MATH 271 MIDTERM WINTER 2015 SOLUTIONS

1. Use the Euclidean algorithm to find gcd (85, 58). Then use your work to write gcd (85, 58) in the form $85a + 58b$ where a and b are integers.
Solution: We have

$$\begin{aligned} 85 &= 1 \times 58 + 27 \\ 58 &= 2 \times 27 + 4 \\ 27 &= 6 \times 4 + 3 \\ 4 &= 1 \times 3 + 1 \\ 3 &= 3 \times 1 + 0 \end{aligned}$$

so $\gcd(85, 58) = 1$, and

$$\begin{aligned} \gcd(85,58) &= 1 \\ &= 4 - 3 \\ &= 4 - (27 - 6 \times 4) \\ &= 7 \times 4 - 27 \\ &= 7 \times (58 - 2 \times 27) - 27 \\ &= 7 \times 58 - 15 \times 27 \\ &= 7 \times 58 - 15 \times (85 - 58) \\ &= -15 \times 85 + 22 \times 58 \end{aligned}$$

Another way is to use the "table method" as follows.

	85	1	0
	58	0	1
$R_1 - R_2$	27	1	-1
$R_2 - 2R_3$	4	-2	3
$R_3 - 6R_4$	3	13	-19
$R_4 - R_5$	1	-15	22

Thus, $\gcd(85, 58) = 1$ and $\gcd(85, 58) = 85 \times (-15) + 58 \times 22$, that is, $a = -15$ and $b = 22$.

2. Prove or disprove each of the following statements.
(a) For all real numbers x and y, $\lceil xy \rceil = \lceil x \rceil \lfloor y \rfloor$.
Solution: This statement is false. Its negation is "There are real numbers x and y so that $\lceil xy \rceil \neq \lceil x \rceil \lfloor y \rfloor$". For example, when $x = 2$ and $y = \frac{1}{2}$, $\lceil xy \rceil = \lceil 1 \rceil = 1$, and $\lceil x \rceil \lfloor y \rfloor = \lceil x \rceil \lfloor \frac{1}{2} \rfloor = \lceil x \rceil \times 0 = 0$. Thus, $\lceil xy \rceil \neq \lceil x \rceil \lfloor y \rfloor$ in this case.

(b) There exists real numbers x and y so that $\lceil xy \rceil = \lceil x \rceil \lfloor y \rfloor$.
Solution: This statement is true. For example, when $x = y = 0$, $\lceil xy \rceil = \lceil 0 \rceil = 0 \times 0 = \lceil 0 \rceil \lfloor 0 \rfloor = \lceil x \rceil \lfloor y \rfloor$.

3. Let $\mathcal{P}$ be the statement:"For all integers a and b, if $a \mid b^2$ then $a \mid b$."
(a) Is $\mathcal{P}$ true? Prove your answer.
Solution: This statement is false. Its negation is: "There are integers a and b so that $a \mid b^2$ but $a \nmid b$." For example, let $a = 4$ and $b = 2$. Then $a \mid b^2$ because $4 \mid 4$, but $a \nmid b$ because $4 \nmid 2$.

(b) Write the converse of $\mathcal{P}$. Is the converse of $\mathcal{P}$ true? Prove your answer.
Solution: The converse of $\mathcal{P}$ is: "For all integers a and b, if $a \mid b$ then $a \mid b^2$." The converse of $\mathcal{P}$ is true and here is a proof. Let a and b be integers so that $a \mid b$. Since $a \mid b$, $a \neq 0$ and there is an integer k so that $b = ak$. Then $b^2 = b \times b = akb = a\,(kb)$ where $a \neq 0$ and kb is an integer. This implies that $a \mid b^2$.

(c) Write the contrapositive of $\mathcal{P}$. Is the contrapositive of $\mathcal{P}$ true? Explain.
Solution: The contrapositive of $\mathcal{P}$ is: "For all integers a and b, if $a \nmid b$ then $a \nmid b^2$." The contrapositive of $\mathcal{P}$ is false because it is logically equivalent to $\mathcal{P}$ which is false as proven in (a).

4. Prove the statement: "For all non-negative real numbers x, if x is irrational then $2\sqrt{x}$ is irrational." by contradiction.
Solution: Suppose that x is a non-negative irrational real number. We prove that $2\sqrt{x}$ is irrational. Suppose that $2\sqrt{x}$ is rational; that is, $2\sqrt{x} = \dfrac{m}{n}$ for some integers m and n where $n \neq 0$. Now, $x = \dfrac{4x}{4} = \dfrac{\left(2\sqrt{x}\right)^2}{4} = \dfrac{\left(\frac{m}{n}\right)^2}{4} = \dfrac{m^2}{4n^2}$ where m^2 and $4n^2$ are integers, and $4n^2 \neq 0$ (because $n \neq 0$).This means x is rational which contradicts the assumption that x is irrational. Thus, $2\sqrt{x}$ is irrational.

