Assignment 4

AMAT 617

Mark Girard

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Problem 1 (Problem 6, Chapter 4.9, p. 269)**.** Let $T_n \in \mathcal{B}(X,Y)$, where $n = 1, 2, \ldots$. To motivate the term "uniform" in Definition 4.9-1 in Kreyszig, show that $T_n \longrightarrow T$ if and only if for every $\varepsilon > 0$ there is an N, depending only on ε, such that for all $n > N$ and all $x \in X$ of norm 1 we have

$$\|T_n x - Tx\| < \varepsilon.$$

Solution. *Proof.* By definition, $T_n \longrightarrow T$ implies that $\|T_n - T\| \longrightarrow 0$. Let $\varepsilon > 0$, then uniform convergence of T_n implies that there exists an $N \in \mathbb{N}$ such that $\|T_n - T\| < \varepsilon$ for all $n \geq N$. For all $x \in X$ with $\|x\| = 1$, we have

$$\begin{aligned} \|T_n x - Tx\| &= \|(T_n - T)x\| \leq \|T_n - T\| \underbrace{\|x\|}_{1} \\ &= \|T_n - T\| \\ &< \varepsilon \end{aligned}$$

as desired.

For the other direction, let $\varepsilon > 0$. By hypothesis, there is an $N_\varepsilon \in \mathbb{N}$ such that $\|T_n x - Tx\| < \varepsilon$ for all $n > N_\varepsilon$ and all $x \in X$ with $\|x\| = 1$. From the definition of the operator norm, we have

$$\|T_n - T\| = \sup_{\substack{x \in X \\ \|x\| = 1}} \|T_n x - Tx\| \leq \varepsilon$$

and thus $T_n \longrightarrow T$ as desired. □

Problem 2 (Problem 8, Chapter 4.9, p. 269)**.** Let $T_n \longrightarrow T$, where $T_n \in \mathcal{B}(X, Y)$. Show that for every $\varepsilon > 0$ and every closed ball $K \subset X$ there is an N such that $\|T_n x - Tx\| < \varepsilon$ for all $n > N$ and $x \in K$.

Solution. *Proof.* Let $K \subset X$ be a closed ball. Then there is a $z \in X$ and $\delta > 0$ such that z is the centre of the ball and δ is the radius. That is,

$$\begin{aligned} K &= \overline{B(z; \delta)} \\ &= \{x \in X \mid \|x - z\| \leq \delta\} . \end{aligned}$$

Let $\varepsilon > 0$. Recall that $T_n \longrightarrow T$ means that $\|T_n - T\| \longrightarrow 0$. Then there is an $N \in \mathbb{N}$ such that

$$\|T_n - T\| < \frac{\varepsilon}{\|z\| + \delta}$$

for all $n > N$. Note that, for all $x \in K$, we have

$$\begin{aligned} \|x\| &= \|x - z + z\| \\ &\leq \|x - z\| + \|z\| \\ &\leq \delta + \|z\| . \end{aligned}$$

Hence, for all $n > N$, we have that

$$\begin{aligned} \|T_n x - Tx\| &\leq \|T_n - T\| \underbrace{\|x\|}_{\leq \|z\| + \delta} \\ &< \frac{\varepsilon}{\|z\| + \delta} (\|z\| + \delta) \\ &< \varepsilon \end{aligned}$$

as desired.

$\square$

Problem 3 (Problem 10, Chapter 4.9, p. 269)**.** Let X be a separable Banach space and $M \subset X'$ a bounded set. Show that every sequence of elements of M contains a subsequence which is weak* convergent to an element of X'.

Solution. Without loss of generality, we may assume that X is a complex Banach space.

Proof. By the theorem in class (see Theorem 4.9-6 in Kreyszig), we have that a sequence of functionals $\{f_n\}$ in X' is weak* convergent to an element of X' if and only if $\{\|f_n\|\}$ is bounded and the sequence $\{f_n(x)\}$ is Cauchy for every x in a total subset of X.

