MATH 271 – Summer 2016
Assignment 2 – solutions

Problem 1.

Let N be your University of Calgary ID number.

(a) Use the Euclidean Algorithm to compute $\gcd(N, 271)$ and use this to find integers x and y so that $\gcd(N, 271) = Nx + 271y$.
Solution. (Answers will differ for different N.) Let's say that $N = 12345678$. Then

$$\begin{aligned} 12345678 &= 45556 \cdot 271 + 2 \\ 271 &= 135 \cdot 2 + 1 \\ 2 &= 2 \cdot 1 + 0. \end{aligned}$$

This means that

$$\begin{aligned} \gcd(12345678, 271) &= \gcd(271, 2) \\ &= \gcd(2, 1) \\ &= \gcd(1, 0) = 1. \end{aligned}$$

Then

$$\begin{aligned} 1 &= 271 - 135 \cdot 2 \\ &= 271 - 135 \cdot (12345678 - 45556 \cdot 271) \\ &= 271 + (135 \cdot 45556) \cdot 271 - 135 \cdot 12345678 \\ &= 271 + (6150060) \cdot 271 - 135 \cdot 12345678 \\ &= (-135) \cdot 12345678 + 6150061 \cdot 271, \end{aligned}$$

so we can set $x = -135$ and $y = 6150061$ so that $\gcd(N, 271) = Nx + 271y$.
Or, using the "table method":

		x	y
R_1	12345678	1	0
R_2	271	0	1
$R_3 = R_1 - 45556 \cdot R_2$	2	1	-45556
$R_4 = R_2 - 135 \cdot R_3$	1	-135	$1 - 135 \cdot (-45556)$

and $1 - 135 \cdot (-45556) = 6150061$.

(b) Suppose that M is an integer such that $\gcd(M, 271) = \gcd(M, 2016)$. Find $\gcd(M, 271)$. Explain how you get the answer.

Solution. $\gcd(M, 271) = 1$.

Explanation. We note that 271 is prime, so its positive divisors are only 1 and 271, thus $\gcd(M, 271)$ can only be 1 or 271. Next we note that $2016 = 2^5 \cdot 3^2 \cdot 7$ and all of the divisors of 2016 are

$$\{d \in \mathbb{Z}^+ \mid d \text{ divides } 2016\} = \{1, 2, 3, 4, 6, 7, 8, 9, 12, 14, 16, 18, 21, 24, 28, 32, 36, 42, 48, 56, 63, 72, 84, 96, 112, 126, 144, 168, 224, 252, 288, 336, 504, 672, 1008, 2016\}.$$

The only divisor that 271 and 2016 have in common is 1. Thus, the only way to have $\gcd(M, 271) = \gcd(M, 2016)$ is when $\gcd(M, 271) = \gcd(M, 2016) = 1$. □

Since $\gcd(K, 2016)$ must be a divisor of 2016 that satisfies $271 > \gcd(K, 2016) > 250$, the only divisor of 2016 that satisfies $271 > d > 250$ is $d = 252$.

(c) Suppose that K is an integer between 800,000 and 900,000 so that $\gcd(K, 271) > \gcd(K, 2016) > 250$. Find all possible values of K. Explain how you get the answers.
Solution. $K = 887,796$.

Explanation. From the condition that $\gcd(K, 271) > \gcd(K, 2016) > 250$ and from part (b), we conclude that $\gcd(K, 271) = 271$ and $\gcd(K, 2016) = 252$. Thus K must be a multiple of both 271 and 252, so K must be a multiple of $271 \cdot 252$. The only multiples of $271 \cdot 252$ that are between 800,000 and 900,000 are

$$12 \cdot 271 \cdot 252 = 819,504 \qquad \text{and} \qquad 13 \cdot 271 \cdot 252 = 887,796.$$

However, $\gcd(12 \cdot 271 \cdot 252, 2016) = 1008$. So K cannot be $12 \cdot 271 \cdot 252$. Hence K must be equal to $13 \cdot 271 \cdot 252$. □

Problem 2.

Consider the sequence of Fibonacci numbers $f_1, f_2, f_3, \ldots$ which are defined as follows: $f_1 = 1$, $f_2 = 1$, and

$$f_n = f_{n-1} + f_{n-2}$$

for all integers $n \geq 3$.

(a) Prove that for all integers $n \geq 3$, $\gcd(f_n, f_{n-1}) = \gcd(f_{n-1}, f_{n-2})$. (You may use Lemma 4.8.2. No induction is needed.)

Proof. Let $n \geq 3$ be an arbitrary integer. Let $a = f_n$, $b = f_{n-1}$, $r = f_{n-2}$, and $q = 1$. Then

$$a = qb + r$$

since $f_n = f_{n-1} + f_{n-2}$ by definition of the sequence. From Lemma 4.8.2, we see that $\gcd(a, b) = \gcd(b, r)$. Thus $\gcd(f_n, f_{n-1}) = \gcd(f_{n-1}, f_{n-2})$. □

(b) Prove by weak induction that $\gcd(f_n, f_{n-1}) = 1$ for all integers $n \geq 2$. (Use part (a).)

