Assignment 3

AMAT 617

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Problem 1 (Problem 4, Chapter 4.1, p. 212)**.** Find all maximal elements of M with respect to the partial ordering $m \leq n$ whenever m divides n, where M is

(a) $\{2, 3, 4, 8\}$

(b) the set of all prime numbers.

Solution. .

(a) The maximal elements of $M = \{2, 3, 4, 8\}$ are 3 and 8. Indeed, 3 does not divide any of the other elements and 2 and 4 both divide 8, wheras 8 does not divide any of the other elements.

(b) Every element of $M = \{p \,|\, p \text{ prime}\}$ is maximal, since no prime number divides any other prime number.

Problem 2 (Problem 8, Chapter 4.2, p. 218)**.** If a subadditive functional defined on a normed space X is nonnegative outside of a sphere $\{x \mid \|x\| = r\}$, show that it is nonnegative for all $x \in X$.

Solution. Let p be a subadditive functional on X such that $p(x)$ is nonnegative for all $x \in X$ with $\|x\| > r$. Note that $p(y) = p(y+0) \le p(y) + p(0)$ for all $y \in X$, and thus

$$0 = p(y) - p(y) \le p(0).$$

Hence $0 \le p(0)$ and thus $p(0)$ is nonnegative. Let $x \in X$ such that $x \neq 0$. If $\|x\| > r$ then $p(x)$ is nonnegative by assumption, so suppose $\|x\| \le r$. Then there is an $n \in \mathbb{N}$ such that $n > \frac{r}{\|x\|}$ and thus $\|nx\| > r$. Then

$$p(\underbrace{x + \cdots + x}_{n \text{ times}}) \le \underbrace{p(x) + \cdots + p(x)}_{n \text{ times}} = np(x),$$

and thus $\frac{1}{n}p(nx) \le p(x)$. But $p(nx) \ge 0$, so $p(x) \ge 0$ as well.

Problem 3 (Problem 8, Chapter 4.3, p. 224)**.** Let X be a nomed space and X' its dual space. If $X \neq \{0\}$, show that X' cannot be $\{0\}$.

Solution. Let $x \in X$ such that $x \neq 0$. By the Hahn-Banach Theorem, there exists a functional $f \in X'$ such that $f(x) = \|x\|$ and $\|f\| = 1$. Hence $X' \neq \{0\}$.

Problem 4 (Problem 8, Chapter 4.5, p. 238)**.** Let X and Y be normed spaces and $T \in \mathcal{B}(X,Y)$ such that $T^{-1} \in \mathcal{B}(Y,X)$ exists. Show that $(T^\times)^{-1} = \left(T^{-1}\right)^\times$.

Solution. First note that $(\mathbb{1}_X)^\times = \mathbb{1}_{X'}$ for any normed space X, where $\mathbb{1}_X$ is the identity mapping on X. Indeed, for any $f \in X'$ and $x \in X$ we have

$$\left((\mathbb{1}_X)^\times f\right)(x) = f(\mathbb{1}_X x) = f(x)$$

and thus $(\mathbb{1}_X)^\times f = f$ for all X'.

Recall that, for operators S and T, we have $(ST)^\times = T^\times S^\times$. Take $S = T^{-1}$, then

$$\mathbb{1}_{X'} = (\mathbb{1}_X)^\times = (T^{-1}T)^\times = T^\times (T^{-1})^\times,$$

so $(T^{-1})^\times$ is the right-inverse of $T^\times$. Similarly,

$$\mathbb{1}_{Y'} = (\mathbb{1}_Y)^\times = (TT^{-1})^\times = (T^{-1})^\times T^\times,$$

so $(T^{-1})^\times$ is also the left-inverse of $T^\times$. Hence, $T^\times$ is invertible and $(T^\times)^{-1} = \left(T^{-1}\right)^\times$.