Since X is separable, there is a sequence $S = \{x_k\}$ that is dense in X. Note that the set S is also total in X. Let $\sigma = \{f_n\}$ be an arbitrary sequence of functionals in M. Since this is bouded, the sequence $\{f_n(x_k)\}$ in $\mathbb{C}$ is bounded for each k. So, by the Bolzano-Weierstrass theorem, there is a convergent subsequence.

Consider a partial ordering of sequences of functionals in X' defined in the following manner. If $\rho = \{g_n\}$ and $\tau = \{h_n\}$ are sequences of functionals with $g_n, h_n \in X'$ for each n, we say

$$\rho \leq \tau \qquad \Longleftrightarrow \qquad \rho \text{ is a subsequence of } \tau.$$

Construct a sequence of subsequences of σ in the following manner. Let $\sigma^{(1)} \leq \sigma$ where $\sigma^{(1)} = \{f_n^{(1)}\}$ is a subsequence of $\sigma = \{f_n\}$ such that $\{f_n^{(1)}(x_1)\}$ is convergent in $\mathbb{C}$. Continue by inductively defining the subsequences

$$\cdots \leq \sigma^{(3)} \leq \sigma^{(2)} \leq \sigma^{(1)} \leq \sigma$$

such that $\sigma^{(k)} = \{f_n^{(k)}\}$ and $\{f_n^{(k)}(x_k)\}$ converges in $\mathbb{C}$ for each k. Finally, use the "Cantor diagonal method" to construct another subsequence of σ in the following manner. Define the sequence $\tau = \{g_j\}$ where

$$g_j = f_k^{(j)}.$$

This is clearly a subsequence of $\sigma = \{f_n\}$. Furthermore, it is also 'eventually' a subsequence of each $\sigma^{(k)} = \{f_n^{(k)}\}$ in the sense that

$$\{g_n\}_{n=k}^{\infty} \qquad \text{is a subsequence of the shifted sequence} \qquad \{f_n^{(k)}\}_{n=k}^{\infty}$$

for each k. Note that each subsequence of a convergent sequence of complex numbers is also convergent to the same limit, hence $\{g_n(x_k)\}$ is convergent in $\mathbb{C}$ for each k. So we have constructed a subsequence $\{g_n\}$ of $\{f_n\}$ that is weak* convergent, as desired. □

Problem 4 (Problem 2, Chapter 4.12, p. 290)**.** Show that an open mapping need not map closed sets onto closed sets.

Solution. *Proof.* Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the projection mapping onto the first coordinate, i.e.

$$f(x, y) = x \qquad \text{for all } (x, y) \in \mathbb{R}^2.$$

This is clearly bounded and linear, and thus an open map by the Open Mapping Theorem. However, it is not closed. Indeed, consider the closed subset $M \subset \mathbb{R}^2$ defined by

$$M = \left\{ \left(x, \frac{1}{x} \right) \,\middle|\, x \neq 0 \right\}.$$

This is indeed closed, since it is the pre-image $M = \mu^{-1}\{0\}$ under the continuous multiplication map

$$\begin{aligned} \mu : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto xy, \end{aligned}$$

and $\{0\} \subset \mathbb{R}$ is closed (and the pre-image of a closed set under a continuous map is closed). However, we have that

$$f(M) = \{x \in \mathbb{R} \,|\, x \neq 0\}$$

which is not closed in $\mathbb{R}$ since 0 is a limit point. □

Problem 5 (Problem 6, Chapter 4.12, p. 290)**.** Let X and Y be Banach spaces and $T : X \longrightarrow Y$ be an injective bounded linear operator. Show that $T^{-1} : \mathcal{R}(T) \longrightarrow X$ is bounded if and only if $\mathcal{R}(T)$ is closed in Y.

Solution. *Proof.* If $\mathcal{R}(T)$ is closed in Y, then it is complete since Y is complete. By the Open Mapping Theorem, we have that the inverse is bounded.