5. Let A and B be sets so that $1 \notin A - B$ and $(1, 2) \in A \times B$.
(a) Which elements must be in A? Explain.
Solution: 1 must be in A. Reason: we know $(1, 2) \in A \times B$, so $1 \in A$.

(b) Which elements must be in B? Explain.
Solution: 1 and 2 must be in B. Reason: we know $1 \notin A - B$, that means $1 \notin A$ or $1 \in B$. However, we know $1 \in A$ in part (a), so it must be the case that $1 \in B$. Next, $2 \in B$ because $(1, 2) \in A \times B$.

(c) What is the smallest possible number of elements in $B - A$? Explain.
Solution: The smallest possible number of elements in $B - A$ is 0. This happens, for example, when $A = B = \{1, 2\}$. In this case $A - B = \emptyset$, so $1 \notin A - B$, and $(1, 2) \in A \times B$ because $1 \in A$ and $2 \in B$.

6. For the following, give a brief explanation on how you get the answers and simplify your answers to a number.
(a) In how many ways can two distinct integers be selected from the set $\{1,\ 2,\ 3, ...,\ 100\}$ so that their sum is even?

Solution: The answer is $2 \times \binom{50}{2} = 2 \times \dfrac{50 \times 49}{2} = 50 \times 49 = 2450$. Reason: the sum of two integers is even when two integers are of the same parity (either both are even, or both are odd). We have 2 ways of choosing the parity and then $\binom{50}{2}$ ways of choosing two integers from the 50 integers of that parity.

Another solution: The answer is $\binom{100}{2} - \binom{50}{1}\binom{50}{1} = \dfrac{100 \times 99}{2} - 50 \times 50 = 50\,(99 - 50) = 50 \times 49 = 2450$. That is the total number of ways to choose 2 integers from 100 integers minus the number of ways to choose 2 integers whose sum is odd (one is even and one is odd).

(b) In how many ways can two distinct integers be selected from the set $\{1,\ 2,\ 3, ...,\ 100\}$ so that their product is even?

Solution: The answer is $\binom{50}{2} + \binom{50}{1}\binom{50}{1} = \dfrac{50 \times 49}{2} + 50 \times 50 = 25 \times (49 + 100) = 25 \times 149 = 3725$. Reason: the product of two integers is even when at least one of the two integers is even. Thus we have two mutually disjoint cases (1) both are even and (2) one is even and one is odd. We have $\binom{50}{2}$ ways of choosing two even integers from the 50 even integers, and $\binom{50}{1}\binom{50}{1}$ ways of choosing one even integer and one odd integer.

Another solution: The answer is $\binom{100}{2} - \binom{50}{2} = \dfrac{100 \times 99}{2} - \dfrac{50 \times 49}{2} = 25\,(2 \times 99 - 49) = 25 \times 149 = 3725$. That is the total number of ways to choose 2 integers from 100 integers minus the number of ways to choose 2 integers whose product is odd (both are odd).

7. Prove by induction that $5^n - 4n - 1$ is divisible by 16 for all integers $n \geq 1$.

Solution: We prove this by induction on n.

Basis: $(n = 1)$

When $n = 1$, $5^n - 4n - 1 = 5^1 - 4 \times 1 - 1 = 0 = 16 \times 0$ where $0 \in \mathbb{Z}$, so $5^n - 4n - 1$ is divisible by 16 when $n = 1$.

Inductive Hypothesis: Suppose that for some integer $k \geq 1$,

$$5^k - 4k - 1 \text{ is divisible by } 16. \qquad (IH)$$

We want to show that $5^{k+1} - 4\,(k+1) - 1$ is divisible by 16.

From (IH), there is an integer m so that $5^k - 4k - 1 = 16m$.

Now,

$$\begin{aligned}
5^{k+1} - 4\,(k+1) - 1 &= 5 \times 5^k - 4k - 4 - 1 \\
&= 5 \times 5^k - 4k - 5 \\
&= 5 \times 5^k - 20k - 5 + 16k \\
&= 5\left(5^k - 4k - 1\right) + 16k \\
&= 5 \times 16m + 16k \qquad \text{because } 5^k - 4k - 1 = 16m \\
&= 16\,(5m + k)\,.
\end{aligned}$$

Since $5^{k+1} - 4\,(k+1) - 1 = 16\,(5m + k)$ where $5m + k$ is an integer, $5^{k+1} - 4\,(k+1) - 1$ is divisible by 16, as required.

Thus, $5^n - 4n - 1$ is divisible by 16 for all integers $n \geq 1$.