Assignment 1

AMAT 617

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Problem 1. Prove Hölder's and the Minkowski inequalies. Use these to show that ℓ^p and L^p are normed spaces.

Solution.

Claim 1. Hölder's inequality for sums

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} \left(\sum_{j=1}^{\infty} |y_j|^q \right)^{1/q}$$

holds for all $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, where $\{x_n\} \in \ell^p$ and $\{y_n\} \in \ell^q$.

Proof. First note that, for any positive numbers α, β, with p and q given as above, we have

$$\begin{aligned} \alpha\beta &= e^{\ln \alpha\beta} \\ &= e^{\ln \alpha + \ln \beta} \\ &= e^{\frac{1}{p}\ln \alpha^p + \frac{1}{q}\ln \beta^q} \\ &\leq \tfrac{1}{p} e^{\ln \alpha^p} + \tfrac{1}{q} e^{\ln \beta^q} \\ &= \frac{\alpha^p}{p} + \frac{\beta^q}{q}, \end{aligned}$$

where the inequality holds due to the fact that the exponential function is convex and that $\frac{1}{p} + \frac{1}{q} = 1$. Then the inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{1}$$

holds for all non-negative numbers, since it holds trivially if either a or b is zero.

If either $x = \{x_n\}$ or $y = \{y_n\}$ are the zero sequence, then Hölder's inequality holds trivially, so suppose that both x and y are nonzero sequences. Then define two new sequences $\alpha = \{\alpha_n\}$ and $\beta = \{\beta_n\}$ by

$$\alpha_n = \frac{x_n}{||x||_p} \qquad \beta_n = \frac{y_n}{||y||_q},$$

such that $||\alpha||_p = ||\beta||_q = 1$. Using the inequality in (1), for each n we have $|\alpha_n \beta_n| \leq \frac{|\alpha_n|^p}{p} + \frac{|\beta_n|^q}{q}$. Summing over n, we obtain

$$\sum_{n=1}^{\infty} |\alpha_n \beta_n| \leq \frac{1}{p} \sum_{i=1}^{\infty} |\alpha_i|^p + \frac{1}{q} \sum_{i=j}^{\infty} |\beta_j|^q = \frac{1}{p} + \frac{1}{q} = 1.$$

Multiplying both sides of $\sum_{n=1}^{\infty} |\alpha_n \beta_n| \leq 1$ by $||x||_p ||y||_q$ yields the desired inequality. □

Claim 2. The Minkowski inequality for sums

$$\left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{1/p} \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{1/p}$$

holds for all $p \geq 1$ where $\{x_n\} \in \ell^p$ and $\{y_n\} \in \ell^p$.

Proof. For $p = 1$, the inequality follows from the triangle inequality for numbers. Consider $p > 1$. Since $|x_n + y_n| \leq |x_n| + |y_n|$ and thus

$$|x_n + y_n|^p = (|x_n| + |y_n|)|x_n + y_n|^{p-1} \leq |x_n + y_n|\, |x_n + y_n|^{p-1},$$

it follows that

$$\sum_{n=1}^{\infty} |x_n + y_n|^p \leq \sum_{n=1}^{\infty} |x_n|\, |x_n + y_n|^{p-1} + \sum_{n=1}^{\infty} |y_n|\, |x_n + y_n|^{p-1}.$$

Define $q = \frac{p}{p-1}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Applying Hölder's inequality, we have

$$\begin{aligned} \sum_{n=1}^{\infty} |x_n|\, |x_n + y_n|^{p-1} &\leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} \left(\sum_{n=1}^{\infty} \left(|x_n + y_n|^{p-1}\right)^q\right)^{1/q} \\ &= \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} \left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{1/q}, \end{aligned}$$

where the final line comes from the fact that $(p-1)q = p$. Similarly, applying Hölder's inequality to the other sum, we have

$$\sum_{n=1}^{\infty} |y_n|\, |x_n + y_n|^{p-1} \leq \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{1/p} \left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{1/q}.$$

Noting that $\frac{1}{q} = 1 - \frac{1}{p}$, combining the inequalities above yields

$$\sum_{n=1}^{\infty} |x_n + y_n|^p \leq \left(\left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{1/p}\right) \left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{1-\frac{1}{p}}.$$

Multiplying both sides by $\left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{\frac{1}{p}-1}$ yields the desired inequality. □

Proposition 1. *For $p \geq 1$, the pair $(\ell^p, ||\cdot||_p)$ with the norm $||x||_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}$ is a normed space.*