Problem 5 (Problem 10, Chapter 4.5, p. 239 – **Annihilator**). Let B be a subset of the dual space X' of a nomed space X. The *annihilator* ${}^aB$ of B is defined to be

$$ {}^aB = \{x \in X \mid f(x) = 0 \text{ for all } f \in B\} . $$

Let $T : X \longrightarrow Y$ be a bounded linear operator. Show that $\mathcal{R}(T) \subset {}^a\mathcal{N}(T^\times)$. What does this mean with respect to the task of solving an equation $Tx = y$?

Solution. Let $f \in \mathcal{N}(T^\times)$, then $T^\times f = 0$. That is,

$$ (T^\times f)(x) = f(Tx) = 0 \qquad \text{for all } x \in X. $$

Since this holds for all $f \in \mathcal{N}(T^\times)$, we have $Tx \in {}^a\mathcal{N}(T^\times)$ for all $x \in X$ and thus $\mathcal{R}(T) \subset {}^a\mathcal{N}(T^\times)$.

This means that if there exists a functional $f \in \mathcal{N}(T^\times)$ such that $f(y) \neq 0$, then $y \notin {}^a\mathcal{N}(T^\times)$ and thus y is not in the range of T. Hence $Tx = y$ has no solution. This is equivalent to:

if $f(y) \neq 0$ for some functional $f \in Y'$
such that $f(Tx) = 0$ for all $x \in X$,
then $Tx = y$ has no solution.

Problem 6 (Problem 4, Chapter 4.6, p. 246)**.** Show that a Banach space X is reflexive if and only if its dual space X' is reflexive. (Hint: Show that a closed subspace of a reflexive Banach space is reflexive.)

Solution.

Lemma 1. *Any closed subspace of a reflexive Banach space is reflexive.*

Proof. Let X be a reflexive Banach space and $Y \subset X$ be a closed subset. Let $\varphi \in Y''$ and define $\tilde{\varphi} \in X''$ by

$$\tilde{\varphi}(f) = \varphi(f|_Y) \qquad \text{for all } f \in X'.$$

Since X is reflexive, $\tilde{\varphi}$ is of the form $\tilde{\varphi} = \psi_x$ for some $x \in X$ where

$$\psi_x(f) = f(x) \qquad \text{for all } f \in X'.$$

So $\varphi(f|_Y) = f(x)$. We claim that $x \in Y$. Indeed, otherwise $x \notin Y$ and there exists a bounded linear functional on X such that $f|_Y = 0$ and $f(x) \neq 0$ (see Lemma 4.6-7 in Kreyszig). But this is a contradiction to the fact that

$$f(x) = \varphi(f|_Y) = \varphi(0) = 0,$$

since $f|_Y = 0$. By the Hahn-Banach Theorem, every linear functional g on $Y \subset X$ can be written as $g = f|_Y$ for some $f \in X'$, and thus

$$\varphi(g) = \varphi(f|_Y) = \tilde{\varphi}(f) = h_x(f) = f(x) = f|_Y(x) = g(x)$$

where $x \in Y$. Hence for each $\varphi \in Y''$ there is an $x \in Y$ such that $\varphi(g) = g(x)$, and thus Y is reflexive. ☐

Proposition 2. *A Banach space X is reflexive if and only if its dual space X' is reflexive.*

Proof. Assume that X is reflexive and let $C_1 : X \longrightarrow X''$ be the canonical isometry. Consider $X''' = (X'')'$ and let $C_2 : X' \longrightarrow X'''$ be the canonical embedding that maps $f \in X'$ to functional $\xi_f \in X'''$ such that $\xi_f(\varphi) = \varphi(f)$ for all $\varphi \in X''$. Let $\xi \in X'''$ and define a functional $f_\xi \in X'$ by

$$f_\xi(x) = \xi(C_1(x)).$$

Since X is reflexive, for each $\varphi \in X''$ there is an $x \in X$ such that $\varphi = C_1(x)$. Hence

$$(C_2(f_\xi))(\varphi) = \varphi(f_\xi) = f_\xi(x) = \xi(C_1(x)) = \xi(\varphi)$$

and thus $\xi = C_2(f_\xi)$. So the embedding C_2 is surjective and is therefore an isometry, so X' is reflexive.

