x\| \leq \|x\| + \|Tx\| = \|(x, Tx)\|$$

and thus $\|P\| \leq 1$. Note that P is bijective. It is clearly surjective, but it is also injective since $P(x, Tx) = 0$ if and only if $x = 0$. So by the bounded inverse theorem P^{-1} exists and is bounded. Hence

$$\|Tx\| \leq \|x\| + \|Tx\| = \|(x, Tx)\| = \left\|P^{-1}(x)\right\| \leq \left\|P^{-1}\right\| \|x\|.$$

□

11 Spectral theory in normed spaces

Definition 11.1. Let X be a normed space and $T\colon \mathcal{D}(T) \longrightarrow X$ be a linear operator with $\mathcal{D}(T) \subset X$. For $\lambda \in \mathbb{C}$, the ***resolvent*** is the linear operator

$$R_\lambda = T_\lambda^{-1} = (T - \lambda I)^{-1}$$

if it exists.

(i) The ***point spectrum*** of T is the set

$$\sigma_p(T) := \{\lambda \in \mathbb{C} \mid R_\lambda \text{ does not exist}\}$$

of eigenvalues.

(ii) The ***continuous spectrum*** of T is the set

$$\sigma_c(T) := \left\{\lambda \in \mathbb{C} \,\middle|\, R_\lambda \text{ exists but is unbounded, and } \overline{\mathcal{D}(R_\lambda)} = X\right\}.$$

(iii) The ***residual spectrum*** of T is the set

$$\sigma_r(T) := \left\{\lambda \in \mathbb{C} \,\middle|\, R_\lambda \text{ exists but } \overline{\mathcal{D}(R_\lambda)} \neq X\right\}.$$

(iv) The ***spectrum*** is the set

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T).$$

(v) The ***resolvent set*** of T is $\rho(T) = \mathbb{C} \setminus \sigma(T)$, and $\lambda \in \rho(T)$ is called a ***regular value***.

Proposition 11.2. *Let $T\colon X \longrightarrow X$ be a linear operator on a Banach space X.*

1. *If T is bounded and $R_\lambda(T)$ exists for some $\lambda \in \mathbb{C}$ such that $\mathcal{D}(R_\lambda(T)) = X$, then $R_\lambda(T)$ is bounded.*
2. *If $\lambda \in \rho(T)$ and T is either closed or bounded, then $\mathcal{D}(R_\lambda) = X$.*
3. $R_\mu - R_\nu = (\mu - \nu)R_\mu R_\nu$
4. *If $[S, T] = 0$ then $[S, R_\mu(T)] = 0$ for all $\mu \in \mathbb{C}$ such that R_μ exists.*
5. *$[R_\mu, R_\nu] = 0$ for all $\mu, \nu \in \mathbb{C}$.*

11.1 Spectral properties of bounded linear operators

Lemma 11.3. *Let X be a Banach space and $T \in \mathcal{B}(X, X)$. If $\|T\| < 1$ then $(I - T)^{-1}$ exists and*

$$(I - T)^{-1} = I + T + T^2 + \cdots = \sum_{k=0}^{\infty} T^k.$$

Theorem 11.4 (Closed spectrum theorem)**.** *Let X be a Banach space and $T\colon X \longrightarrow X$ be a bounded linear operator. Then the resolvent set $\rho(T)$ is open and the spectrum $\sigma(T)$ is closed in $\mathbb{C}$.*

Theorem 11.5. *Let X be a Banach space and $T \in \mathcal{B}(X, X)$. THen $\sigma(T)$ is compact and $|\lambda| \leq \|T\|$ for all $\lambda \in \sigma(T)$.*

Definition 11.6. Let X be a Banach space and T a bounded linear operator on X. The ***spectral radius*** of T is defined as

$$r_\sigma(T) = \sup_{\lambda \in \sigma(T)} |\lambda|.$$

Theorem 11.7. *Let X be a Banach space and $T \in \mathcal{B}(X,T)$. Then $r_\sigma(T) = \lim_{n\to\infty} \sqrt[n]{\|T^n\|} \leq \|T\|$.*