Now suppoose that T^{-1} is bounded. We want to show that $\mathcal{R}(T)$ is closed, so let $\{y_n\}$ be a Cauchy sequence in $\mathcal{R}(T)$. Then there is a sequence $\{x_n\}$ in X such that $y_n = Tx_n$ and thus $x_n = T^{-1}y_n$. But $\{x_n\}$ is also Cauchy. Indeed, we have

$$\begin{aligned} \|x_n - x_m\| &= \left\|T^{-1}y_n - T^{-1}y_m\right\| \\ &= \left\|T^{-1}\right\| \|y_n - y_m\| \longrightarrow 0 \end{aligned}$$

since T^{-1} is bounded and $\{y_n\}$ is Cauchy. Since X is complete, we have that $x_n \longrightarrow x$ for some $x \in X$. Since T is bounded, we have that

$$Tx_n \longrightarrow Tx$$

and thus $y_n = Tx_n \longrightarrow Tx$. So the Cauchy sequence $\{y_n\}$ converges in $\mathcal{R}(T)$ and thus $\mathcal{R}(T)$ is closed. ☐

Problem 6 (Problem 8, Chapter 4.12, p. 291)**.** Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on a vector space X such that $X_1 = (X, \|\cdot\|_1)$ and $X_2 = (X, \|\cdot\|_2)$ are complete. If $\|x_n\|_1 \longrightarrow 0$ always implies $\|x_n\|_2 \longrightarrow 0$, show that convergence in X_1 implies convergence X_2 and conversely, and there are positive numbers a and b such that for all $x \in X$

$$a\,\|x\|_1 \leq \|x\|_2 \leq b\,\|x\|_1 .$$

Solution. *Proof.* Suppose that there is no number b such that $\|x\|_2 \leq b\,\|x\|_1$ for all $x \in X$. Then for each $n \in \mathbb{N}$ we can find an $x_n \in X$ such that $\|x_n\|_2 > n\,\|x_n\|_1$. Without loss of generality, we may assume that $\|x_n\|_2 = 1$, and thus

$$\|x_n\|_1 < \frac{1}{n}$$

for all n. Hence $\|x_n\|_1 \longrightarrow 0$, but $\|x_n\|_2 = 1$ and thus $\|x_n\|_1 \not\longrightarrow 0$, a contradiction to the assumption that $\|x_n\|_1 \longrightarrow 0$ always implies $\|x_n\|_2 \longrightarrow 0$.

Now consider the identity map $T : X_1 \longrightarrow X_2$, that is, $Tx = x$ for all $x \in X$. This is clearly bounded, since $\|Tx\|_2 = \|x\|_2 \leq b\,\|x\|_1$ for all $x \in X$. Since T is clearly invertible, namely $T^{-1}x = x$, the Bounded Inverse Theorem tells us that T^{-1} is also bounded. Hence we have

$$\|x\|_1 = \left\|T^{-1}x\right\|_1 \leq \left\|T^{-1}\right\| \|x\|_2 ,$$

and thus $\frac{1}{\|T^{-1}\|}\,\|x\|_1 \leq \|x\|_2$ for all $x \in X$, so we may chose $a = \frac{1}{\|T^{-1}\|}$.

Next we show that convergence in one implies convergence in the other. Let $\{x_n\}$ be a Cauchy sequence in X_1 such that $x_n \longrightarrow x$ for some $x \in X$. This implies that $\|x_n - x\|_1 \longrightarrow 0$. From the arguments above, we have that

$$\|x_n - x\|_2 \leq b\,\|x_n - x\|_1 \longrightarrow 0$$

so $\{x_n\}$ also converges to x in X_2. Similarly, if we have the convergent sequence $\|x_n - x\|_2 \longrightarrow 0$ in X_2, then

$$\|x_n - x\|_1 \leq \frac{1}{a}\,\|x_n - x\|_2 \longrightarrow 0,$$

so $\{x_n\}$ converges in X_1. □

Problem 7 (Problem 10, Chapter 4.12, p. 291)**.** Each norm on a vector space X defines a topology on X. If we have two norms on X such that $X_1 = (X, \|\cdot\|_1)$ and $X_2 = (X, \|\cdot\|_2)$ are Banach spaces and the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ defined by $\|\cdot\|_1$ and $\|\cdot\|_2$ satisfy $\mathcal{T}_1 \supseteq \mathcal{T}_2$, show that $\mathcal{T}_1 = \mathcal{T}_2$.

