MATH 271 – Summer 2016
Assignment 1 – solutions

Problem 1.

You are wandering in a fictional forest that is inhabited by trolls. Each troll is either a knight or a knave. Knights always tell the truth and knaves always speak falsehoods. For each of the following situations, determine which trolls are knights and which are knaves. Write a proof using complete sentences that confirms your assertion.

(a) You encounter two trolls (X and Y) who make the following statements.

X: "Both of us are knaves."
Y: "Exactly one of us is a knave."

Solution: X is a knave and Y is a knight. We prove this as follows.

Proof (*that X is a knave*). Suppose instead that X is a knight. Then X's statement is true, so both X and Y are knaves. In particular, X is a knave. But we assumed X is a knight. This is a contradiction, so the assumption that X is a knight is wrong. Therefore X must be a knave.

Proof (*that Y is a knight*). Suppose instead that Y is a knave. We already know that X is a knave, so X and Y are both knaves. But X is a knave, so X's statement is false. That is, they are not both knaves and they are both knaves. This is a contradiction, so the assumption that Y is a knave must be wrong. Therefore Y is a knight.

(b) You encounter three trolls (Q, R, and S) who make the following statements.

Q: "Exactly one of us is a knight."
R: "Exactly one of us is a knave."
S: "We are all knaves."

Solution: Q is a knight and R and S are knaves. We prove this as follows.

Proof (*that S is a knave*). Suppose instead that S is a knight. Then S's statement is true. That is, all of Q, R, and S are knaves. In particular, S is a knave. Hence S is a knight and S is a knave, a contradiction. So the assumption that S is a knight is wrong. Therefore, S must be a knave.

Proof (*that R is a knave*). Suppose instead that R is a knight. Then R's statement that there is exactly one knave is true. Thus Q must be a knight, since S is a knave and there can only by one knave. Then Q's statement that there is exactly one knight is true. So there is exactly one knight and there are two knights, a contradiction. Therefore the assumption that R is a knight is wrong. Therefore R must be a knave.

Proof (*that Q is a knight*). Suppose instead that Q is a knave. We know that R and S are knaves, so all of them are knaves. But S is a knave, so his statement that all of them are knaves must be false. So all of them are knaves and it is not the case that all of them are knaves, a contradiction. Thus the assumption that Q is a knave is wrong. Therefore Q must be a knight.

(c) You encounter three trolls (A, B, and C) who make the following statements.

A: "If I am a knave, then exactly two of us are knights."
B: "Troll A is a knave."
C: "At least one of us is a knight."

Solution: A and C are knights and B is a knave. We prove this as follows.

Proof (*that B is a knave*). Suppose instead that B is a knight. Then B's statement is true, so A is a knave. Then A's statement must be false, which means that the negation of A's statement must be true. The negation of A's statement is "I am a knave and not exactly two of us are knights." But A is a not a knight and B is a knight. Since there cannot be exactly two knights, C cannot be a knight. So C is a knave, which means that C's statement that there is exactly one knight is false. So B is the only knight and there is not exactly one knight. This is a contradiction, so the assumption that B is a knight must be wrong. Therefore B is a knave.

Proof (*that A is a knight*). Since B is a knave, B's statement must be false. So A is not a knave. Thus A is a knight.

Proof (*that C is a knight*). Since B is a knight, there is at least one knight. Thus C's statement must be true. Therefore C must be a knight.

Problem 2.

For each true statement, give a proof. For each false statement, write the negation and prove that.

(a) $\forall x, y \in \mathbb{R}$, if $\lfloor xy \rfloor = \lfloor x \rfloor \lfloor y \rfloor$ then $x \in \mathbb{Z}$ or $y \in \mathbb{Z}$.
This statement is false.
Negation: $\exists x, y \in \mathbb{R}$ so that $\lfloor xy \rfloor = \lfloor x \rfloor \lfloor y \rfloor$ but $x \notin \mathbb{Z}$ and $y \notin \mathbb{Z}$.
Proof (*of the negation*). Let $x = \frac{1}{2}$ and $y = \frac{1}{3}$. Then $xy = \frac{1}{2}\frac{1}{3} = \frac{1}{6}$ and $\lfloor xy \rfloor = \lfloor \frac{1}{6} \rfloor = 0$. Furthermore, $\lfloor x \rfloor \lfloor y \rfloor = 0 \cdot 0 = 0$. Thus $\lfloor xy \rfloor = \lfloor x \rfloor \lfloor y \rfloor$, but $\frac{1}{2} \notin \mathbb{Z}$ and $\frac{1}{3} \notin \mathbb{Z}$. □

(b) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ so that $\lfloor xy \rfloor = \lfloor x \rfloor \lfloor y \rfloor$.
Proof. Let $x \in \mathbb{R}$ be arbitrary. Choose $y = 0$. Then $xy = 0$ and $\lfloor xy \rfloor = 0$. Furthermore, $\lfloor y \rfloor = 0$ so $\lfloor x \rfloor \lfloor y \rfloor = \lfloor x \rfloor \cdot 0 = 0$ and thus $\lfloor xy \rfloor = \lfloor x \rfloor \lfloor y \rfloor$. □