Proof. We prove this by induction.

Base case ($n = 2$): Note that $\gcd(f_2, f_1) = \gcd(1, 1) = 1$.

Inductive step: Let $k \geq 2$ be an integer. Suppose that

$$\gcd(f_k, f_{k-1}) = 1. \tag{IH}$$

(We want to show that $\gcd(f_{k+1}, f_k) = 1$.) Now $k + 1 \geq 2$, so using part (a) we see that

$$\begin{aligned} \gcd(f_{k+1}, f_k) &= \gcd(f_k, f_{k-1}) && \text{from part (a)} \\ &= 1 && \text{by IH,} \end{aligned}$$

which is what we wanted to show.

Therefore, by induction, $\gcd(f_n, f_{n-1}) = 1$ for all integers $n \geq 2$ □

(c) Prove by weak induction that $\displaystyle\sum_{i=1}^{n} (f_i)^2 = f_{n+1} f_n$ for all integers $n \geq 1$.

Proof. We prove this by induction.

Base case ($n = 1$): Note that $\sum_{i=1}^{1}(f_i)^2 = (f_1)^2 = 1$ and $f_2 f_1 = 1 \cdot 1 = 1$. Hence $\sum_{i=1}^{1}(f_i)^2 = f_{1+1}f_1$.

Inductive step: Let $k \geq 1$ be an integer. Suppose that

$$\sum_{i=1}^{k}(f_i)^2 = f_{k+1}f_k. \tag{IH}$$

(We want to show that $\sum_{i=1}^{k+1}(f_i)^2 = f_{k+2}f_{k+1}$.) Now

$$\begin{aligned}
\sum_{i=1}^{k+1}(f_i)^2 &= \sum_{i=1}^{k}(f_i)^2 + (f_{k+1})^2 \\
&= f_{k+1}f_k + (f_{k+1})^2 && \text{by IH} \\
&= f_{k+1}(f_k + f_{k+1}) \\
&= f_{k+1}(f_{k+2}) && \text{since } f_{k+2} = f_{k+1} + f_k \\
&= f_{k+2}f_{k+1}
\end{aligned}$$

which is what we wanted to show.

Therefore, by induction, $\sum_{i=1}^{n}(f_i)^2 = f_{n+1}f_n$ for all integers $n \geq 1$. □

Problem 3.

The sequence $s_0, s_1, s_2, \ldots,$ is defined by $s_0 = 1$, and for all integers $n > 0$,

$$s_n = s_{\lfloor \frac{n}{2} \rfloor} + s_{\lfloor \frac{2n}{3} \rfloor} + n.$$

The sequence $t_0, t_1, t_2, \ldots,$ is defined by $t_0 = 2$, $t_1 = 3$, and for all integers $n > 0$, $t_n = 3t_{n-1} - 2t_{n-2}$.

(a) Find s_1, s_2, s_3, s_4, s_5, s_6, s_7, and s_8. Guess the smallest integer a so that $s_n > 4n$ for all integers $n \geq a$.
Solution: We see that

$$\begin{aligned}
s_1 &= s_{\lfloor \frac{1}{2} \rfloor} + s_{\lfloor \frac{2}{3} \rfloor} + 1 = s_0 + s_0 + 1 = 1 + 1 + 1 && =3 \\
s_2 &= s_{\lfloor \frac{2}{2} \rfloor} + s_{\lfloor \frac{4}{3} \rfloor} + 2 = s_1 + s_1 + 2 = 3 + 3 + 2 && =8 \\
s_3 &= s_{\lfloor \frac{3}{2} \rfloor} + s_{\lfloor \frac{6}{3} \rfloor} + 3 = s_1 + s_2 + 3 = 3 + 8 + 3 && =14 \\
s_4 &= 20 \\
s_5 &= 27 \\
s_6 &= 40 \\
s_7 &= 41 \\
s_8 &= 55,
\end{aligned}$$

so we can guess that $s_n > 4n$ for all $n \geq 3$.

(b) Prove by strong induction that $s_n > 4n$ for all integers $n \geq a$, where a is the integer you chose in part (a).

Proof. We prove this by induction.

Base cases:

($n=3$): Note that $s_3 = 14$ and $4 \cdot 3 = 12$. Thus $s_3 > 4 \cdot 3$ since $14 > 12$.
($n=4$): Note that $s_4 = 20$ and $4 \cdot 4 = 16$. Thus $s_3 > 4 \cdot 4$ since $20 > 16$.
($n=5$): Note that $s_5 = 27$ and $4 \cdot 5 = 20$. Thus $s_4 > 4 \cdot 5$ since $27 > 20$.