Theorem 11.8 (Spectral mapping theorem for polynomials)**.** *Let X be a Banach space, $T \in \mathcal{B}(X,X)$ and $p(t) = \alpha_0 + \alpha_1 t + \cdots + \alpha_n t^n$ be a polynomial of degree n. Then $\sigma(p(T)) = p(\sigma(T))$.*

Proof. I should probably know how to prove.... □

11.2 Banach algebras

Definition 11.9. An ***algebra*** $\mathcal{A}$ is a vector space with an associative binary operation $x \cdot y \in \mathcal{A}$ for all $x, y \in \mathcal{A}$. That is $x \cdot (y+z) = x \cdot y + x \cdot z$ for all $x, y, z \in \mathcal{A}$. The algebra has an ***identity*** if there is an element $e \in \mathcal{A}$ such that $e \cdot x = x \cdot e = x$ for all $x \in \mathcal{A}$.

A ***normed*** algebra is an algebra $\mathcal{A}$ that is normed and a vector space and satisfies

$$\|x \cdot y\| \leq \|x\| \, \|y\|$$

for all $x, y \in \mathcal{A}$. A ***Banach*** algebra is a normed algebra whose underlying normed space is Banach.

Definition 11.10. Let $\mathcal{A}$ be a complex Banach algebra with identity and let $x \in \mathcal{A}$. The ***resolvent set*** of x is the set $\rho(x)$ of all $\lambda \in \mathbb{C}$ such that $x - \lambda e$ is invertible. The ***spectrum*** of x is the set $\sigma(x) = \mathbb{C} \setminus \rho(x)$.

Theorem 11.11. *Let X be a Banach space and consider the Banach algebra $\mathcal{A} = \mathcal{B}(X,X)$. Then the notions of resolvent set and spectrum coincide.*

Theorem 11.12. *Let $\mathcal{S}$ be a complex Banach algebra $\mathcal{A}$ with identity and $x \in \mathcal{A}$. If $\|x\| < 1$ then $e - x$ is invertible with*

$$(e-x)^{-1} = e + \sum_{k=1}^{\infty} x^k.$$

Theorem 11.13. *Let $\mathcal{A}$ be a complex Banach algebra with identity. The group $G \subset \mathcal{A}$ of all invertible elements is open in $\mathcal{A}$.*

Definition 11.14. Let $\mathcal{A}$ be a Banach algebra and $x \in \mathcal{A}$. Then the ***spectral radius*** of x is

$$r_\sigma(x) := \sup_{\lambda \in \sigma(x)} |\lambda| \, .$$

Theorem 11.15 (Spectral radius in a Banach algebra)**.** *Let $\mathcal{A}$ be a complex Banach algebra with identity. Then $r_\sigma(x) \leq \|x\|$ and the spectrum $\sigma(x)$ is compact.*

Theorem 11.16 (Nonempty spectrum)**.** *The spectrum of every element of a complex Banach algebra is nonempty.*

11.3 Spectral theory of Self-Adjoint operators

Theorem 11.17. *Let H be a Hilbert space and $T \colon H \longrightarrow H$ a self-adjoint linear operator. Then T is bounded.*

Proposition 11.18. *All eigenvalues of self-adjoint operators are real and eigenvectors of self-adjoint linear operators corresponding to different eigenvalues are orthogonal.*

Theorem 11.19. *Let H be a complex Hilbert space and $T \colon H \longrightarrow H$ be a self-adjoint operator. Then $\sigma(T)$ is real.*

Lemma 11.20. *Let H be a complex Hilbert space and $T \colon H \longrightarrow H$ a self-adjoint linear operator. Then $\lambda \in \rho(T)$ if and only if there is a constant $c > 0$ such that $\|T_\lambda x\| > c \|x\|$ for all $x \in H$.*

Theorem 11.21 (Properties of the sectrum of self-adjoint linear operators)**.** *Let H be a Hilbert space and $T\colon H \longrightarrow H$ a bounded self-adjoint linear operator, and define*

$$m := \inf_{\substack{x\in H\\ \|x\|=1}} \langle Tx, x\rangle \qquad \textit{and} \qquad M := \sup_{\substack{x\in H\\ \|x\|=1}} \langle Tx, x\rangle .$$

1. *$\sigma(T) \subset [m, M]$.*
2. *$\|T\| = \max\{|m|, |M|\}$.*
3. *$m, M \in \sigma(T)$.*
4. *$\sigma_r(T) = \emptyset$.*

Definition 11.22. Let H be a Hilbert space and consider the set of self-adjoint linear operators in H. Then we can defing a partial order in the following manner:

$$T_1 \leq T_a \quad \text{if and only if} \quad |T_1x, x| \leq \langle T_2x, x\rangle \quad \text{for all } x \in H.$$

A self-adjoint operator T is ***positive*** if $0 \leq T$.

