# Assignment 2

AMAT 617

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**Problem 1** (Problem 8, Chapter 3.2, p. 141 in Kreyszig)**.** Show that in an inner product space, $x \perp y$ if and only if $||x + \alpha y|| \geq ||x||$ for all scalars $\alpha$.

**Solution.** First suppose that $x \perp y$. Then $\langle x, y\rangle = 0$. For any $\alpha$,

$$\begin{aligned} ||x + \alpha y||^2 &= \langle x + \alpha y, x + \alpha y\rangle \\ &= ||x||^2 + \alpha \underbrace{\langle y, x\rangle}_{=0} + \overline{\alpha} \underbrace{\langle x, y\rangle}_{=0} + \underbrace{|\alpha|^2 ||y||^2}_{\geq 0} \\ &\geq ||x||^2. \end{aligned}$$

Hence $||x|| \leq ||x + \alpha y||$.

Now suppose that $||x|| \leq ||x + \alpha y||$ for all $\alpha$. Assume that $\langle y, x\rangle \neq 0$ and let $\alpha = -\frac{\langle x,y\rangle}{||y||^2}$. Then

$$\begin{aligned} ||x + \alpha y||^2 &= \langle x + \alpha y, x + \alpha y\rangle \\ &= ||x||^2 + \overline{\alpha} \langle x, y\rangle + \alpha \underbrace{\left(\langle y, x\rangle + \overline{\alpha}||y||^2\right)}_{=0} \\ &= ||x||^2 - \frac{|\langle x, y\rangle|^2}{||y||^2} \\ &< ||x||^2 \end{aligned}$$

since $\frac{|\langle x,y\rangle|^2}{||y||^2} > 0$, a contradiction to the fact that $||x|| \leq ||x + \alpha y||$ for all $\alpha$.

**Problem 2** (Problem 10, Chapter 3.3, p. 150 in Kreyszig)**.** If $M \neq \emptyset$ is any subset of a Hilbert space $H$, show that $M^{\perp\perp}$ is the smallest closed subspace of $H$ which contains $M$. That is, $M^{\perp\perp}$ is contained in any closed subspace $Y \subset H$ such that $M \subset Y$.

**Solution.** We first show that for any $M \subset H$, $M^{\perp}$ is closed. Indeed, for any $x \in H$ the inverse image of $\{0\} \subset H$ under the continuous mapping

$$y \longmapsto \langle y, x\rangle$$

is closed since $\{0\}$ is closed. That is, $\{y \in H \mid \langle y, x\rangle = 0\}$ is closed for all $x \in H$. Since $M^{\perp}$ is the intersection of a family of such closed sets, namely

$$M^{\perp} = \bigcap_{x \in M} \{y \in H \mid \langle y, x\rangle = 0\},$$

it is also closed. Hence $M^{\perp\perp}$ is closed as well. Since $\overline{\text{span}(M)}$ is the smallest closed subspace that contains $M$, it remains to show that $\overline{\text{span}(M)} = M^{\perp\perp}$.

By the lemma in class, for a nonempty subset $N \subset H_1$ of a Hilbert space, the set $\overline{\text{span}(N)}$ is dense in $H_1$ if and only if $N^{\perp} = \{0\}$. Note that $M^{\perp\perp}$ itself is a Hilbert space since it is closed, and $M \subset M^{\perp\perp}$ is a subset. Denote $H_1 = M^{\perp\perp}$ and note that $\text{span}(M) \subset H_1$. Since we have that $M^{\perp} \cap M^{\perp\perp} = \{0\}$, this implies that $\overline{\text{span}(M)} = M^{\perp\perp}$.

**Problem 3** (Problem 4, Chapter 3.4, p. 159 in Kreyszig)**.** Give an example of an $x \in \ell^2$ such that the Bessel inequality

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq ||x||^2$$

is a strict inequality.