Now suppose that X' is reflexive. By the argument above, we have that X'' is reflexive as well. Consider $\mathcal{R}(C_1) \subset X''$, which is isometric to X. Since $\mathcal{R}(C_1)$ is closed in X'', by the lemma above we have that $\mathcal{R}(C_1)$ and thus X are also reflexive. ☐

Problem 7 (Problem 6, Chapter 4.6, p. 246)**.** Show that different closed subspaces Y_1 and Y_2 of a normed space X have different annihilators.

Solution. *Proof.* Let $Y_1, Y_2 \subset X$ be two closed subspaces of X with $Y_1 \neq Y_2$. Without loss of generality, we may assume that $Y_1 \setminus Y_2 \neq \emptyset$, then let $y \in Y_1 \setminus Y_2$. Since $y \notin Y_2$, by the Hahn-Banach Theorem (see Lemma 4.6-7) there exists a functional $f \in X'$ such that $f(y) \neq 0$ and $f|_{Y_2} = 0$. Thus $f \in (Y_2)^a$ but $f \notin (Y_1)^a$, hence $(Y_1)^a \neq (Y_2)^a$. □

Problem 8 (Problem 8, Chapter 4.6, p. 246)**.** Let M be any subset of a normed space X. Show that an $x_0 \in X$ is an element of $A = \overline{\operatorname{span} M}$ if and only if $f(x_0) = 0$ for every $f \in X'$ such that $f|_M = 0$.

Solution. Using the 'annihilator' notation, this is equivalent to saying that $\overline{\operatorname{span} M} = {}^a(M_a)$.

Proof. Now suppose that $x_0 \notin A = \overline{\operatorname{span} M}$. By the Hahn-Banach Theorem (see Lemma 4.6-7), there is a linear functional $f \in X'$ such that $f(x_0) \neq 0$ and $f|_A = 0$. But M is a subset of A, so $f|_M = 0$ with $f(x_0) \neq 0$ and thus $x_0 \notin {}^a(M_a)$. $\square$

Problem 9 (Problem 4, Chapter 4.7, p. 254)**.** Find a meager dense subset in $\mathbb{R}^2$.

Solution. Note that $\mathbb{Q}$ is meager and dense in $\mathbb{R}$. Futhermore, $\mathbb{Q}$ is countable and we may enumerate it as $\mathbb{Q} = \{q_1, q_2, q_3, \dots\}$. Consider the subset $M \subset \mathbb{R}^2$ defined by

$$M = \bigcup_{i=1}^{\infty} M_i, \qquad \text{where} \quad M_i = \{(q_i, x) \,|\, x \in \mathbb{R}\}\,.$$

That is, M consists of the union of parallel lines intersecting the x-axis at each rational point. Then M is clearly dense in $\mathbb{R}^2$, and each M_i is rare is $\mathbb{R}^2$. Since M is a countable union of rare sets, it is meager in $\mathbb{R}^2$.

Problem 10 (Problem 6, Chapter 4.7, p. 255)**.** Show that the complement M^c of a meager subset M of a complete metric space X is nonmeager.

Solution. If $X = \emptyset$ then this is true vacuously. Let $X \neq \emptyset$ be a complete metric space and let $M \subset X$ be a meager subset. Note that we may write X as a union

$$X = M \cup M^c.$$

If M^c were meager, then it may be written as a countable union of rare sets. Hence X is a countable union of rare sets (since it is a union of two sets that are countable unions of rare sets), and thus X is meager in itself. But this is a contradiction to Baire's Category Theorem, which states that a nonempty complete normed space must be non-meager in itself.

Problem 11 (Problem 8, Chapter 4.7, p. 255)**.** Show that completeness of X is essential in the statement of the Uniform Boundedness Theorem. (Hint: Consider $X \subset \ell^\infty$ consisting of all finite sequences $x = \{x_j\}$, that is, x_j nonzero for only finitely many j, and consider the operator $T_n x = f_n x = n x_n$.)

Solution. Recall the statement of the theorem.