Solution. *Proof.* As in the previous problem, consider the identity map $T : X_1 \longrightarrow X_2$ that takes $x \overset{T}{\longmapsto} x$. Given an open set $U \in \mathcal{T}_2$, we have that $T^{-1}(U) = U$ is in $\mathcal{T}_1$. Hence T is continuous and thus bounded. By the Open Mapping Theorem, we have that T is also an open map, and thus $T(V) \in \mathcal{T}_2$ for each open set $V \in \mathcal{T}_1$, and thus $\mathcal{T}_1 = \mathcal{T}_2$. □

Problem 8 (Problem 6, Chapter 4.13, p. 296)**.** Let T be a closed linear operator. If two sequences $\{x_n\}$ and $\{\tilde{x}_n\}$ in $\mathcal{D}(T)$ converge with the same limit x and if $\{Tx_n\}$ and $\{T\tilde{x}_n\}$ both converge, show that $\{Tx_n\}$ and $\{T\tilde{x}_n\}$ have the same limit.

Solution. *Proof.* We have the linear operator $T : \mathcal{D}(T) \longrightarrow Y$ with $\mathcal{D}(T) \subset X$. From the theorem in class (see Theorem 4.13-3 in Kreyszig), we have the following: since T is closed, the sequence $\{x_n\}$ in $\mathcal{D}(T)$ converges to $x \in X$ and $Tx_n \longrightarrow y$ for some $y \in Y$, we have that $x \in \mathcal{D}(T)$ and $Tx = y$. Similarly, by convergence of $\tilde{x}_n \longrightarrow x$ and $T\tilde{x}_n \longrightarrow \tilde{y}$ for some $\tilde{y} \in Y$, we have $Tx = \tilde{y}$. Thus, we have $y = \tilde{y}$ as desired. □

Problem 9 (Problem 8, Chapter 4.13, p. 296)**.** Let X and Y be normed spaces and let $T : X \longrightarrow Y$ be a closed linear operator.

(a) Show that the image $T(C)$ of a compact subset $C \subset X$ is closed in Y.

(b) Show that the inverse image of $T^{-1}(K)$ of a compact subset $K \subset Y$ is closed in X.

Solution. Since this was not covered in the lecture, I'll include Kreyszig's definition of compact here.

Definition. Let X be a metric space. A subset $M \subset X$ is said to be (*sequentially*) *compact* if every sequence of elements in M has a convergent subsequence that converges to an element in M.

(a) Let $\{y_n\}$ be a Cauchy sequence in $T(C)$, such that $y_n \longrightarrow y$ for some $y \in Y$. Then for each n we have $y_n = Tx_n$ for some $x_n \in C$. Since C is compact, the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ such that $x_{n_k} \xrightarrow{k\to\infty} x$ for some $x \in C$. Then $\{Tx_{n_k}\} = \{y_{n_k}\}$ is a subsequence of $\{y_n\}$ and thus converges to the same limit, i.e.

$$Tx_{n_k} \xrightarrow{k\to\infty} y.$$

Since T is closed, by the theorem in class (see Theorem 4.13-3 in Kreyszig) we have that $Tx = y$. Since $x \in C$, we have that $y \in T(C)$ and thus every Cauchy sequence in $T(C)$ converges to something in $T(C)$. So $T(C)$ is closed, as desired.

(b) Let $\{x_n\}$ be a Cauchy sequence in $T^{-1}(K)$ such that $x_n \longrightarrow x$ for some $x \in X$. Since $x_n \in T^{-1}(K)$, we have that $y_n = Tx_n \in K$ for each n. Since K is compact, the sequence $\{y_n\}$ in K has a convergent subsequence $\{y_{n_k}\}$ that converges to some element

$$Tx_{n_k} = y_{n_k} \xrightarrow{k\to\infty} y$$

with $y \in K$. Since $x_{n_k} \xrightarrow{k\to\infty} x$ and $Tx_{n_k} \xrightarrow{k\to\infty} y$, by closedness of T we have that $Tx = y$. Then $x \in T^{-1}(K)$ since $y \in K$, and thus every Cauchy sequence in $T^{-1}(K)$ converges to something in $T^{-1}(K)$. Hence $T^{-1}(K)$ is closed, as desired.