Theorem 11.23. *Let H be a Hilbert space and $T_1, T_2 \geq 0$ two positive operators on H such that $[T_1, T_2] = 0$. Then $T_1T_2 \geq 0$.*

Theorem 11.24. *Let H be a complex Hilbert space and $T \geq 0$ a positive operator in H. Then there exists a unique operator $A \geq 0$ such that $A^2 = T$.*

Definition 11.25. Let H be a Hilbert space and $Y \subset H$ a closed subspace. Then $H = Y \oplus Y^\perp$ and any element $x \in H$ can be uniqely represented as $x = y + y'$ where $y \in Y$ and $y' \in Y^\perp$, and the ***projection*** operator into Y is the operator

$$\begin{aligned} P\colon H &\longrightarrow Y \\ y + y' &\longmapsto y. \end{aligned}$$

Theorem 11.26. *Let H be a Hilbert space and $P\colon H \longrightarrow H$ a bounded linear operator. Then P is a projection if and only if it is self-adjoint and idempotent.*

Proposition 11.27 (Properties of projections)**.** *Let H be a Hilbert space and P_1, P_2 and P projections on H. Then the followng hold.*

1. *$\langle Px, x\rangle = \|Px\|^2$ for all $x \in H$, and thus $P \geq 0$*
2. *$\|P\| = 1$ if $P(H) \neq \{0\}$*
3. *$Q = P_1P_2$ is a projection if and only if $[P_1, P_2] = 0$, and $Q(H) = P_1(H) \cap P_2(H)$.*
4. *$Q = P_1 + P_2$ is a projection if and only if $P_1(H) \perp P_2(H)$, and $Q(H) = P_1(H) \oplus P_2(H)$.*
5. *Given vectors $v, w \in H$, we have $v \perp w$ if and only if $P_vP_w = 0$.*

Theorem 11.28 (Partial order theorem for projections)**.** *Let H be a Hilbert space with P_1 and P_2 projections. Then the following are equivalent:*

1. *$P_1P_2 = P_2P_1 = P_1$,*
2. *$\|P_1x\| \leq \|P_2x\|$ for all $x \in H$,*
3. *$P_1 \leq P_2$,*
4. *$\mathcal{N}(P_2) \subset \mathcal{N}(P_1)$,*
5. *$P_1(H) \subset P_2(H)$.*

Theorem 11.29 (Difference of projections)**.** *Let H be a Hilbert space with P_1 and P_2 projections. Then $P = P_2 - P_1$ is a projection if and only if $P_1(H) \subset P_2(H)$. Furthermore, if P is a projection then $P(H) = P_2(H) \cap \big(P_1(H)\big)^\perp$.*

11.4 The spectral family

Definition 11.30. Let H be a Hilbert space. A ***real spectral family*** is a one-parameter family $\mathcal{F} = (E_\lambda)_{\lambda\in\mathbb{R}}$ of projections E_λ on H which satisfies the following properties.

(i) $E_\lambda \leq E_\mu$ for all $\lambda \leq \mu$,

(ii) $\lim\limits_{\lambda\to-\infty} E_\lambda x = 0$ and $\lim\limits_{\lambda\to+\infty} E_\lambda x = x$ for all $x \in H$,

(iii) Continuity from the right. That is

$$E_{\lambda+} := \lim_{\mu\to\lambda+} E_\mu x = E_\lambda x$$

for all $x \in H$.

A ***spectral family on an interval*** $[a, b] \subset \mathbb{R}$ is a real spectral family that satisfies properties (i) and (iii) above and the modified property

(ii*) $E_\lambda = 0$ for all $\lambda < a$ and $E_\lambda = I$ for all $\lambda \geq b$.

