# MATH 271 WINTER 2016
# MIDTERM SOLUTIONS

**1**. Use the Euclidean algorithm to find $\gcd(271, 98)$. Then use your work to write $\gcd(271, 98)$ in the form $271a + 98b$ where $a$ and $b$ are integers.
*Solution*: We have

$$\begin{aligned} 271 &= 2 \times 98 + 75 \\ 98 &= 1 \times 75 + 23 \\ 75 &= 3 \times 23 + 6 \\ 23 &= 3 \times 6 + 5 \\ 6 &= 1 \times 5 + 1 \\ 5 &= 5 \times 1 + 0 \end{aligned}$$

and so $\gcd(271, 98) = 1$, and
$\gcd(271, 98) = 1 = 6 - 5 = 6 - (23 - 3 \times 6) = 4 \times 6 - 23 = 4(75 - 3 \times 23) - 23 = 4 \times 75 - 13 \times 23 = 4 \times 75 - 13(98 - 75) = 17 \times 75 - 13 \times 98 = 17(271 - 2 \times 98) - 13 \times 98 = 17 \times 271 - 47 \times 98$.

Another way is to use the "table method" as follows.

| | | 271 | 98 |
|---|---|---|---|
| | 271 | 1 | 0 |
| | 98 | 0 | 1 |
| $R_1 - 2R_2$ | 75 | 1 | $-2$ |
| $R_2 - R_3$ | 23 | $-1$ | 3 |
| $R_3 - 3R_4$ | 6 | 4 | $-11$ |
| $R_4 - 3R_5$ | 5 | $-13$ | 36 |
| $R_5 - R_6$ | 1 | 17 | $-47$ |

Thus, $\gcd(271, 98) = 1$ and $\gcd(271, 98) = 271 \times 17 + 98 \times (-47)$, that is, $a = 17$ and $b = -47$.

**2**. Prove the statement: "For all real numbers $x$, if $x$ is irrational then $271x$ is irrational." by contradiction.
*Solution*: Suppose that $x$ is an irrational real number. We prove that $271x$ is irrational by contradiction. Suppose that $271x$ is rational; that is, $271x = \dfrac{m}{n}$ where $m, n \in \mathbb{Z}$ and $n \neq 0$. Then $x = \dfrac{m}{271n}$ where $m, 271n \in \mathbb{Z}$ and $271n \neq 0$, which implies that $x$ is rational. This contradicts the assumption that $x$ is irrational. Thus, $271x$ is irrational.

**3**. Let $\mathcal{P}$ be the statement: "For all sets $A$, $B$ and $C$, if $A \cap B \subseteq C$ then $A - C = \emptyset$."
(a) Is $\mathcal{P}$ true? Prove your answer.
*Solution*: $\mathcal{P}$ is false. We prove the negation of $\mathcal{P}$ which is "There are sets $A$, $B$ and $C$ so that $A \cap B \subseteq C$, but $A - C \neq \emptyset$."

For example, let $A = \{1\}$ and $B = C = \emptyset$. Then, $A \cap B = A \cap \emptyset = \emptyset \subseteq C$, but $A - C = A - \emptyset = A = \{1\} \neq \emptyset$.

(b) Write the converse of $\mathcal{P}$. Is the converse of $\mathcal{P}$ true? Prove your answer.
*Solution*: The converse of $\mathcal{P}$ is "For all sets $A$, $B$ and $C$, if $A - C = \emptyset$ then $A \cap B \subseteq C$."

The converse of $\mathcal{P}$ is true and here is a proof. Let $A$, $B$ and $C$ be sets so that $A - C = \emptyset$. We prove that $A \cap B \subseteq C$. Suppose that $x \in A \cap B$. We prove that $x \in C$ by contradiction. Suppose that $x \notin C$. Since $x \in A \cap B$, we know $x \in A$. Since $x \in A$ and $x \notin C$, we know that $x \in A - C$ which contradicts the assumption that $A - C = \emptyset$. Thus, $x \in C$.