Theorem 3 (Uniform Boundedness Theorem)**.** *Let $\{T_n\}$ be a sequence of bounded linear operators $T_n : X \longrightarrow Y$ from a Banach space X into a normed space Y such that $\{\|T_n x\|\}$ is bounded for every $x \in X$. That is, for every x there exists a c_x such that $\|T_n x\| \leq c_x$ for all n. Then the sequence of the norms $\|T_n\|$ is bounded.*

As in the problem statement, let $X \subset \ell^\infty$ be the subspace consisting of finite sequences. Then X is not complete. For each n let $T_n = f_n$ be the linear functional defined by $f_n(x) = n x_n$. Let $x \in X$, then $x = \{x_j\}$ and there exists an $N \in \mathbb{N}$ such that $x_j = 0$ for all $j > N$. Then

$$\|T_n x\| = |f_n(x)| = |n x_n| = n|x_n| \leq n \, \|x\|$$

where $c_x = \|x\|$ depends on the choice of x, and thus $\{\|T_n x\|\}$ is bounded for each $x \in X$. However, $\|T_n\| = n$ for each n. Indeed, from the above analysis we see that $\|T_n\| \leq n$, and choosing $x = \{\underbrace{1, 1, \ldots, 1}_{n}, 0, 0, \cdots\}$ with $\|x\| = 1$ we have $\|T_n x\| = n$ so $\|T_n\| \geq n$ for each n. Thus $\{\|T_n\|\}$ is not bounded.

Problem 12 (Problem 12, Chapter 4.7, p. 255)**.** Let X and Y be Banach spaces and $T_n \in \mathcal{B}(X,Y)$ such that $\{T_n x\}$ is Cauchy in Y for every $x \in X$. Show that $T_n x \longrightarrow Tx$ for some $T \in \mathcal{B}(X,Y)$.

Solution. Since every Cauchy sequence is bounded, we have that $\{\|T_n x\|\}$ is bounded for all $x \in X$ and thus $\{\|T_n\|\}$ is bounded by the Uniform Boundedness Theorem. Thus $\sup\limits_{n\in\mathbb{N}} \|T_n\|$ is finite.

Since Y is complete and the sequence $\{T_n x\}$ is Cauchy in Y for each $x \in X$, each $\{T_n x\}$ converges to an element in Y. Define a mapping $T : X \longrightarrow Y$ by

$$Tx = \lim_{n\to\infty} T_n x.$$

This is clearly linear, since by the linearity of T_n we have

$$T(ax+by) = \lim_{n\to\infty} T_n(ax+by) = \lim_{n\to\infty} (aT_n x + bT_n y) = aTx + bTy$$

for all $x, y \in X$ and a, b in the field $\mathbb{F}$. This operator is also bounded, since for all $x \in X$ we have

$$\begin{aligned} \|Tx\| = \left\|\lim_{n\to\infty} T_n x\right\| &= \lim_{n\to\infty} \|T_n x\| \\ &\le \sup_{n\in\mathbb{N}} \|T_n\| \, \|x\| \, , \end{aligned}$$

where we use the fact that the norm is continuous, so it commutes with limits. Hence $\frac{\|Tx\|}{\|x\|} \le \sup\limits_{n\in\mathbb{N}} \|T_n\| < \infty$ for all $x \ne 0$, but this is finite and thus $\|T\| \le \sup\limits_{n\in\mathbb{N}} \|T_n\| < \infty$, so T is also bounded and thus $T \in \mathcal{B}(X,Y)$ as desired.

Problem 13 (Problem 14, Chapter 4.7, p. 255)**.** If X and Y are Banach spaces and $T_n \in \mathcal{B}(X, Y)$ for each n, show that the following statements are equivalent:

(a) $\{\|T_n\|\}$ is bounded,

(b) $\{\|T_n x\|\}$ is bounded for all $x \in X$,

(c) $\{|g(T_n x)|\}$ is bounded for all $x \in X$ and $g \in Y'$.

Solution. We first prove the following lemma.