**Solution.** Let $\{\tilde{e}_j\}_{j=1}^{\infty}$ be the standard orthonormal basis of $\ell^2$, i.e.

$$\tilde{e}_j = (0, \dots, 0, \underbrace{1}_{j^{\text{th}}\text{position}}, 0, \dots),$$

and let $\{e_k\}_{k=1}^{\infty}$ be the orthonormal set defined by $e_k = \tilde{e}_{k+1}$ (namely, we have 'thrown away' the first element). Let $x \in \ell^2$ be $\tilde{e}_1$,

$$x = \tilde{e}_1 = (1, 0, 0, \dots).$$

Then $||x|| = 1$, but clearly $\langle x, e_k \rangle = 0$ for all $k$. So

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 = 0 < 1 = ||x||^2.$$

**Problem 4** (Problem 6, Chapter 3.4, p. 159 in Kreyszig — **Minimum property of Fourier coefficients**). Let $\{e_1, \dots, e_n\}$ be an orthonormal set in an inner product space $X$, where $n$ is fixed. Let $x \in X$ be any fixed element and $y = \beta_1 e_1 + \cdots + \beta_n e_n$. Then $||x - y||$ depends on $\beta_1, \dots, \beta_n$. Show by direct calculations that $||x - y||$ is minimum if and only if $\beta_j = \langle x, e_j \rangle$, where $j = 1, \dots, n$.

**Solution.** Let $\alpha_i = \langle x, e_i \rangle$ and define $\tilde{x} = x - (\alpha_1 e_1 + \cdots + \alpha_n e_n)$. Then $x - y = \tilde{x} + \sum_{i=1}^{n} (\alpha_i - \beta_i) e_i$ and we have $\langle \tilde{x}, e_j \rangle = 0$ for all $j$. So

$$\begin{aligned}
||x - y||^2 &= \langle x - y, x - y \rangle \\
&= \left\langle \tilde{x} + \sum_{i=1}^{n} (\alpha_i - \beta_i) e_i, \tilde{x} + \sum_{j=1}^{n} (\alpha_j - \beta_j) e_j \right\rangle \\
&= \langle \tilde{x}, \tilde{x} \rangle + \sum_i (\alpha_i - \beta_i) \underbrace{\langle \tilde{x}, e_i \rangle}_{=0} + \sum_j \overline{(\alpha_j - \beta_j)} \underbrace{\langle e_j, \tilde{x} \rangle}_{=0} + \sum_{i,j} (\alpha_i - \beta_i) \overline{(\alpha_j - \beta_j)} \underbrace{\langle e_j, e_i \rangle}_{\delta_{ij}} \\
&= ||\tilde{x}||^2 + \sum_i ||\alpha_i - \beta_i||^2 \\
&\geq ||\tilde{x}||^2,
\end{aligned}$$

with equality if and only if $\alpha_i = \beta_i$ for all $i$. So this is at a minimum if and only if $\beta_i = \alpha_i$.

**Problem 5** (Problem 4, Chapter 3.5, p. 166 in Kreyszig)**.** If $\{x_j\}$ is a sequence in an inner product space $X$ such that the series $||x_1||+||x_2||+\cdots$ converges, show that $\{s_n\}$ is a Cauchy sequence, where $s_n = x_1 + \cdot + x_n$.

**Solution.** (This follows from the fact that inner product spaces are normed spaces.)

Define the sequence $\{t_n\}$ with $t_n = ||x_1|| + \cdots + ||x_n||$. Since $\{t_n\}$ converges, let $\varepsilon > 0$ and let $N$ such that $||t_m - t_n|| < \varepsilon$ for all $n, m > N$. Let $m, n > N$ and assume without loss of generality that $n \leq m$. Then

$$\begin{aligned} ||s_m - s_n|| &= ||x_{n+1} + x_{n+2} + \cdots + x_m|| \\ &\leq ||x_{n+1}|| + \cdots + ||x_m|| \\ &= \big|||x_{n+1}|| + \cdots + ||x_m||\big| \\ &= ||t_m - t_n|| \\ &< \varepsilon, \end{aligned}$$

so the sequence $\{s_n\}$ is Cauchy.