Proof. We need to check that $||\cdot||_p$ satisfies the the conditions to be a norm. Conditions (i)-(iii) are trivial, so it remains to check the triangle inequality. That is, $||x+y||_p \leq ||x||_p + ||y||_p$ for all $x, y \in \ell^p$. This follows directly from the Minkowski inequality. Indeed, we have

$$||x+y||_p = \left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{1/p} \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{1/p} = ||x||_p + ||y||_p.$$

□

Claim 3. Hölder's inequality for integrals

$$\int_a^b |f(x)g(x)|dx \leq \left(\int_a^b |f(x)|dx\right)^{1/p} \left(\int_a^b |g(x)|dx\right)^{1/q}$$

holds for all $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, where f and g are integrable functions on the interval $[a, b]$.

Proof. If either $||f||_p = \left(\int |f(x)|^p dx\right)^{1/p} = 0$ or $||g||_q = \left(\int |g(x)|^q dx\right)^{1/q} = 0$, then the inequality holds trivially. So suppose that both f and g have nonzero norm. Then define the functions u and v on $[a, b]$ by

$$u(x) = \frac{f(x)}{||f||_p} \qquad v(x) = \frac{g(x)}{||g||_q}$$

such that $||u||_p = ||v||_q = 1$. Using the inequality (1) from Claim 1 above, we have that at each $x \in [a, b]$

$$|u(x)v(x)| \leq \frac{|u(x)|^p}{p} + \frac{|v(x)|^q}{q}.$$

Integrating this inequality over x yields

$$\int_a^b |u(x)v(x)|dx \leq \frac{1}{p}\int_a^b |u(x)|^p dx + \frac{1}{q}\int_a^b |v(x)|^q dx = \frac{1}{p} + \frac{1}{q} = 1.$$

As in the proof of Claim 1, multiplying both sides of the above inequality by $||f||_p||g||_q$ yields the desired result. □

Claim 4. The Minkowski inequality for integrals

$$\left(\int_a^b |f(x) + g(x)|dx^p\right)^{1/p} \leq \left(\int_a^b |f(x)|dx^p\right)^{1/p} + \left(\int_a^b |g(x)|dx^p\right)^{1/p}$$

holds for all $p \geq 1$ where f and g are integrable functions on the interval $[a, b]$.

Proof. The proof is analogous. □

Proposition 2. *For $p \geq 1$, the pair $(L^p[a, b], ||\cdot||_p)$ with the norm $||f||_p = \left(\int_a^b |f(x)|^p dx\right)^{1/p}$ is a normed space.*

Proof. We need to check that $||\cdot||_p$ satisfies the the conditions to be a norm. Conditions (i)-(iii) are trivial, so it remains to check the triangle inequality. That is, $||f + g||_p \leq ||f||_p + ||g||_p$ for all $f, g \in L^p[a, b]$. This follows directly from the Minkowski inequality. Indeed, we have

$$||f + g||_p = \left(\int_a^b |f(x) + g(x)|^p dx\right)^{1/p} \leq \left(\int_a^b |f(x)|^p dx\right)^{1/p} + \left(\int_a^b |g(x)|^p dx\right)^{1/p} = ||f||_p + ||g||_p.$$

□

Problem 2 (Problem 14, Chapter 2.2, p. 66 in Kreyszig)**.** If d is a metric on a vector space $X \neq \{0\}$ which is obtained from a norm, and $\tilde{d}$ is defined by

$$\tilde{d}(x, y) = \begin{cases} 0, & y = x \\ d(x, y) + 1, & y \neq x, \end{cases}$$

show that $\tilde{d}$ cannot be obtained from a norm.

Solution. Suppose there is a norm $||\cdot||' : X \to \mathbb{R}$ such that $\tilde{d}(x, y) = ||x - y||'$ for all $x, y \in X$. Then for all $a \in \mathbb{R}$ and $x, y \in X$, $\tilde{d}$ satisfies

$$\tilde{d}(ax, ay) = ||ax - ay||' = |a| \, ||x - y||' = |a|\tilde{d}(x, y). \tag{2}$$

But, since d is obtained from a norm, we have $d(ax, ay) = |a|d(x, y)$ for all $x, y \in X$ and $a \in \mathbb{R}$. Let $x \neq y$ and $a \neq 0$, then

$$\begin{aligned} \tilde{d}(ax, ay) = d(ax, ay) + 1 &= |a|d(x, y) + 1 \\ &= |a|\big(d(x, y) + 1\big) + 1 - |a| \\ &= |a|\tilde{d}(x, y) + 1 - |a|. \end{aligned} \tag{3}$$

In particular for $a = 2$, the result in (2) is not equal to that in (3).