Problem 10 (Problem 10, Chapter 4.13, p. 296)**.** Let X and Y be normed spaces and X compact. If $T : X \longrightarrow Y$ is a bijective closed linear operator, show that T^{-1} is bounded.

Solution. (I'm confused why we are assuming that X is compact... Isn't $\{0\}$ the only compact normed space? That must be a typo in the book. I'm just going to ignore this assumption, since it is not needed.)

Proof. Since T is closed, we have that $T(M)$ is closed in Y for each closed subset $M \subset X$. Since T is bijective, its inverse T^{-1} exists, and furthermore we have that $\left(T^{-1}\right)^{-1} = T$. Hence, for each closed set $M \in X$, we have that its preimage under T^{-1}, namely $\left(T^{-1}\right)^{-1}(M) = T(M)$, is closed in Y. This is the definition of continuity of T^{-1}, and thus T^{-1} is also bounded. □

Problem 11 (Problem 12, Chapter 4.13, p. 296)**.** Let X and Y be normed spaces. If $T_1 \colon X \longrightarrow Y$ is a closed linear operator and $T_2 \in \mathcal{B}(X, Y)$, show that $T_1 + T_2$ is a closed linear operator.

Solution. *Proof.* Let $x_n \longrightarrow x$ be a convergent sequence in X such that $(T_1 + T_2)x_n \longrightarrow y$ for some $y \in Y$. Since T_2 is bounded, we have that $x_n \longrightarrow x$ implies $T_2 x_n \longrightarrow T_2 x$. Therefore, we have that

$$T_1 x_n = \underbrace{(T_1 + T_2)x_n}_{\to y} - \underbrace{T_2 x_n}_{\to T_2 x} \longrightarrow y - T_2 x,$$

so $T_1 x = y - T_2 x$ since T_1 is closed. Hence

$$(T_1 + T_2)x = y,$$

and thus $T_1 + T_2$ is closed by Theorem 4.13-3 (in Kreyszig), since $x_n \longrightarrow x$ and $(T_1 + T_2)x_n \longrightarrow y$ implies $(T_1 + T_2)x = y$. $\square$

Problem 12 (Problem 14, Chapter 4.13, p. 297)**.** Assume that the terms of the series $u_1 + u_2 + \cdots$ are continuously differentiable on the interbal $[0, 1]$ and that the series is uniformly convergent on $[0, 1]$ and has the sum x. Furthermore, suppose that $u_1' + u_2' + \cdots$ also converges unigivenformly on $[0, 1]$. Show that then x is continuously differentiable on $[0, 1]$ and $x' = u_1' + u_2' + \cdots$.

Solution. *Proof.* Note that the differential operator

$$\begin{aligned} T : \mathcal{D}(T) &\longrightarrow X \\ x &\longmapsto x' \end{aligned}$$

where $X = \mathcal{C}[0, 1]$ and $\mathcal{D}(T) \subset X$ is the subspace of differentiable functions on $[0, 1]$, is a closed operator. Hence, if $x_n \longrightarrow x$ and $x_n' \longrightarrow y$ for some $y \in \mathcal{C}[0, 1]$, we have that $x \in \mathcal{D}(T)$ and $Tx = y$.

By the notation in the problem statement, we have $u_i : [0, 1] \longrightarrow \mathbb{R}$ is continuously differentiable for each i, and that $x : [0, 1] \longrightarrow \mathbb{R}$ is the function

$$x(t) = \sum_{i=1}^{\infty} u_i(t),$$

and this is uniformly convergent. Hence, we have that the partial sums

$$x_n(t) := \sum_{i=1}^{n} u_i(t)$$

converge to x, that is $x_n \longrightarrow x$. By hypothesis, we have that the partial sums of the derivatives, given by

$$x_n'(t) = \sum_{i=1}^{n} u_i'(t),$$

converges uniformly to some function $y \in \mathcal{C}[0, 1]$. That is, $x_n' = Tx_n \longrightarrow y$. Since the differential operator is closed, we have that

$$y = Tx = x',$$

as desired. □