**Problem 6** (Problem 8, Chapter 3.5, p. 166 in Kreyszig)**.** Let $\{e_k\}$ be an orthonormal sequence in a Hilbert space $H$, and let $M = \text{span}\{e_k\}$. Show that for any $x \in H$ we have $x \in \overline{M}$ if and only if $x$ can be represented by

$$\sum_{k=1}^{\infty} \alpha_k e_k$$

with coefficients $\alpha_k = \langle x, e_k \rangle$.

**Solution.** To say that $x$ can be *represented by* the series in $\sum_{k=1}^{\infty} \alpha_k e_k$ means that $x$ is equal to the limit of $\sum_{k=1}^{n} \alpha_k e_k$. Namely, for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$||x - \sum_{k=1}^{n} \alpha_k e_k|| < \varepsilon \qquad \text{for all } n > N.$$

Suppose $x = \sum_{k=1}^{\infty} \alpha_k e_k$ and define the sequence of partial sums with $x_n = \sum_{k=1}^{n} \alpha_k e_k$ which is in $M$. Then $\{x_n\}$ is a sequence in $M$ and $\lim_{n\to\infty} x_n = x$. So $x \in \overline{M}$. Furthermore,

$$\langle x, e_k \rangle = \lim_{n\to\infty} \langle x_n, e_k \rangle = \alpha_k.$$

Now suppose that $x \in \overline{M}$ and define $\alpha_k = \langle x, e_k \rangle$. Since $x$ is in the closure of $M$, for each $\varepsilon > 0$ there exists a $z \in M$ such that $||x - z|| < \varepsilon$. We can define a sequence $\{y_n\}$ in $M$ in the following manner. For each $n \in \mathbb{N}$, there exists a $z_n \in M$ such that $||x - z_n|| < \frac{1}{n}$. Since $z_n$ is in $\text{span}\{e_k\}$, it must be written as a finite sum of the form

$$z_n = \sum_{k=1}^{N_n} \beta_k^{(n)} e_k.$$

Then define the sequence $\{y_n\}$ by $y_n = \sum_{k=1}^{N_n} \alpha_k e_k$. Now $\{e_k\}_{k=1}^{N_n}$ is an orthonormal set in $H$, so by Problem 4 (Minimum Fourier Coefficients) we have that $||x - y_n||$ is minimal over all $y \in \text{span}\{e_k\}_{k=1}^{N_n}$ such that

$$||x - y_n|| \leq ||x - z_n||,$$

hence $||x - y_n|| < \frac{1}{n}$. Thus $\{y_n\}$ converges to $x$. Note that $\{y_n\}$ is a subsequence of the sequence $\{x_m\}$ defined by

$$x_m = \sum_{k=1}^{m} \alpha_k e_k,$$

but $\sum_{k=1}^{\infty} |\alpha_k|^2 \leq ||x||^2$ due to Bessel's inequality. Thus the sum $\sum_{k=1}^{\infty} |\alpha_k|^2$ converges, so that the sum $\sum_{k=1}^{\infty} \alpha_k e_k$ converges as well. Since $\{y_n\}$ and $\{x_m\}$ are both Cauchy sequences and one is a subsequence of the other, they must converge to the same point and thus

$$x = \sum_{k=1}^{\infty} \alpha_k e_k$$

as desired.

**Problem 7** (Problem 10, Chapter 3.6, p. 175 in Kreyszig)**.** Let $M$ be a subset of a Hilbert space $H$, and let $v, w \in H$. Suppose that $\langle v, x \rangle = \langle w, x \rangle$ for all $x \in M$ implies $v = w$. If this holds for all $v, w \in H$, show that $M$ is total in $H$.

