Assignment 5

AMAT 617

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7 April 2014

Problem 1 (Problem 8, Chapter 7.2, p. 374)**.** If T is a bounded linear operator and T_1 is a linear extension of T, show that $\sigma_c(T) \subset \sigma_c(T_1) \cup \sigma_p(T_1)$.

Solution. *Proof.* Let $\lambda \in \sigma_c(T)$. If ${T_1}_\lambda^{-1}$ does not exist, then $\lambda \in \sigma_p(T_1)$, so suppose that ${T_1}_\lambda^{-1}$ exists. Note that $\mathcal{D}\big(T_\lambda^{-1}\big) = \mathcal{R}\big(T_\lambda\big)$ and $\mathcal{D}\big({T_1}_\lambda^{-1}\big) = \mathcal{R}\big(T_{1\lambda}\big)$. Furthermore, since T_1 is an extension of T, we have that $\mathcal{R}\big(T_\lambda\big) \subset \mathcal{R}\big(T_{1\lambda}\big)$. Hence $T_{1\lambda}$ is unbounded since T_λ is unbounded, and

$$X = \overline{\mathcal{D}\big(T_\lambda^{-1}\big)} \subset \overline{\mathcal{D}\big({T_1}_\lambda^{-1}\big)}$$

so $\mathcal{D}\big({T_1}_\lambda^{-1}\big)$ is dense in X, and thus $\lambda \in \sigma_c(T_1)$ as desired. □

Problem 2 (Problem 4, Chapter 7.3, p. 379)**.** Let $X = \ell^2$ and $T\colon X \longrightarrow X$ be defined by $y = Tx$, $x = (\xi_i)$ and $y = (\eta_i)$ where $\eta_i = \alpha_i \xi_i$ and (α_i) is dense in $[0,1]$. Find $\sigma_p(T)$ and $\sigma(T)$.

Solution. We have $\sigma_p(T) = \{\alpha_i \mid i \in \mathbb{N}\}$, $\sigma(T) = [0,1]$ and $\rho(T) = (-\infty, 0) \cup (1, +\infty)$.

Proof. We first note that $\sigma_p(T) = \{\alpha_i \mid i \in \mathbb{N}\}$. Indeed, for each α_j, consider the sequence

$$x_j = (0, \dots, 0, 1, 0, \dots)$$

with a 1 in the j^{th} position and zeros elsewhere. Then $Tx_j = \alpha_j x_j$ and thus $\alpha_j \in \sigma_p(T)$. Furthermore, if $\lambda \in \sigma_p(T)$, then there is a nonzero $x = (\xi_i)$ such that $Tx = \lambda x$. In particular, this means that $\alpha_i \xi_i = \lambda \xi_i$ for each i where $\xi_j \neq 0$ for at least one j. Hence $\lambda = \alpha_j$ and thus $\lambda \in \{\alpha_i \mid i \in \mathbb{N}\}$.

Note that T is bounded with $\|T\| \leq 1$, so by the theorem in class (see Theorem 7.3-2 in Kreyszig) we have that $\sigma(T)$ is closed. So $[0,1] = \overline{\sigma_p(T)} \subset \sigma(T)$.

Finally, we note that $(-\infty, 0) \cup (1, +\infty) \subset \rho(T)$. Indeed, suppose $\lambda \in (-\infty, 0) \cup (1, +\infty)$, then there exists a $\delta > 0$ such that $|\alpha_i - \lambda| > \delta$ for all $\alpha_i \in [0,1]$. Let $y \in X$ with $y = (\eta_i)$ and define $\xi_i = \frac{1}{\alpha_i - \lambda}\eta_i$. Then for each i we have $|\xi_i| = \frac{1}{|\alpha_i - \lambda|}|\eta_i| < \frac{1}{\delta}|\eta_i|$ and thus

$$\sqrt{\sum_{i=1}^{\infty} |\xi_i|^2} < \sqrt{\frac{1}{\delta^2}\sum_{i=1}^{\infty} |\eta_i|^2} = \frac{1}{\delta}\|y\|.$$