(c) Write the contrapositive of $\mathcal{P}$. Is the contrapositive of $\mathcal{P}$ true? Explain.
*Solution*: The converse of $\mathcal{P}$ is "For all sets $A$, $B$ and $C$, if $A - C \neq \emptyset$ then $A \cap B \nsubseteq C$."

The contrapositive of $\mathcal{P}$ is false because it is logically equivalent to $\mathcal{P}$ and $\mathcal{P}$ is false as proven in part (a).

**4**. Let $S = \{1, 2, 3, ..., 10\}$ and $T = \{1, 2, 3, ..., 20\}$
(a) Is there a subset $X$ of $T$ so that $4 \in X$ and $X$ has exactly 4 elements? Prove your answer.
*Solution*: Yes, there is such a set $X$. For example, $X = \{1, 2, 3, 4\}$. Clearly, $4 \in X$ and $X$ has exactly 4 elements.

(b) Is there a subset $Y$ of $T$ so that $4 \in Y$, and both $Y - S$ and $S - Y$ have exactly 4 elements? Prove your answer.
*Solution*: Yes, there is such a set $Y$. For example, $Y = \{4, 5, 6, 7, 8, 9, 11, 12, 13, 14\}$. Clearly, $4 \in Y$. Now, $Y - S = \{11, 12, 13, 14\}$ and $S - Y = \{1, 2, 3, 10\}$, so both $Y - S$ and $S - Y$ have exactly 4 elements.

For the following, give a brief explanation on how you get the answers and simplify your answers to a number.
(c) How many subsets $X$ of $T$ are there so that $4 \in X$ and $X$ has exactly 4 elements?
*Solution*: The answer is $\binom{19}{3} = \dfrac{19 \times 18 \times 17}{3 \times 2 \times 1} = 969$. We choose elements for $X$ as follow: First choose 4 and then we choose 3 more from the remaining 19 elements of $T$.

(d) How many subsets $Y$ of $T$ are there so that $4 \in Y$, and both $Y - S$ and $S - Y$ have exactly 4 elements?
*Solution*: The answer is $\binom{9}{5} \times \binom{10}{4} = \binom{9}{4} \times \binom{10}{4} = \dfrac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} \times \dfrac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 126 \times 210 = 26460$. We choose elements for $Y$ as follow: First choose 4, then we choose 5 more from the 9 elements of the set $S - \{4\}$, and finally, we choose 4 more from the 10 elements of $T - S$.

**6**. Prove by induction that $5^n - 4n - 1$ is divisible by 16 for all integers $n \geq 1$.
*Solution*: We prove this by induction on $n$.
*Basis:* $(n = 1)$

When $n = 1$, $5^n - 4n - 1 = 5^1 - 4 \times 1 - 1 = 0 = 16 \times 0$ where $0 \in \mathbb{Z}$, so $5^n - 4n - 1$ is divisible by 16 when $n = 1$.
*Inductive Hypothesis:* Suppose that for some integer $k \geq 1$,

$$5^k - 4k - 1 \text{ is divisible by 16.} \qquad (IH)$$

We want to show that $5^{k+1} - 4(k+1) - 1$ is divisible by 16.

From $(IH)$, there is an integer $m$ so that $5^k - 4k - 1 = 16m$.

Now,

$$\begin{aligned}
5^{k+1} - 4(k+1) - 1 &= 5 \times 5^k - 4k - 4 - 1 \\
&= 5 \times 5^k - 4k - 5 \\
&= 5 \times 5^k - 20k - 5 + 16k \\
&= 5\left(5^k - 4k - 1\right) + 16k \\
&= 5 \times 16m + 16k \qquad \text{because } 5^k - 4k - 1 = 16m \\
&= 16(5m + k).
\end{aligned}$$

Since $5^{k+1} - 4(k+1) - 1 = 16(5m+k)$ where $5m+k$ is an integer, $5^{k+1} - 4(k+1) - 1$ is divisible by 16, as required.

Thus, $5^n - 4n - 1$ is divisible by 16 for all integers $n \geq 1$.