**Solution.** By the first totality theorem stated in class, a subset $M$ is total in $H$ if and only if $M^\perp = \{0\}$. Let $y \in M^\perp$, then $\langle y, x \rangle = 0$ for all $x \in M$ and

$$\langle \alpha y, x \rangle = \langle y, x \rangle = 0 \qquad \text{for all scalars } \alpha \text{ and all } x \in M.$$

Hence, by assumption, $\alpha y = y$ for all scalars $\alpha$, which can only occur if $y = 0$. So $M^\perp = \{0\}$ and thus $M$ is total in $H$.

**Problem 8** (Problem 8, Chapter 3.8, p. 194 in Kreyszig)**.** Show that any Hilbert space $H$ is isomprphic to its second dual space $H'' = (H')'$. (Hint: see question 7 on the same page.)

**Solution.** We first show that $H'$ and $H''$ are Hilbert spaces.

**Lemma 1.** *If $H$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, then $H'$ is Hilbert space with inner product $\langle \cdot, \cdot \rangle_1$ given by*

$$\langle f_z, f_w \rangle_1 = \overline{\langle z, w \rangle} = \langle w, z \rangle ,$$

*where $f_z(x) = \langle x, z \rangle$ and $f_w(x) = \langle x, w \rangle$.*

*Proof.* By the Riesz representation, for all $f \in H'$ there is a vector $z \in H$ such that $f = f_z$ (that is $f(x) = f_z(x) = \langle x, z \rangle$) and $||f|| = ||x||$. So we can write $H' = \{f_z \mid z \in H\}$. Furthermore we have

$$f_z(x) + f_w(x) = \langle x, z \rangle + \langle x, w \rangle = \langle x, z + w \rangle = f_{z+w}(x)$$

so $f_z + f_w = f_{z+w}$, and similarly

$$\alpha f_z(x) = \alpha \langle x, z \rangle = \langle z, \overline{\alpha} z \rangle = f_{\overline{\alpha} z}(x)$$

so $\alpha f_z = f_{\overline{\alpha} z}$. Using $\langle \cdot, \cdot \rangle_1$ defined as above on $H' \times H'$, we claim that this is an inner product. Indeed, for any $f_z, f_{z'}, f_w \in H'$ we have

1. $\langle f_z + f_{z'}, f_w \rangle_1 = \langle f_{z+z'}, f_w \rangle = \langle w, z + z' \rangle = \langle w, z \rangle + \langle w, z' \rangle = \langle f_z, f_w \rangle_1 + \langle f_{z'}, f_w \rangle_1$
2. $\langle \alpha f_z, f_w \rangle_1 = \langle f_{\overline{\alpha} z}, f_w \rangle = \langle w, \overline{\alpha} z \rangle = \alpha \langle w, z \rangle = \alpha \langle f_z, f_w \rangle_1$
3. $\langle f_z, f_w \rangle_1 = \langle w, z \rangle = \overline{\langle z, w \rangle} = \overline{\langle f_w, f_z \rangle_1}$
4. $\langle f_z, f_z \rangle_1 = \langle z, z \rangle = ||z||^2 \geq 0$ and $\langle f_z, f_z \rangle_1 = \langle z, z \rangle = ||z||^2 = 0$ if and only if $z = 0$ and thus $f_z = 0$.

Furthermore, we have $\sqrt{\langle f_z, f_z \rangle_1} = \sqrt{\langle z, z \rangle} = ||z|| = ||f_z||$. Hence $\langle \cdot, \cdot \rangle_1$ satisfies all the requirements for being an inner product and this inner product gives the same norm. Since the dual space $H'$ is also complete, this is also a Hilbert space. ☐

**Corollary 2.** *If $H$ is a Hilbert space, the second dual space $H''$ is a Hilbert space with inner product given by*

$$\langle F_f, F_g \rangle_2 = \overline{\langle f, g \rangle_1} = \langle g, f \rangle_1$$