So $x = (\xi_i)$ is in X and $\|x\| < \frac{1}{\delta}\|y\|$ such that $T_\lambda x = y$. Hence $y \in \mathcal{D}(T_\lambda^{-1})$ so we have that $\mathcal{D}(T_\lambda^{-1}) = X$. Furthermore,

$$\|T_\lambda^{-1} y\| = \|x\| < \frac{1}{\delta}\|y\|$$

so $\|T_\lambda^{-1}\| < \frac{1}{\delta}$ and thus T_λ^{-1} is bounded. Hence $\lambda \in \rho(T)$. Since $[0,1] \subset \sigma(T)$ and $(-\infty,0) \cup (1,+\infty) \subset \rho(T)$, we must have that $\sigma(T) = [0,1]$ and $\rho(T) = (-\infty, 0) \cup (1, +\infty)$. □

Problem 3 (Problem 6, Chapter 7.3, p. 379)**.** With $X = \ell^2$, find a linear operator $T\colon X \longrightarrow X$ whose eigenvalues are dense in a given compact set $K \subset \mathbb{C}$ and $\sigma(T) = K$.

Solution. As in the previous problem, let (α_i) be dense in K and define T analogously, i.e.

$$y = Tx \quad \text{with } x = (\xi_i),\ y = (\eta_i) \text{ and } \eta_i = \alpha_i \xi_i.$$

Then the eigenvalues of T are $\sigma_p(T) = \{\alpha_i \mid i \in \mathbb{N}\}$ and we have $K = \overline{\sigma_p(T)} \subset \sigma(T)$ since $\sigma(T)$ is closed. If $\lambda \notin K$, then there exists a $\delta > 0$ such that $|\alpha_i - \lambda| > \delta$ for all α_i. Let $y \in X$ with $y = (\eta_i)$ and define $\xi_i = \frac{1}{\alpha_i - \lambda}\eta_i$ such that $x = (\xi_i) \in X$ with $T_\lambda x = y$ and

$$\|T_\lambda {-} 1 y\| = \|x\| < \frac{1}{\delta}\|y\|,$$

so $y \in \mathcal{D}(T_\lambda^{-1})$. Furthermore, T_λ^{-1} is bounded and $\mathcal{D}(T_\lambda^{-1}) = X$, so $\lambda \in \rho(T)$. Hence $\sigma(T) = K$.

Problem 4 (Problem 8, Chapter 7.3, p. 379)**.** Let $X = \mathcal{C}[0,\pi]$ and define $T\colon \mathcal{D}(T) \longrightarrow X$ by $x \longmapsto x''$, where

$$\mathcal{D}(T) = \{x \in X \mid x', x'' \in X,\, x(0) = x(\pi) = 0\}\,.$$

Show that $\sigma(T)$ is not compact.

Solution. Consider the sequence $\{x_n\}$ of functions in $\mathcal{D}(T)$ given by $x_n(t) = \sin(nt)$. Then for $y_n := Tx_n$ we have

$$y_n(t) = -n^2 \sin(nt) = -n^2 x_n(t).$$

Hence $-n^2 \in \sigma_p(T)$ for all $n \in \mathbb{N}$, so $\sigma(T)$ is unbounded and thus not compact.

Problem 5 (Problem 10, Chapter 7.3, p. 379)**.** Let $T\colon \ell^p \longrightarrow \ell^p$ be defined by $x \longmapsto (\xi_2, \xi_3, \dots)$ where x is given by $x = (\xi_1, \xi_2, \dots)$., and $1 \le p \le +\infty$. If $|\lambda| = 1$, is λ an eigenvalue of T?

Solution. No. Suppose to the contrary that there is a λ with $|\lambda| = 1$ such that λ is an eigenvalue. Then there is a nonzero sequence $x = (\xi_i)$ such that $\xi_{k+1} = \lambda^k \xi_k$ for each k, and thus $\xi_k = \lambda^{k-1}\xi_1$. However, the norm

$$\begin{aligned} \|x\|_p &= \left(\sum_{j=1}^{\infty} |\xi_j|^p \right)^{1/p} \\ &= \left(\sum_{j=1}^{\infty} \underbrace{|\lambda|^{k-1}}_{=1} |\xi_1|^p \right)^{1/p} \\ &= |\xi_1| \left(\sum_{j=1}^{\infty} 1 \right)^{1/p} \end{aligned}$$

does not converge so $x \notin \ell^p$.