(c) There exists a real number x so that x is not an integer, $x > 2016$, and $\lfloor x^2 \rfloor = \lfloor x \rfloor^2$.
Proof. Let $x = 2016.0001$. It is clear that $x > 2016$ and $x \notin \mathbb{Z}$. (We need to show that $\lfloor x^2 \rfloor = \lfloor x \rfloor^2$.) Now $\lfloor x \rfloor^2 = 2016^2 = 4064256$ and

$$\lfloor x^2 \rfloor = \lfloor (2016.0001)^2 \rfloor = \lfloor 4064256.40320001 \rfloor = 4064256,$$

so $\lfloor x^2 \rfloor = \lfloor x \rfloor^2$. □

(d) $\forall N \in \mathbb{Z}^+, \exists x \in \mathbb{R}$ so that $x \notin \mathbb{Z}$, $x > N$, and $\lfloor x^2 \rfloor = \lfloor x \rfloor^2$.
Proof. Let $N \geq 1$ be an integer. Choose $x = N + \frac{1}{4N}$. Note that $0 < \frac{1}{4N} < 1$. Adding N to all parts of the inequality yields $N < N + \frac{1}{4N} < N + 1$, and thus

$$N < x < N + 1.$$

Therefore $\lfloor x \rfloor = N$ by definition of floor, and it is clear that x is not an integer. (We want to show that $\lfloor x^2 \rfloor = \lfloor x \rfloor^2$.) Now we show that $x^2 < N^2 + 1$, since

$$\begin{aligned}
x^2 = \left(N + \frac{1}{4N}\right)^2 &= N^2 + 2N\frac{1}{4N} + \frac{1}{16N^2} \\
&= N^2 + \frac{1}{2} + \frac{1}{16N^2} \\
&\leq N^2 + \frac{1}{2} + \frac{1}{16} && \text{where } \frac{1}{16N^2} \leq \frac{1}{16} \text{ since } N \geq 1 \\
&= N^2 + \frac{9}{16} \\
&< N^2 + 1 && \text{since } \frac{9}{16} < 1.
\end{aligned}$$

Furthermore, it is clear that $N^2 \leq x^2$. Thus

$$N^2 \leq x^2 < N^2 + 1.$$

Therefore $\lfloor x^2 \rfloor = N^2$ by the definition of floor. Hence $\lfloor x^2 \rfloor = \lfloor x \rfloor^2$. □

Problem 3.

Prove the following statements.

(a) $\forall a, b, d \in \mathbb{Z}$, if $d \mid a$ and $d \mid b$ then $d \mid (3a + 2b)$ and $d \mid (2a + b)$.

Proof. Let a, b, and d be arbitrary integers. Assume that $d \mid a$ and $d \mid b$. Then there exist integers k and m so that $a = kd$ and $b = md$. (We want to show that $d \mid (3a + 2b)$ and $d \mid (2a + b)$.) Now

$$\begin{aligned} 3a + 2b &= 3(kd) + 2(md) \\ &= (3k + 2m)d \end{aligned}$$

and $3k + 2m$ is an integer, so $d \mid (3a + 2b)$. Similarly,

$$\begin{aligned} 2a + b &= 2(kd) + (md) \\ &= (2k + m)d \end{aligned}$$

and $2k + m$ is an integer, so $d \mid (2a + b)$. □

(b) $\forall a, b, d \in \mathbb{Z}$, if $d \mid (3a + 2b)$ and $d \mid (2a + b)$ then $d \mid a$ and $d \mid b$.

Proof. Let a, b, and d be arbitrary integers. Assume that $d \mid (3a + 2b)$ and $d \mid (2a + b)$. Then there exist integers s and t so that $3a + 2b = sd$ and $2a + b = td$. (We want to show that $d \mid a$ and $d \mid b$.) Now

$$\begin{aligned} a &= 2(2a + b) - (3a + 2b) \\ &= 2td - sd \\ &= (2t - s)d \end{aligned}$$

and $2t - s$ is an integer, so $d \mid a$. Similarly,

$$\begin{aligned} b &= 2(3a + 2b) - 3(2a + b) \\ &= 2sd - 3td \\ &= (2s - 3t)d \end{aligned}$$

and $2s - 3t$ is an integer, so $d \mid b$. □

(c) $\forall a, b \in \mathbb{Z}^+$, $\gcd(a, b) = \gcd(3a + 2b, 2a + b)$.

Proof. Let $a, b \in \mathbb{Z}^+$ be arbitrary. Let $x = \gcd(a, b)$ and let $y = \gcd(3a + 2b, 2a + b)$.

By definition of Greatest Common Divisor, x is a common divisor of a and b. Thus $x \mid a$ and $x \mid b$. From part (a), we know that $x \mid (3a + 3b)$ and $x \mid (2a + b)$. Then x is a common divisor of $3a + 2b$ and $2a + b$. By definition of gcd, it must be the case that $x \leq \gcd(3a + 2b, 2a + b)$. Hence $\gcd(a, b) \leq \gcd(3a + 2b, 2a + b)$.

By definition of Greatest Common Divisor, y is a common divisor of $3a + 2b$ and $2a + b$. Thus $y \mid (3a + 2b)$ and $y \mid (2a + b)$. From part (b), we know that $y \mid a$ and $y \mid b$. Then y is a common divisor of a and b. By definition of gcd, it must be the case that $y \leq \gcd(a, b)$. Hence $\gcd(3a + 2b, 2a + b) \leq \gcd(a, b)$.

Since $\gcd(a, b) \leq \gcd(3a + 2b, 2a + b)$ and $\gcd(3a + 2b, 2a + b) \leq \gcd(a, b)$, it follows that $\gcd(a, b) = \gcd(3a + 2b, 2a + b)$. □

Proof (*alternate*). We can make use of Lemma 4.8.2 and use a method similar to the Euclidian Algorithm. Note that

$$\begin{aligned}(3a + 2b) &= 1 \cdot (2a + b) + (a + b)\\ (2a + b) &= 1 \cdot (a + b) + (a)\\ (a + b) &= 1 \cdot (a) + (b).\end{aligned}$$

From Lemma 4.8.2, this implies that

$$\begin{aligned}\gcd(3a + 2b, 2a + b) &= \gcd(2a + b, a + b)\\ &= \gcd(a + b, a)\\ &= \gcd(a, b),\end{aligned}$$

as desired. □