Definition 11.31 (positive and negative components, absolute value)**.** Let H be a Hilbert space and $T\colon H \longrightarrow H$ a self-adjoint linear operator. The ***absolute value*** of T is the operator

$$|T| = \sqrt{TT^*}.$$

The positive and negative components of T are the operators

$$T^+ := \frac{1}{2}\left(|T| + T\right) \qquad \text{and} \qquad T^- := \frac{1}{2}\left(|T| - T\right).$$

Proposition 11.32. *Let H be a Hilbert space and $T\colon H \longrightarrow H$ a self-adjoint linear operator. For each $\lambda \in \mathbb{R}$, define the projection*

$$E_\lambda\colon H \overset{onto}{\longrightarrow} \mathcal{N}(T_\lambda^+).$$

Then the family $(E_\lambda)_{\lambda\in\mathbb{R}}$ is a real spectral family.

Definition 11.33. The spectral family defined in the above proposition is the ***real spectral family associated with*** T.

Proposition 11.34 (Properties of self-adjoint operators)**.** *Let H be a Hilbert space and $T\colon H \longrightarrow H$ a self-adjoint operator. Denote $E\colon H \overset{onto}{\longrightarrow} \mathcal{N}(T^+)$ the projection onto the nullspace of T^+. Then the following hold.*

1. $[T, |T|] = [T, T^\pm] = 0$,
2. $T^+T^- = 0$,
3. $[E, T] = [E, |T|] = 0$,
4. $T^+E = ET^+ = 0$ *and* $T^-E = ET^- = T^-$,
5. $TE = -T^-$ *and* $T(I - E) = T^+$,
6. $T^\pm \geq 0$.

Lemma 11.35. *Let H be a Hilbert space and $T\colon H \longrightarrow H$ a self-adjoint operator. For $\mu > \lambda$ and the operators T_λ^+ and T_μ^+, we have $T_\mu^+ T_\lambda^+ = (T_\mu^+)^2$.*

Theorem 11.36. *Let H be a Hilbert space and $T\colon H \longrightarrow H$ a bounded self-adjoint linear operator. Let $m = \inf_{\lambda\in\sigma(T)} \lambda$ and $M = \sup_{\lambda\in\sigma(T)} \lambda$. Then the spectral family associated to T given by (E_λ) is a spectral family on the interval $[m, M]$.*

Lemma 11.37. *Let H be a Hilbert space and (T_n) a sequence of self-adjoint operators K a bounded self-adjoint operator satisfying*

$$T_1 \leq T_2 \leq \cdots \quad \text{and} \quad T_n \leq K \text{ for all } n \in \mathbb{N},$$

with $[T_i, T_j] = 0$ and $[T_i, K] = 0$ for all i. Then (T_n) is strongly convergent to a bounded self-adjoint linear operator T such that $T \leq K$.

...

Theorem 11.38 (Spectral representation)**.** *Let H be a Hilbert space and $T\colon H \longrightarrow H$ a bounded self-adjoint linear operator. Then T has the spectral representation*

$$T = \int_{m-0}^{M} \lambda dE_\lambda$$

where $m \inf_{\lambda\in\sigma(T)} |\lambda|$ and $M \sup_{\lambda\in\sigma(T)} |\lambda|$, Furthermore, for all $x, y \in H$ we have the representation

$$\langle Tx, y\rangle = \int_{m-0}^{M} \lambda dw(\lambda)$$

where $w(\lambda) = \langle E_\lambda x, y\rangle$.

Theorem 11.39 (Properties of $(E_\lambda)_{\lambda\in\mathbb{R}}$)**.** *Let H be a Hilbert space, $T\colon H \longrightarrow H$ a self-adjoint linear operator and $(E_\lambda)_{\lambda\in\mathbb{R}}$ the associated spectral family. Let $\lambda_0 \in \mathbb{R}$.*

1. *E_λ is discontinuous at $\lambda_0 \in \mathbb{R}$ if and only if $\lambda_0 \in \sigma_p(T)$.*
2. *$\lambda_0 \in \rho(T)$ if and only if there is a $c > 0$ such that the family of projectors E_λ is constant on the interval $J = [\lambda_0 - c, \lambda_0 + c]$.*

12 Compactness

Definition 12.1. A metric space X is ***compact*** if every sequence in X has a convergent subsequence. A subset $M \subset X$ of a metric space is compact if every sequence in M has a convergent subsequence that converges in M.