Problem 6 (Problem 4, Chapter 7.4, p. 385)**.** Let X be a complete Banach space, $T \in \mathcal{B}(X, X)$ and p a polynomial. Show that the equation

$$p(T)x = y$$

has a unique solution x for every $y \in X$ if and only if $p(\lambda) \neq 0$ for all $\lambda \in \sigma(T)$.

Solution. (For one direction, I'm not sure how to prove this without using complex analysis and the Spectral Theorem...)

Proof. Suppose that $p(T)x = y$ has a unique solution, then $p(T)$ is ivertible. Suppose that $p(\lambda) = 0$ for some $\lambda \in \sigma(T)$. Then $p(z) = (z - \lambda)q(z)$ for some polynomial q, but

$$p(T) = (T - \lambda I)q(T) = q(T)(T - \lambda I)$$

would not be invertible, since $T - \lambda I$ is not invertible, a contradiction to the assumption.

Now suppose that $p(\lambda) \neq 0$ for all $\lambda \in \sigma(T)$. Then the function $q(z) = \frac{1}{p(z)}$ is holomorphic on some domain Ω that contains $\sigma(T)$, so we may define $q(T)$ and $\sigma(q(T)) = q(\sigma(T))$. Furthermore, we have

$$q(T)p(T) = p(T)q(T) = (p \cdot q)(T)$$

but $p \cdot q = 1$ and is defined on $\Omega \supset \sigma(T)$. Hence $(p \cdot q)(T) = I$, so $q(T) = [p(T)]^{-1}$ and thus $p(T)$ is invertible. $\square$

(Addendum: the correct solution is:

Solution. Note that $p(\lambda) \neq 0$ for all $\lambda \in \sigma(T)$ is equivalent to $0 \notin p(\sigma(T)) = \sigma(p(T))$. This in turn means $0 \in \rho(p(T))$, and thus $p(T)^{-1}$ exists and is defined everywhere. So $x = p(T)^{-1}y$ is unique.

Conversely suppose that $p(T)x = y$ has a unique $x \in X$ for all y. Then $p(T)$ is bijective and bounded, so $p(T)^{-1}$ exists and is bounded by the open mapping theorem. Hence $0 \in \rho(T)$.

Problem 7 (Problem 10, Chapter 7.5, p. 394)**.** Show that the existence of the limit in

$$r_\sigma(T) = \lim_{n\to\infty} \sqrt[n]{\|T^n\|}$$

already follows from $\|T^{m+n}\| \le \|T^m\|\|T^n\|$.

(Hint: set $a_n = \|T^n\|$, $b_n = \ln a_n$, $\alpha = \inf(b_n/n)$ and show that $b_n/n \longrightarrow \alpha$. See eq. (7) in Sec 2.7.)

Solution. Define the sequences $a_n = \|T^n\|$ and $b_n = \frac{1}{n}\ln a_n$. We want to show that the sequence b_n is decreasing. (I honestly have no idea where to go from here.....)

Problem 8 (Problem 8, Chapter 7.6, p. 403)**.** Let $\mathcal{A}$ be a complex Banach algebra with identity and let G be the set of all invertible elements of $\mathcal{A}$. Show that the mapping $G \longrightarrow G$ given by $x \longmapsto x^{-1}$ is continuous.

Solution. We first prove the following lemma.

Lemma 1. *Let $x_0 \in G$. Suppose that $x \in \mathcal{A}$ is an element such that $\|x - x_0\| < \frac{1}{\|x_0^{-1}\|}$. Then $x \in G$ and*

$$\|x^{-1} - x_0^{-1}\| < \frac{\|x_0^{-1}\|^2 \|x - x_0\|}{1 - \|x_0^{-1}\| \|x - x_0\|}.$$