*where $F_f(h) = \langle h, f \rangle_1$ and $F_g(h) = \langle h, g \rangle_1$ for $f, g, h \in H'$. Furthermore, $H''$ can be written as*

$$H'' = \{F_f \mid f \in H'\} = \{F_{f_z} \mid z \in H\} .$$

*Proof.* The proof follows exactly as above. In particular, we have $F_{\alpha f} = \overline{\alpha} F_f$ and $F_{f+g} = F_f + F_g$. ☐

**Proposition 3.** *Any Hilbert space $H$ is isomorphic to its second dual space $H''$.*

*Proof.* By the lemma and corollary above, $H''$ is indeed a Hilbert space. Define the map

$$\begin{aligned} C : H &\longrightarrow H'' \\ z &\longmapsto F_{f_z} . \end{aligned}$$

Clearly, $C$ is surjective (by the Riesz Representation Theorem). It is also linear, since

$$C(\alpha z + \beta w) = F_{f_{\alpha z + \beta w}} = F_{\overline{\alpha} f_z + \overline{\beta} f_w} = F_{\overline{\alpha} f_z} + F_{\overline{\beta} f_w} = \alpha F_{f_z} + \beta F_{f_w} = \alpha C(z) + \beta C(w).$$

It is injective, since $C(z) = 0$ means $F_{f_z} = 0$ and thus $f_z = 0$ which implies $z = 0$. Finally, $C$ preserves the inner product, since

$$\langle C(z), C(w) \rangle_2 = \langle F_{f_z}, F_{f_w} \rangle_2 = \langle f_w, f_z \rangle_1 = \langle z, w \rangle .$$

Since $C$ is an isomorphism that preserves the inner product, we have that $H \cong H''$. ☐

**Problem 9** (Problem 4, Chapter 3.9, p. 200 in Kreyszig)**.** Let $H_1$ and $H_2$ be Hilbert spaces and $T : H_1 \longrightarrow H_2$ be a bounded linear operator. If $M_1 \subset H_1$ and $M_2 \subset H_2$ are such that $T(M_1) \subset M_2$, show that $T^*({M_2}^\perp) \subset {M_1}^\perp$.

**Solution.** Let $x \in T^*({M_2}^\perp)$. Then there exists a $y \in {M_2}^\perp$ such that $T^*y = x$. For all $z \in M_1$,

$$\begin{aligned} \langle x, z \rangle &= \langle T^*y, z \rangle \\ &= \langle y, Tz \rangle \\ &= 0 \end{aligned}$$

since $Tz \in M_2$ and $y \in {M_2}^\perp$. Hence $x \in {M_1}^\perp$.

**Problem 10** (Problem 6, Chapter 3.9, p. 200 in Kreyszig)**.** If $M_1 = \mathcal{N}(T) = \{x \,|\, Tx = 0\}$, where $T$ is the linear operator in the previous problem, show that

(a) $T^*(H_2) \subset {M_1}^{\perp}$

(b) $(T(H_1))^{\perp} \subset \mathcal{N}(T^*)$

(c) $M_1 = (T^*(H_2))^{\perp}$.

**Solution.** .

(a) This follows from the previous problem. Indeed, we have $M_1 \subset H_1$ and let $M_2 = H_2 \subset H_2$. Then $T(M_1) = \{0\} \subset M_2$. By the previous problem, $T^*(M_2) \subset {M_1}^{\perp}$ as desired.

(b) Let $x \in (T(H_1))^{\perp}$. Then $\langle x, y \rangle = 0$ for all $y \in T(H_1)$ and thus $\langle x, Tz \rangle = 0$ for all $z \in H_1$. And thus

$$\langle T^*x, z \rangle = \langle x, Tz \rangle = 0$$

for all $z$ in $H_1$. This can only be true if $T^*x = 0$, hence $x \in \mathcal{N}(T^*)$.