Problem 3 (Problem 6, Chapter 2.3, p. 70 in Kreyszig)**.** Show that the closure $\bar{Y}$ of a subspace Y of a normed space is again a subspace.

Solution. Let $x, y \in \overline{Y}$ and $\alpha, \beta \in \mathbb{K}$. Then there exist sequences $\{x_n\}$ and $\{y_n\}$ in Y such that

$$x = \lim_{n\to\infty} x_n \qquad \text{and} \qquad y = \lim_{n\to\infty} y_n.$$

Define $z_n = \alpha x_n + \beta y_n$ for each n, and define $z = \alpha x + \beta y$. Since Y is a subspace, $z_n \in Y$ for each n. Then

$$\begin{aligned}\lim_{n\to\infty} ||z - z_n|| &= \lim_{n\to\infty} ||\alpha(x - x_n) - \beta(y - y_n)|| \\ &\leq |\alpha| \lim_{n\to\infty} ||x - x_n|| + |\beta| \lim_{n\to\infty} ||y - y_n|| \\ &= 0,\end{aligned}$$

so $z = \lim_{n\to 0} z_n$, and thus $z \in \overline{Y}$. Hence, $\overline{Y}$ is a subspace.

Problem 4 (Problem 8, Chapter 2.3, p. 71 in Kreyszig)**.** If in a normed space X, absolute convergence of any series always implies convergence of that series, show that X is complete.

Solution. Let $\{x_j\}$ be a Cauchy sequence in X. Construct another sequence $\{y_j\}$ in the following manner. For each $j \in \mathbb{N}$ there exists an $N_j \in \mathbb{N}$ such that $||x_n - x_m|| < \frac{1}{2^{-j}}$ for all $n, m > N_j$. Define $y_j = x_{N_j}$. In particular, $||y_{j+1} - y_j|| < \frac{1}{2^j}$ for each j.

Let $\{z_j\}$ be the sequence defined by $z_1 = y_1$ and $z_j = y_{j+1} - y_j$ for $j > 1$. Consider the series $\sum_{j=1}^{\infty} ||z_j||$. This series converges, since $||z_j|| = ||y_{j+1} - y_j|| < \frac{1}{2^j}$ for each j and thus

$$\sum_{j=1}^{\infty} ||z_j|| < \sum_{j=1}^{\infty} \frac{1}{2^j} = 2.$$

By assumption, absolute convergence implies convergence, so the sequence $z = \sum_{j=1}^{\infty} z_j$ converges to some element $z \in X$. Note that the partial sums of this series are

$$s_n = \sum_{j=1}^{n} z_j = y_1 + (y_2 - y_1) + \cdots + (y_n - y_{n-1}) = y_n.$$

This implies that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} s_n = z$. So the sequence $\{y_j\}$ is convergent and converges to $\lim_{n\to\infty} y_n = z$.

Finally, we need to show that $\{x_n\}$ also converges to z. Indeed, for all $\epsilon > 0$ there exists an $n \in \mathbb{N}$ such that $\frac{1}{2^{n-1}} < \epsilon$. By definition of the sequence $\{y_j\}$, we have $||y_n - y_m|| < \frac{1}{2^n}$ for each $m > n$, and thus

$$||y_n - z|| = \lim_{m\to\infty} ||y_n - y_m|| \leq \frac{1}{2^n}.$$

Similarly, by definition of $\{y_j\}$, there exists an $M \in \mathbb{N}$ such that $||x_m - y_n|| < \frac{1}{2^n}$ for all $m > M$. Hence

$$\begin{aligned} ||x_m - z|| = ||x_m - y_n + y_n - z|| &\leq ||x_m - y_n|| + ||y_n - z|| \\ &< \frac{1}{2^n} + \frac{1}{2^n} \\ &= \frac{1}{2^{n-1}} \\ &< \epsilon, \end{aligned}$$

and so $\{x_j\}$ converges to z. We have shown that every Cauchy sequence in X converges to something in X, and thus X is complete.

Problem 5 (Problem 10, Chapter 2.3, p. 71 in Kreyszig - Schauder basis)**.** Show that if a normed space has a Schauder basis, it is separable.