Proof. Define the element $y = e - x_0^{-1}x$. We have that

$$\|y\| = \|e - x_0^{-1}x\| \leq \|x_0^{-1}\| \|x_0 - x\| < 1.$$

By the theorem in class (see Them. 7.7-1 in Kreyszig), we have that $e - y$ is invertible and

$$(e - y)^{-1} = e + \sum_{i=1}^{\infty} y^n$$

and this series converges. Since $\|y\| < 1$, we have that $\|y^n\| < \|y\|^n$ for all n. Note that $e - y = x_0^{-1}x$ and thus

$$\begin{aligned} \|e - (x_0^{-1}x)^{-1}\| = \left\|e - (e - y)^{-1}\right\| &\leq \Big\|\sum_{i=1}^{\infty} y^n\Big\| \\ &\leq \sum_{i=1}^{\infty} \|y\|^n \\ &\leq \sum_{i=1}^{\infty} \left(\|x_0^{-1}\| \|x - x_0\|\right)^n \\ &= \frac{\|x_0^{-1}\| \|x - x_0\|}{1 - \|x_0^{-1}\| \|x - x_0\|} \end{aligned}$$

since this is a geometric sum and $\|y\| < \|x_0^{-1}\| \|x - x_0\| < 1$. Finally, we have

$$\begin{aligned} \left\|x_0^{-1} - x^{-1}\right\| &= \left\|x_0^{-1}\left(e - x_0 x^{-1}\right)\right\| \\ &\leq \|x_0^{-1}\| \|e - x_0 x^{-1}\| \\ &< \frac{\|x_0^{-1}\|^2 \|x - x_0\|}{1 - \|x_0^{-1}\| \|x - x_0\|} \end{aligned}$$

as desired. □

We now show that the inverse is continuous.

Proof. Fix $x_0 \in G$ and let $\varepsilon > 0$. Then let $\delta = \frac{\varepsilon}{\|x_0^{-1}\|\left(\|x_0^{-1}\| + \varepsilon\right)}$ and note that $\delta < \frac{1}{\|x_0^{-1}\|}$. Suppose that $x \in \mathcal{A}$ is an element such that

$$\|x - x_0\| < \delta.$$

Since $\|x - x_0\| < \delta < \frac{1}{\|x_0^{-1}\|}$, we have that

$$0 < 1 - \|x_0^{-1}\|\delta < 1 - \|x_0^{-1}\| \|x - x_0\|$$

and thus

$$\frac{1}{1-\|x_0^{-1}\|\|x-x_0\|} < \frac{1}{1-\|x_0^{-1}\|\delta}.$$

From the lemma, we have that

$$\begin{aligned}
\|x^{-1}-x_0^{-1}\| &< \frac{\|x_0^{-1}\|^2\|x-x_0\|}{1-\|x_0^{-1}\|\|x-x_0\|} \\
&< \frac{\|x_0^{-1}\|^2\delta}{1-\|x_0^{-1}\|\delta} \\
&= \|x_0^{-1}\|\frac{\varepsilon}{\|x_0^{-1}\|+\varepsilon}\frac{1}{\left(1-\frac{\varepsilon}{\|x_0^{-1}\|+\varepsilon}\right)} \\
&= \frac{\varepsilon\|x_0^{-1}\|}{\|x_0^{-1}\|+\varepsilon-\varepsilon} \\
&= \varepsilon.
\end{aligned}$$

Hence, for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|x - x_0\| < \delta$ implies $\|x^{-1} - x_0^{-1}\| < \varepsilon$. So the inverse is continuous. □

Problem 9 (Problem 10, Chapter 9.1, p. 465)**.** Let T be a linear operator on a Hilbert space H that satisfies

$$\langle Tx, y\rangle = \langle x, Ty\rangle \qquad \text{for all } x, y \in H.$$

Then T is bounded. (Use the uniform bounded theorem to prove)

Solution. Suppose to the contrary that T is unbounded. Then there is a sequence $\{y_n\}$ in H such that $\|y_n\| = 1$ and $\|Ty_n\| \longrightarrow \infty$. Consider the sequence of functionals f_n defined by $f_n(x) = \langle Tx, y_n\rangle$. Then f_n is bounded for each n since

$$|f_n(x)| = |\langle Tx, y_n\rangle| \leq \|Tx\| \underbrace{\|y_n\|}_{=1} = \|Tx\|.$$

Furthermore, the sequence $\{f_n(x)\}$ is bounded for all $x \in H$ since $|f_n(x)| = |\langle Tx, y_n\rangle| \leq \|Tx\|$. By the Uniform Boundedness Theorem, the sequence $\|f_n\|$ is bounded. That is there exists a $c > 0$ such that $\|f_n\| < c$ for all n. Finally, note that

$$\|Ty_n\|^2 = \langle Ty_n, Ty_n\rangle = |f_n(Ty_n)| \leq c\|Ty_n\|$$

for all n and thus $\|Ty_n\| \leq c$, a contradiction to the assumption that $\|Ty_n\| \longrightarrow \infty$.