Inductive step: Let $k \geq 5$ be an integer. Suppose that

$$s_i > 4i \qquad \text{for all integers } i \quad 3 \leq i \leq k. \tag{IH}$$

(We want to show that $s_{k+1} > 4(k+1)$.) First note that $\lfloor x \rfloor + 1 > x$ is true for all $x \in \mathbb{R}$ by definition of the floor. Thus

$$\lfloor x \rfloor > x - 1 \qquad \text{for all } x \in \mathbb{R}. \tag{$*$}$$

Furthermore, note that

$$3 \leq \left\lfloor \frac{k+1}{2} \right\rfloor \leq k \quad \text{and} \quad 3 \leq \left\lfloor \frac{2k+2}{3} \right\rfloor \leq k \quad \text{are both true since} \quad k \geq 5, \tag{$**$}$$

so we may use the induction hypothesis. Now

$$\begin{aligned}
s_{k+1} &= s_{\lfloor \frac{k+1}{2} \rfloor} + s_{\lfloor \frac{2k+2}{3} \rfloor} + k + 1 && \text{by definition of the sequence} \\
&> 4\left\lfloor \frac{k+1}{2} \right\rfloor + 4\left\lfloor \frac{2k+2}{3} \right\rfloor + k + 1 && \text{by IH and } (**) \\
&> 4\left(\frac{k+1}{2} - 1\right) + 4\left(\frac{2k+2}{3} - 1\right) + k + 1 && \text{by } (*) \\
&= 2(k-1) + \frac{4}{3}(2k-1) + k + 1 \\
&= \left(2 + \frac{4}{3} + 1\right)k - 2 - \frac{4}{3} + 1 \\
&= \frac{17}{3}k - \frac{7}{3} \\
&= \frac{12}{3}k + 4 + \frac{5}{3}k - 4 - \frac{7}{3} \\
&= 4k + 4 + \frac{5}{3}k - \frac{19}{3} \\
&= 4(k+1) + \frac{1}{3}(5k - 19) \\
&> 4(k+1) && \text{where } \frac{1}{3}(5k-19) > 0 \text{ since } k \geq 5,
\end{aligned}$$

which is what we wanted to show.

Therefore, by induction, $s_n > 4n$ for all integers $n \geq 3$. $\square$

(c) Find t_2, t_3, t_4, t_5, and t_6. Guess a formula for t_n.
Solution: We see that

$$\begin{aligned}
t_2 &= 3t_1 - 2t_0 = 3 \cdot 3 - 2 \cdot 2 = 9 - 4 && = 5 && = 4 + 1 && = 2^2 + 1 \\
t_3 &= 3t_2 - 2t_1 = 3 \cdot 5 - 2 \cdot 3 = 15 - 6 && = 9 && = 8 + 1 && = 2^3 + 1 \\
t_4 &= 3t_3 - 2t_2 = 3 \cdot 9 - 2 \cdot 5 = 27 - 10 && = 17 && = 16 + 1 && = 2^4 + 1 \\
t_5 &= 33 && && = 32 + 1 && = 2^5 + 1 \\
t_6 &= 65 && && = 64 + 1 && = 2^6 + 1
\end{aligned}$$

so we can guess that $t_n = 2^n + 1$ for all $n \geq 0$.

(d) Prove by strong induction that your guess in part (c) is correct for all integers $n \geq 0$.

Proof. We prove this by induction.

Base cases: (Note that first base case is $n = 0$.)

($n = 0$): Note that $t_0 = 2$ and $2^0 + 1 = 1 + 1 = 2$. Thus $t_0 = 2^0 + 1$.
($n = 1$): Note that $t_1 = 3$ and $2^1 + 1 = 2 + 1 = 3$. Thus $t_1 = 2^1 + 1$.

Inductive step: Let $k \geq 1$ be an integer. Suppose that

$$t_i = 2^i + 1 \qquad \text{for all integers } i \quad 0 \leq i \leq k. \tag{IH}$$

(We want to show that $t_{k+1} = 2^{k+1} + 1$.) Now

$$\begin{aligned}
t_{k+1} &= 3t_k - 2t_{k-1} && \text{by definition of the sequence} \\
&= 3\left(2^k + 1\right) - 2\left(2^{k-1} + 1\right) && \text{by IH} \\
&= 3 \cdot 2^k + 3 - 2 \cdot 2^{k-1} - 2 \\
&= 3 \cdot 2^k - 2^k + 1 \\
&= (3 - 1) \cdot 2^k + 1 \\
&= 2 \cdot 2^k + 1 \\
&= 2^{k+1} + 1,
\end{aligned}$$

which is what we wanted to show.

Therefore, by induction, $t_n = 2^n + 1$ for all integers $n \geq 0$. □