Solution. First note that $\mathbb{K}$ (either $\mathbb{R}$ or $\mathbb{C}$) is separable. Indeed, $\mathbb{Q}$ is countably dense in $\mathbb{R}$, and

$$Q = \{a + bi \,|\, a, b \in \mathbb{Q}\}$$

is countably dense in $\mathbb{C}$. Without loss of generality let $\mathbb{K} = \mathbb{C}$, and define the countable set $Y \subset X$

$$Y = \left\{ \sum_{i=1}^{k} \beta_i e_i \,\middle|\, k \in \mathbb{N},\, \beta_i \in Q \right\}$$

of all finite sums of linear combinations of e_i with coefficients in Q. We will show that Y is dense in X.

Let X be a normed space X over $\mathbb{C}$ and let $\{e_n\}$ a Schauder basis of X. Let $x \in X$ and $\epsilon > 0$. Then there exists a sequence of scalars $\{\alpha_n\}$ in $\mathbb{C}$ and an $N \in \mathbb{N}$ such that

$$||x - \sum_{j=1}^{n} \alpha_j e_j|| < \frac{\epsilon}{2}$$

for all $n \geq N$. In particular, for $n = N$,

$$||x - \sum_{j=1}^{N} \alpha_j e_j|| < \frac{\epsilon}{2}.$$

By density of Q in $\mathbb{C}$, there exist constants $\beta_1, \ldots, \beta_N$ in Q such that

$$||\sum_{j=1}^{N} (\beta_j - \alpha_j) e_j|| < \frac{\epsilon}{2}.$$

By the triangle inequality, we have

$$\begin{aligned} ||\sum_{j=1}^{N} \beta_j e_j - x|| &\leq ||\sum_{j=1}^{N} (\beta_j - \alpha_j) e_j|| + ||\sum_{j=1}^{N} \alpha_j e_j - x|| \\ &\leq \epsilon \end{aligned}$$

Note that $\sum_{j=1}^{N} \beta_j e_j \in Y$. Thus, for all $x \in X$ and $\epsilon > 0$ there exists a $y \in Y$ such that $||y - x|| \leq \epsilon$. Hence Y is dense in X, as desired.

Problem 6 (Problem 12, Chapter 2.3, p. 71 in Kreyszig - seminorm)**.** A *seminorm* on a vector space X is a mapping $p : X \to \mathbb{R}$ satisfying

(N1) $p(x) \geq 0$

(N3) $p(\alpha x) = |\alpha| p(x)$

(N4) $p(x + y) \leq p(x) + p(y)$ (triangle inequality).

Show that

$$p(0) = 0, \tag{*}$$

$$|p(y) - p(x)| \leq p(y - x). \tag{**}$$

Solution. For equation (*), note that $0x = 0$ for each $x \in X$, i.e. zero times an element is equal to the zero vector. Thus, from (N3),

$$p(0) = p(0x) = 0p(x) = 0.$$

For equation (**), note that

$$p(y) = p(x + y - x) \leq p(x) + p(y - x) \qquad \text{and} \qquad p(x) = p(y + x - y) \leq p(y) + p(x - y)$$

from (N4). Hence

$$p(y) - p(x) \leq p(y - x) \qquad \text{and} \qquad p(x) - p(y) \leq p(x - y).$$

From (N3), $p(x - y) = p(y - x)$. Thus

$$|p(y) - p(x)| \leq p(y - x).$$

Problem 7 (Problem 6, Chapter 2.7, p. 101 in Kreyszig - Range)**.** Show that the range $\mathcal{R}(T)$ of a bounded linear operator $T : X \to Y$ need not be closed in Y.

Solution. Let X be a proper dense subspace of a space Y and consider the identity operator on X $T : \mathcal{D}(T) \to \mathcal{R}(T)$ with $\mathcal{D}(T) = \mathcal{R}(T) = X \subset Y$. Clearly $||T|| = 1$ so T is bounded. But $\mathcal{R}(T)$ is not closed in Y, since $\mathcal{D}(T) = X \neq Y$ by assumption.

It remains to show that there exists such a space Y with a proper dense subspace X. Indeed, let $Y = \ell^p$ for $p > 1$ and let X be the space of all sequences that terminate after a finite number of entries, i.e.

$$X = \{\{\alpha_j\} \,|\, \alpha_j \neq 0 \text{ for only finitely many } j\}\,.$$

Then X is clearly a subspace of Y. Furthermore, X is dense in Y, since for any sequence $y = \{y_j\}$ in Y we can construct a sequence $\{x^{(n)}\}$ of sequences in X given by

$$x^{(n)} = \{y_1, y_2, \ldots, y_n, 0, 0, \ldots\}$$

such that $\{x^{(n)}\}$ converges to y.