(c) First note that, for any subsets $M \subset N$, we have $N^{\perp} \subset M^{\perp}$. From part (a), we have $T^*(H_2) \subset M^{\perp}$, so we have $M^{\perp\perp} = \left(M^{\perp}\right)^{\perp} \subset (T^*(H_2))^{\perp}$. Furthermore, we know that $M_1 \subset {M_1}^{\perp\perp}$, and thus we have

$$M_1 \subset M^{\perp\perp} \subset (T^*(H_2))^{\perp}.$$

Finally, let $x \in (T^*(H_2))^{\perp}$. Then $\langle x, T^*z \rangle = 0$ for all $z \in H_2$. Hence

$$\langle Tx, z \rangle = \langle x, T^*z \rangle = 0$$

for all $z \in H_2$. This is true if and only if $Tx = 0$ and thus $x \in \mathcal{N}(T) = M_1$. Hence $(T^*(H_2))^{\perp} \subset M_1$ and thus $M_1 = (T^*(H_2))^{\perp}$

**Problem 11** (Problem 8, Chapter 3.9, p. 201 in Kreyszig)**.** Let $S = I + T^*T : H \longrightarrow H$, where $T$ is linear and bounded. Show that $S^{-1} : S(H) \longrightarrow H$ exists.

**Solution.** Recall that $S^{-1}$ exists if and only if $Sx = 0$ implies $x = 0$. Let $x \in H$ and suppose that $Sx = 0$. Then $Sx = (I+T^*T)x = x+T^*Tx = 0$, and in particular this means that the inner product $\langle Sx, x\rangle$ vanishes. Then

$$\begin{aligned} 0 &= \langle Sx, x\rangle \\ &= \langle x + T^*Tx, x\rangle \\ &= \langle x, x\rangle + \langle T^*Tx, x\rangle \\ &= ||x||^2 + \langle Tx, Tx\rangle \\ &= ||x||^2 + ||Tx||^2. \end{aligned}$$

Since both terms in the last line are non-negative, this can only vanish if $||x||^2 = 0$ and thus $x = 0$.

**Problem 12** (Problem 6, Chapter 3.10, p. 207 in Kreyszig)**.** If $T : H \longrightarrow H$ is a bounded self-adjoint linear operator and $T \neq 0$, show that

(a) $T^n \neq 0$ for every $n = 2, 4, 8, 16, \ldots$

(b) $T^n \neq 0$ for every $n \in \mathbb{N}$.

**Solution.** Note that, since $T \neq 0$, there exists an $x \in H$ such that $||Tx||^2 = \langle Tx, Tx \rangle \neq 0$.

(a) The proof follows by induction. As a base case, we have that $T = T^{2^0} \neq 0$. Assume that $T^{2^n} \neq 0$ for some $n \in \mathbb{N} \cup \{0\}$ and suppose that $T^{2^{n+1}} = 0$. Then $T^{2^{n+1}} x = 0$ for all $x \in H$. Therefore $\left\langle T^{2^{n+1}} x, x \right\rangle = \langle 0, x \rangle$ vanishes for all $x \in H$ and thus

$$\begin{aligned} 0 &= \left\langle T^{2^{n+1}} x, x \right\rangle \\ &= \left\langle T^{2^n} x, T^{2^n} x \right\rangle \\ &= ||T^{2^n} x||^2, \end{aligned}$$

so $T^{2^n} x = 0$ for all $x$, a contradiction to $T^{2^n} \neq 0$. So $T^{2^{n+1}} \neq 0$. Hence, by induction, $T^{2^n} \neq 0$ for all $n \in \mathbb{N} \cup \{0\}$.

(b) Suppose $T^N = 0$ for some $N \in \mathbb{N}$. Then $T^m = 0$ for all $m \geq N$. But there exists an $n \in \mathbb{N}$ such that $2^n > N$, and by part (a) we have that $T^{2^n} \neq 0$, a contradiction.