Lemma 4. *If $\{x_n\}$ is a sequence in a Banach space X such that $\{f(x_n)\}$ is bounded for each functional $f \in Y'$, then $\{\|x_n\|\}$ is bounded.*

Proof. For each $f \in X'$, define the sequence $\{\varphi_n\}$ in X'' by $\varphi_n(f) = f(x_n)$. Then the sequence $\{\varphi_n(f)\}$ is bounded for each $f \in X'$ by assumption, and hence $\{\|\varphi_n(f)\|\}$ is bounded for each f. By the Uniform Boundedness Theorem, we have that $\{\|\varphi_n\|\}$ is also bounded, but $\|\varphi_n\| = \|x_n\|$ for each n and thus $\{\|x_n\|\}$ is bounded. □

We now proceed to prove the problem statement.

- $(a) \Rightarrow (b)$. Since $\{\|T_n\|\}$ is bounded, there is a constant $c \geq 0$ such that $\|T_n\| \leq c$ for all n. Define the constant $c_x = c\,\|x\|$ for each $x \in X$. Then

$$\|T_n x\| \leq \|T_n\|\,\|x\| \leq c_x,$$

  and thus $\{\|T_n x\|\}$ is bounded for each $x \in X$.

- $(b) \Rightarrow (a)$. This follows from the Uniform Boundedness Theorem.

- $(a) \Rightarrow (c)$. Let $g \in Y'$ and define the functionals $f_n \in X'$ for each n by $f_n(x) = g(T_n x)$. Since $\{\|T_n\|\}$ is bounded, there is a constant c such that $\|T_n\| \leq c$ for all n. Then we have

$$\begin{aligned} |f_n(x)| = |g(T_n x)| &\leq \|g\|\,\|T_n\|\,\|x\| \\ &\leq \|g\|\,c\,\|x\| \end{aligned}$$

  for each $x \in X$ and thus $\|f_n\| \leq c\,\|g\|$, so the sequence $\{\|f_n\|\}$ is bounded. From the $(a) \Rightarrow (b)$ part of this problem, this implies that $\{|f_n(x)|\} = \{|g(T_n x)|\}$ is bounded for all x.

- $(c) \Rightarrow (b)$. Let $x \in X$ and define the sequence $\{y_n\}$ in Y by $y_n = T_n x$. Then the sequence $\{g(y_n)\}$ is bounded for each $g \in Y'$. Since Y is a Banach space, Lemma 4 from above says that $\{\|y_n\|\}$ is bounded, and thus $\{\|T_n x\|\}$ is bounded.

Problem 14 (Problem 2, Chapter 4.8, p. 262)**.** Let X and Y be normed spaces, $T \in \mathcal{B}(X,Y)$ and $\{x_n\}$ a sequence in X. If $x_n \xrightarrow{w} x$, show that $Tx_n \xrightarrow{w} Tx$.

Solution. Let $g \in Y'$ and define a functional $f \in X'$ by $f(x) = g(Tx)$. Then $g(Tx_n) = f(x_n)$ and thus $g(Tx_n) \longrightarrow g(Tx)$ if and only if $f(x_n) \longrightarrow f(x)$, which is true since $x_n \xrightarrow{w} x$. Hence $g(Tx_n) \longrightarrow g(Tx)$ for all $g \in Y'$ and thus $Tx_n \xrightarrow{w} Tx$.

Problem 15 (Problem 4, Chapter 4.8, p. 262)**.** Show that $x_n \xrightarrow{w} x$ implies $\liminf_{n\to\infty} \|x_n\| \geq \|x\|$.

Solution. By the Hahn-Banach theorem, there exists a functional $f \in X'$ such that $\|f\| = 1$ and $f(x) = \|x\|$. Since x_n weakly converges to x, we have that $f(x_n) \longrightarrow f(x)$, hence $|f(x_n)| \longrightarrow |f(x)| = \|x\|$. Thus

$$\begin{aligned} \|x\| = |f(x)| &= \lim_{n\to\infty} |f(x_n)| \\ &= \liminf_{n\to\infty} |f(x_n)| \\ &\leq \liminf_{n\to\infty} \underbrace{\|f\|}_{=1} \|x_n\| \\ &= \liminf_{n\to\infty} \|x_n\|, \end{aligned}$$

as desired.