Problem 8 (Problem 8, Chapter 2.7, p. 101 in Kreyszig)**.** Show that the inverse $T^{-1} : \mathcal{R}(T) \to \mathcal{D}(T)$ of a bounded linear operator $T : \mathcal{D}(T) \to \mathcal{R}(T)$ need not be bounded.

Solution. Consider the opertor $T : \ell^\infty \to \ell^\infty$ defined by

$$Tx = \left\{ \frac{x_n}{n} \right\} \qquad \text{where } x = \{x_n\}.$$

This is indeed a linear operator, since

$$T(x+y) = \left\{ \frac{x_n + y_n}{n} \right\} = \left\{ \frac{x_n}{n} \right\} + \left\{ \frac{y_n}{n} \right\} = Tx + Ty.$$

and $T(\alpha x) = \left\{ \frac{\alpha x_n}{n} \right\} = \alpha \left\{ \frac{x_n}{n} \right\} + \alpha Tx$. Furthermore, T is bounded, since

$$||Tx||_\infty = \sup_n \frac{|x_n|}{n} \leq \sup_n |x_n| = ||x||_\infty,$$

since $\frac{|x_n|}{n} \leq |x_n|$ for all $n \geq 1$. Since $Tx = 0$ if and only if x is the zero sequence, T^{-1} exists.

We now show that T^{-1} is not bounded. Let $c > 0$. Then there exists a $N \in \mathbb{N}$ such that $N > c$. Let z be the sequence $z = \{z_n\}$ defined by

$$z_n = \begin{cases} 0, & n \neq N \\ 1, & n = N. \end{cases}$$

Then $z \in \mathcal{R}(T)$. Indeed, $z = T\,(Nz)$ where $Nz = \{Nz_n\}$ is the sequence consisting of N in the n^{th} spot and zeros elsewhere and thus $||\frac{1}{N}z||_\infty = N$. So $Nz \in \ell^\infty = \mathcal{D}(T)$. Then $T^{-1}z = Nz$, but

$$||T^{-1}z||_\infty = N||z||_\infty = N > c.$$

So T^{-1} is unbounded.

Problem 9 (Problem 10, Chapter 2.8, p. 110 in Kreyszig)**.** Let $f \neq 0$ be a linear functional on a vector space X and let $\mathcal{N}(f)$ be the null space of f. Show that two elements $x_1, x_2 \in X$ belong to the same element of the quotient space $X/\mathcal{N}(f)$ if and only if $f(x_1) = f(x_2)$. Show that $\operatorname{codim} \mathcal{N}(f) = 1$.

Solution. Suppose $f(x_1) = f(x_2)$ and set $y = x_1 - x_2$. Then $y \in \mathcal{N}(f)$, since $f(y) = f(x_1) - f(x_2) = 0$. So $x_1 = x_2 + y$ and thus $x_2 \in x_1 + \mathcal{N}(f)$. Clearly $x_1 \in x_1 + \mathcal{N}(f)$. Thus x_1 and x_2 belong to the same element of $X/\mathcal{N}(f)$.

Now suppose that x_1 and x_2 belong to the same element in $X/\mathcal{N}(f)$. Then $x_1, x_2 \in x_0 + \mathcal{N}(f)$ for some element $x_0 \in X$. That is, $x_1 = x_0 + y_1$ and $x_2 = x_0 + y_2$ for some $y_1, y_2 \in \mathcal{N}(f)$. Then

$$f(x_1) = f(x_0 + y_1) = f(x_0) + f(y_1) = f(x_0) \quad \text{and} \quad f(x_2) = f(x_0 + y_2) = f(x_0) + f(y_1) = f(x_0),$$

so $f(x_1) = f(x_2)$.

We now show that $\operatorname{codim} \mathcal{N}(f) = 1$. Note that $\operatorname{codim} \mathcal{N}(f) = \dim X/\mathcal{N}(f)$.

Since $f \neq 0$, there is an element $x_0 \in X$ such that $f(x_0) = 1$. Indeed, there is at least one element $x' \in X$ such that $f(x') \neq 0$, so set $x_0 = \frac{x'}{f(x')}$. Define $\hat{x}_0 = x_0 + \mathcal{N}(f)$ as an element in $X/\mathcal{N}(f)$. Let $\hat{x} \in X/\mathcal{N}(f)$ not be the zero vector in $X/\mathcal{N}(f)$. Then $\hat{x} = x + \mathcal{N}(f)$ for some $x \in X \setminus \mathcal{N}(f)$, i.e. $f(x) \neq 0$. Let $\alpha = f(x)$. Then $\alpha\hat{x}_0 = \hat{x}$. Indeed,

$$f(\alpha x_0) = \alpha f(x_0) = f(x)$$

so, due to the arguments above, αx_0 and x define the same element of $X/\mathcal{N}(f)$ and thus $\hat{x} = \alpha\hat{x}_0$. Hence, $X/\mathcal{N}(f)$ is a 1-dimensional vector space.