Problem 10 (Extra problem 1)**.** Show that the product AB of two operators A and B is positive if A and B are positive and $[A, B] = 0$.

Solution. *Proof.* Note that A and B must be self-adjoint linear operators since they are positive. If $A = 0$ then the statement is trivial, so suppose that $\|A\| \neq 0$. Define the operator $A_1 = \frac{1}{\|A\|}A$ such that $0 \leq A_1 \leq I$ and A_1 also commutes with B. The goal is to construct a sequence of operators positive self-adjoint operators $(A_n)_{n\in\mathbb{N}}$ such that $0 \leq A_n \leq I$ and $[A_n, B] = 0$ for each n and

$$\sum_{k=1}^{\infty} {A_k}^2 x = A_1 x$$

for all $x \in H$. Then we would have that

$$\begin{aligned}
\langle ABx, x\rangle = \|A\|\langle A_1 B, x, x\rangle &= \|A\|\left\langle \sum_{k=1}^{\infty} {A_k}^2(Bx), x \right\rangle \\
&= \|A\| \sum_{k=1}^{\infty} \langle A_k Bx, A_n x\rangle \\
&= \|A\| \sum_{k=1}^{\infty} \underbrace{\langle B(A_k x), (A_k x)\rangle}_{\geq 0} \\
&\geq 0
\end{aligned}$$

and thus $AB \geq 0$.

Indeed, we construct the sequence of operators $(A_n)_{n\in\mathbb{N}}$ defined by $A_1 = \frac{1}{\|A\|}A$ and

$$A_{n+1} = A_n - {A_n}^2 \quad \text{for } n \geq 1.$$

For each n, the operator A_n is a polynomial in A and thus A_n is self-adjoint. Similarly, we have $[A_n, B] = 0$ for all n since A commutes with B.

We show by induction that $0 \leq A_n \leq I$ for all n. Suppose that $0 \leq A_k \leq I$ for some $k \geq 1$ and thus we have $0 \leq I - A_k$ as well. We will show that $0 \leq A_{k+1} \leq I$.

- We first show that $0 \leq A_{k+1}$. Observe that

  $$\begin{aligned}
  A_{k+1} = A_k - {A_k}^2 &= A_k + {A_k}^2 - 2A_n + {A_k}^3 - {A_k}^3 \\
  &= {A_k}^2(I - A_k) + A_k(I - A_k)^2.
  \end{aligned}$$

  Note that $(I - A_k)$ and A_k commute. Hence, for all $x \in H$ we have

  $$\begin{aligned}
  \langle A_{k+1}x, x\rangle &= \left\langle {A_k}^2(I - A_k)x, x\right\rangle + \left\langle A_n(I - A_k)^2 x, x\right\rangle \\
  &= \langle A_k(1 - A_k)x, A_k x\rangle + \langle (I - A_k)A_k x, (I - A_k)x\rangle \\
  &= \underbrace{\langle (I - A_k)A_n x, A_k x\rangle}_{\geq 0} + \underbrace{\langle A_k(I - A_k)x, (I - A_k)x\rangle}_{\geq 0} \\
  &\geq 0
  \end{aligned}$$

  since $0 \leq (I - A_k)$ and $0 \leq A_k$ by the induction hypothesis. So $0 \leq A_{k+1}$ as desired.

- We now show that $0 \leq I - A_{k+1}$ and hence $A_{k+1} \leq I$. Note that

  $$I - A_{k+1} = I - (A_k - {T_k}^2) = (I - A_k) + {A_k}^2.$$

So for all $x \in H$ we have

$$\begin{aligned}\langle (I - A_{k+1})x, x\rangle &= \langle (I - A_k)x, x\rangle + \left\langle {A_k}^2 x, x\right\rangle \\ &= \underbrace{\langle (I - A_k)x, x\rangle}_{\geq 0} + \underbrace{\langle A_k x, A_k x\rangle}_{\geq 0} \\ &\geq 0\end{aligned}$$

since $0 \leq (I - A_k)$ and $0 \leq A_k$ by the induction hypothesis. So $I - A_{k+1} \geq 0$ and thus $A_{k+1} \leq I$ as desired.