Problem 16 (Problem 6, Chapter 4.8, p. 262)**.** If $\{x_n\}$ is a weakly convergent sequence in a normed space X, say $x_n \xrightarrow{w} x$, show that there is a sequence $\{y_m\}$ of linear combinations of elements of $\{x_n\}$ which converges strongly to x.

Solution. We first prove the following lemma (see problem 4.8.1 in Kreeyszig).

Lemma 5. *If $x_n \xrightarrow{w} x$ in a normed space X, then $x \in \overline{Y}$ where $Y = \operatorname{span}\{x_n\}$.*

Proof. Suppose otherwise that $x \notin \overline{Y}$. Then the distance

$$\delta = \inf_{y \in \overline{Y}} \|x - y\|$$

from x to $\overline{Y}$ is positive. By the theorem in class (Theorem 4.6-7 in Kreyszig), there exists a functional $f \in X'$ such that $\|f\| = 1$, $f(x) = \delta$ and $f(y) = 0$ for all $y \in Y$. Hence $f(x_n) = 0$ for each n, and thus $f(x_n)$ does not converge to $f(x) = \delta$, a contradiction to $x_n \xrightarrow{w} x$. □

The proof of the problem statement follows trivially from this lemma. Indeed, since $x \in \overline{Y}$, there is a sequence $\{y_m\}$ in Y such that $y_m \longrightarrow x$. But $Y = \operatorname{span}\{x_n\}$, and thus each y_m is a linear combination of elements in $\{x_n\}$.

Problem 17 (Problem 8, Chapter 4.8, p. 262 – **Weak Cauchy sequence**). A *weak Cauchy sequence* in a real or complex normed space X is a sequence $\{x_n\}$ in X such that for every $f \in X'$ the sequence $\{f(x_n)\}$ is Cauchy in $\mathbb{R}$ or $\mathbb{C}$, respectively. Show that every weak Cauchy sequence is bounded.

Solution. For clarity, denote $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Define the sequence $\{\varphi_n\}$ in X'' such that $\varphi_n(f) = f(x_n)$ for all $f \in X'$. Let $f \in X'$ be arbitrary. Since $\{x_n\}$ is weak Cauchy, we have that $\{f(x_n)\}$ is Cauchy in $\mathbb{F}$ and thus the sequence $\{|f(x_n)|\}$ is bounded. Hence $\{|\varphi_n(f)|\}$ is bounded for each $f \in X'$. But X' is Banach, so we have that $\{\|\varphi_n\|\}$ is bounded by the Uniform Boundedness Theorem. Then $\{x_n\}$ is bounded since $\|x_n\| = \|\varphi_n\|$.

Problem 18 (Problem 10, Chapter 4.8, p. 263 – **Weak completeness**). A normed space X is said to be *weakly complete* if each Cauchy sequence in X converges weakly in X. If X is reflexive, show that X is weakly complete.

Solution. Let $\{x_n\}$ be a Cauchy sequence in X. For each $x_n \in X$ let $\varphi_n \in X''$ be the functional defined by $\varphi_n(f) = f(x_n)$ for all functionals $f \in X'$. Since X is reflexive, we have $X \cong X''$ so $\{\varphi_n\}$ is a Cauchy sequence in X''. Let $\varepsilon > 0$, then there exists an $N \in \mathbb{N}$ such that $\|\varphi_n - \varphi_m\| \leq \frac{\varepsilon}{\|f\|}$ for all $n, m \geq N$ since $\{\varphi_n\}$ is Cauchy. Then for each $n, m \geq N$ we have

$$\begin{aligned}
|f(x_n) - f(x_m)| &= |\varphi_n(f) - \varphi_m(f)| \\
&\leq \|\varphi_n - \varphi_m\| \, \|f\| \\
&< \varepsilon,
\end{aligned}$$

so the sequence $\{f(x_n)\}$ is Cauchy in $\mathbb{R}$ and thus $\{x_n\}$ is weakly convergent. Then X is weakly complete, since every Cauchy sequence in X is weakly convergent.