Problem 10 (Problem 12, Chapter 2.8, p. 111 in Kreyszig)**.** If Y is a subspace of a vector space X and $\operatorname{codim} Y = 1$, then every element of X/Y is called a *hyperplane parallel to* Y.

Show that for any linear functional $f \neq 0$ on X, the set $H_1 = \{x \in X \mid f(x) = 1\}$ is a hyperplane parallel to the null space $\mathcal{N}(f)$ of f.

Solution. Let $Y = \mathcal{N}(f)$. From the previous problem, we have that $\operatorname{codim} Y = 1$, and that there exists an $x_0 \in X$ such that $f(x_0) = 1$, since $f \neq 0$. We show that $H_1 = x_0 + \mathcal{N}(f)$.

Indeed, for $x_0 + y \in x_0 + \mathcal{N}(f)$ we have $f(x_0 + y) = f(x_0) + f(y) = 1$, so $x_0 + \mathcal{N}(f) \subset H_1$. Now let $x \in H_1$ and set $y = x - x_0$. Then $y \in \mathcal{N}(f)$ since

$$f(y) = f(x) - f(x_0) = 1 - 1 = 0.$$

So $x = x_0 + y$ and $x \in x_0 + \mathcal{N}(f)$. Hence $H_1 \subset x_0 + \mathcal{N}(f)$. We conclude that $H_1 = x_0 + \mathcal{N}(f)$.

Problem 11 (Problem 10, Chapter 2.9, p. 117 in Kreyszig). Let Z be a proper subspace of an n-dimensional vector space X, and let $x_0 \in X \setminus Z$. Show that there is a linear functional f on X such that $f(x_0) = 1$ and $f(x) = 0$ for all $x \in Z$.

Solution. Since Z is finite dimensional, there is an orthonormal basis $\{e_1, \ldots, e_k\}$ of Z. Since X is n-dimensional, the dimension of $Z^\perp$ is $n - k$, so there is an orthonormal basis $\{e_{k+1}, \ldots, e_n\}$ of $Z^\perp$ such that $\{e_1, \ldots, e_n\}$ is an orthonormal basis of X. Then there exists unique coefficients α_j such that

$$x_0 = \alpha_1 e_1 + \cdots \alpha_n e_n.$$

Since $x_0 \notin Z$, there is at least one $t \in \{k+1, \ldots, n\}$ such that $\alpha_t \neq 0$. Define a linear functional f by its action on the basis elements in the following manner:

$$f(e_j) = \begin{cases} \frac{1}{\alpha_t}, & j = t \\ 0, & j \neq t. \end{cases}$$

Then for all $z \in Z$, we have $z = \sum_{j=1}^{k} \beta_j e_j$ and thus

$$f(z) = \sum_{i=1}^{k} \beta_j f(e_j) = 0.$$

So f vanishes on Z. However, $f(x_0) = \frac{1}{\alpha_t} \alpha_t = 1$.

Problem 12 (Problem 10, Chapter 2.10, p. 126 in Kreyszig)**.** Let X and $Y \neq \{0\}$ be normed spaces, where $\dim X = \infty$. Show that there is at least one unbounded linear operator $T : X \longrightarrow Y$.

Solution. Let $y \in Y$ $y \neq 0$. Let $E = \{e_\kappa\}_{\kappa \in I}$ be a Hamel basis of X, where I is some index set with $|I| = \dim X$. Let $E' = \{e_{\kappa_n}\}_{n\in\mathbb{N}}$ be a countable subset of E. Then define an operator $T : X \longrightarrow Y$ by

$$Te_{\kappa_n} = n\frac{||e_{\kappa_n}||}{||y||}y$$

and $Te_\kappa = 0$ if $\kappa \neq E'$. This is a linear operator.

For each $c > 0$ there exists an $N > c$ such that

$$||Te_{\kappa_N}|| = N\frac{||e_{\kappa_N}||}{||y||}||y|| = N||e_{\kappa_N}|| > c||e_{\kappa_N}||,$$

and thus T is unbounded.