Proposition 12.2 (Properties of compactness)**.** *Let X be a normed space.*

1. *If $M \subset X$ is a compact set, then it is closed and bounded.*
2. *There are sets that are closed and bounded but not compact.*
3. *In finite dimensions, a subset $M \subset X$ is compact if and only if it is closed and bounded.*
4. *If $B_X(0; 1)$ is compact, then $\dim X < \infty$.*

Lemma 12.3 (Riesz's lemma)**.** *Let X be a normed space, $Z \subset X$ a subspace and $Y \subsetneq Z$ a proper closed subspace. Then for any $0 < t < 1$ there exists a $z \in Z$ such that $\|z\| = 1$ and $\|z - y\| \geq t$ for all $y \in Y$.*

Definition 12.4. Let X be a normed space and $M \subset X$ a subset. Then M is ***relatvely compact*** in X if its closure in X is compact.

Definition 12.5. Let X and Y be normed spaces. A linear operator $T\colon X \longrightarrow Y$ is called ***compact*** if for every bounded subset $M \subset X$, $T(M)$ is relatively compact in Y.

Lemma 12.6. *Let X and Y be normed spaces and $T\colon X \longrightarrow Y$ a linear operator. If T is compact, then it is bounded. If $\dim X = \infty$, then the identity operator $I\colon X \longrightarrow X$ is not compact.*

Theorem 12.7. *Let X and Y be normed spaces. A linear operator $T\colon X \longrightarrow Y$ is compact if and only if every bounded sequence (x_n) in X gets mapped to a sequence (Tx_n) that has a convergent subsequence.*

Theorem 12.8. *Let X and Y be normed spaces and $T\colon X \longrightarrow Y$ a linear operator. If T is bounded and $\dim T(X) < \infty$ then T is compact. if $\dim(X) < \infty$ then T is compact.*

Theorem 12.9. *Let X be a normed space, Y a Banach space and (T_n) a sequence of compact linear operators $T_n\colon X \longrightarrow Y$. If $\|T_n - T\| \longrightarrow 0$ as for some linear operator $T\colon X \longrightarrow Y$, then T is compact.*

Theorem 12.10. *Let X and Y be normed spaces and $T\colon X \longrightarrow Y$ a compact linear operator. If $x_n \overset{w}{\longrightarrow} x$ in X then $Tx_n \longrightarrow Tx$ in Y.*

Proposition 12.11. *Let X be a metric space. A subset $B \subset X$ is relatively compact in X if and only if every sequence (x_n) in B has a convergent subsequence in X.*

Definition 12.12. Let X be a metric space, $B \subset X$ and $\varepsilon > 0$. A set $M_\varepsilon \subset X$ is an ***ε-net*** for B if for every point $z \in B$ there is a point in M_ε that is a distance less that ε away from z. The set B is called ***totally bounded*** if for every $\varepsilon > 0$ there exists a finite ε-net for B.

Proposition 12.13 (Propertoes of ε-nets and total boundedness)**.** *Let X be a metric space and $B \subset X$ a subset.*

1. *If B is relatively compact then B is totally bounded.*
2. *if B is totally bouned and X Banach then B is relatively compact.*
3. *If B is totally bouned then for all $\varepsilon > 0$ there exists a finite ε-net M_ε for B such that $M_\varepsilon \subset B$.*
4. *If B is totally bounded then B is separable.*
5. *Total boundedness implies boundedness, but not vice versa.*

Theorem 12.14. *Let X and Y be normed spaces and $T\colon X \longrightarrow Y$ a compact linear operator. Then the range of T is separable.*

Theorem 12.15. *Let X and Y be normed spaces. If $T\colon X \longrightarrow Y$ is a compact linear operator, then $T^\times\colon Y' \longrightarrow X'$ is also compact.*