Next, we note that ${A_k}^2 = A_k - A_{k+1}$ for each k and thus

$$\begin{aligned}\sum_{k=1}^{n} {A_k}^2 = {A_1}^2 + \sum_{k=2}^{n}(A_k - A_{k+1}) &= {A_1}^2 + \sum_{k=2}^{n} A_k - \sum_{k=2}^{n} A_{k+1} \\ &= \underbrace{{A_1}^2 + A_2}_{A_1} - A_{n+1} \\ &= A_1 - A_{n+1}\end{aligned}$$

so we have $\sum_{k=1}^{n} {A_k}^2 = A_1 - A_{n+1}$. Furthermore, note that $A_1 - A_{n+1} \leq A_1$. Hence for all $x \in H$ we have

$$\begin{aligned}\sum_{k=1}^{n} \|A_n x\|^2 = \sum_{k=1}^{n} \langle A_k x, A_k x\rangle &= \left\langle \sum_{k=1}^{n} ({A_k}^2)x, x\right\rangle \\ &= \langle (A_1 - A_{n+1})x, x\rangle \\ &\leq \langle A_1 x, x\rangle\end{aligned}$$

and thus thes series $\sum_{k=1}^{n} \|A_n x\|^2$ converges. This implies that $\|T_n x\| \longrightarrow 0$ and thus $A_n x \longrightarrow 0$ for all x. Hence

$$\left(\sum_{k=1}^{n} {A_k}^2\right)x = A_1 x - A_{n+1}x \longrightarrow A_1 x,$$

so we may write $\displaystyle\sum_{k=1}^{\infty} {A_k}^2 x = A_1 x$ for all $x \in H$. □

Problem 11 (Extra problem 2)**.** Show that a positive self-adjoint linear operator has a unique positive square root.

Solution.

Claim. *Let X be a Banach space and T a positive self-adjoint bounded linear operator. Then there exists a unique positive operator A such that $A^2 = T$.*

Proof. If $T = 0$ then $A = 0$, so we may assume that $\|T\| \neq 0$. Without loss of generality, we may assume that $T \leq I$. Otherwise, we may define $S = \frac{1}{\|T\|}T$ such that $S \leq I$. If B is the unique positive linear operator such that $B^2 = S$, then $A = \sqrt{\|T\|}B$ is the unique operator such that $A^2 = \|T\|B^2 = \|T\|S = T$.

To show the existence of a positive square root, we construct a sequence of operators $(A_n)_{n\in\mathbb{N}}$ in the following manner. Define $A_0 = 0$ and

$$A_{n+1} = A_n + \frac{1}{2}(T - {A_n}^2) \quad \text{for } n = 1, 2, \ldots.$$

We show the following:

(i) $A_n \leq I$ for all n;

(ii) $A_n \leq A_{n+1}$ for ann n;

(iii) for all $x \in H$, the sequence $(A_n x)$ converges to Ax where A is an operator such that $A^2 = T$;

(iv) $[A, S] = 0$ for all bounded linear operators S on H such that $[S, T] = 0$.

This proves the existence of a positive square root of T.

(i) We first show that $A_n \leq I$ for each n. Indeed, for $n = 0$ we have $T_0 = 0 \leq I$ and for $n = 1$ we have $A_1 = \frac{1}{2}T \leq I$ since we assumed that $T \leq I$. So suppose that $A_k \leq I$ for some $k \geq 1$, then $0 \leq I - T_k$ by the induction hypothesis and $0 \leq I - T$. By the previous problem, we also have that $0 \leq (I - T_k)^2$. Since $A_{k+1} = A_k + \frac{1}{2}(T - {A_k}^2)$, we have

$$\begin{aligned} I - T_{k+1} &= I - T_k - \tfrac{1}{2}(T - {T_k}^2) \\ &= \tfrac{1}{2}I + \tfrac{1}{2}I - T_k - \tfrac{1}{2}T + \tfrac{1}{2}{T_k}^2 \\ &= \tfrac{1}{2}(I - 2T_k + {T_k}^2) + \tfrac{1}{2}(I - T) \\ &= \tfrac{1}{2}\underbrace{(I - T_k)^2}_{\geq 0} + \tfrac{1}{2}\underbrace{(I - T)}_{\geq 0} \\ &\geq 0 \end{aligned}$$

and thus $T_{k+1} \leq I$.

(ii) Next, note that $A_0 = 0 \leq \frac{1}{2}T = A_1$ and thus $A_0 \leq A_1$. Suppose that $A_{k-1} \leq A_k$ for some $k \geq 1$. Since $A_k \leq I$ and $A_{k-1} \leq I$, we have $\frac{1}{2}(A_k + A_{k-1}) \leq I$. Then

$$\begin{aligned} A_{k+1} - A_k &= A_k + \tfrac{1}{2}(T - {A_k}^2) - \left[A_{k-1} - \tfrac{1}{2}(T - {A_{k-1}}^2)\right] \\ &= \underbrace{(A_k - A_{k-1})}_{\geq 0}\underbrace{\left(I - \tfrac{1}{2}(A_k - A_{k-1})\right)}_{\geq 0} \qquad (1) \\ &\geq 0. \end{aligned}$$

Indeed, each A_k is a polynomial in T, hence all the A_k's and T all commute with one anothe. So the two positive operators in (1) commute with each other and thus their product is another postive operator. Hence $A_k \leq A_{k+1}$ as desired.

(iii) We have the monotone sequence of self-adjoint operators

$$A_0 \leq A_1 \leq A_2 \leq \cdots \leq I$$

and I is bounded. Hence, the Monotone Sequence Theorem (Theorem 9.3-1 in Kreyszig) implies the existence of a bounded self-adjoint linear operator A such that $A_n x \longrightarrow Ax$ for all $x \in H$. Since $A_{n+1}x - A_n x = \frac{1}{2}(Tx - {A_n}^2 x)$, we have

$$\tfrac{1}{2}(Tx - {A_n}^2 x) = A_{n+1}x - A_n x \longrightarrow 0$$

and thus $Tx = A^2 x$ for all x.

Furthermore, note that $0 \leq A$ since $\langle A_n x, x \rangle \geq 0$ for all n and x implies $\langle Ax, x \rangle \geq 0$ for all x.

(iv) Suppose that S is a bounded linear operator on H such that $[S, T] = 0$. Since each A_n is a polynomial in T, we have that $[S, A_n] = 0$ for each n. Noting that $A_n x \longrightarrow Ax$ for each x yields $[S, A] = 0$.

Lastly, we prove uniqueness of A. Suppose that B is another positive self-adjoint operator such that $A^2 = B^2 = T$. Then $BT = BB^2 = B^2 B = TB$ and thus $[B, T] = 0$ such that $[A, B] = 0$. Let $x \in H$ and define $y = (A - B)x$ such that $\langle Ay, y \rangle \geq 0$ and $\langle By, y \rangle \geq 0$ by positivity of A and B. Note that $(A + B)(A - B) = (A^2 - B^2)$ and thus

$$0 = \langle Ay, y \rangle + \langle By, y \rangle = \langle (A + B)y, y \rangle = \langle (A + B)(A - B)x, y \rangle = \left\langle (A^2 + B^2)x, y \right\rangle.$$

Hence $\langle Ay, y \rangle = \langle By, y \rangle = 0$. Since $A \geq 0$ is self-adjoint, there is a self-adjoint linear operator $0 \leq C$ such that $C^2 = A$. Then

$$0 = \langle Ay, y \rangle = \left\langle C^2 y, y \right\rangle = \langle Cy, Cy \rangle = \|Cy\|^2$$

such that $Cy = 0$ and thus $Ay = C^2 y = 0$. Analogously, we can find a self-adjoint linear operator D such that $D^2 = B$ to find that $Dy = 0$ and thus $By = 0$. Hence $(A - B)y = 0$ and thus

$$\|(A - B)x\|^2 = \langle (A - B)x, (A - B)x \rangle = \left\langle (A - B)^2 x, x \right\rangle = \langle \underbrace{(A - B)y}_{=0}, x \rangle = 0.$$

So $(A - B)x = 0$ and thus $Ax = Bx$ for all $x \in H$. $\square$