MATH 271 – Summer 2016
Assignment 3 – solutions

Problem 1. For any sets X and Y, we define the *symmetric difference* of X and Y as

$$X \triangle Y = (X \cup Y) - (X \cap Y).$$

Note that it is also true that $X \triangle Y = (X - Y) \cup (Y - X)$. For each of the following statements, determine whether the statement is true or false. Prove the true statements using the element method. Prove the false statements false by giving a counterexample.

(a) "For all sets A, $A \triangle A = \emptyset$."
Solution. This statement is true.
Proof. Let A be a set. We will show that $A \triangle A = \emptyset$. Assume instead that $A \triangle A \neq \emptyset$. Then there exists an element $x \in A \triangle A$. By definition of $\triangle$, this means that $x \in A - A$. However $A - A = \emptyset$, so $x \in \emptyset$. This is a contradiction, so the assumption that $A \triangle A \neq \emptyset$ must be wrong. Therefore $A \triangle A = \emptyset$. ☐

(b) "For all sets A, B, and C, if $A \subseteq B \cup C$ and $B \subseteq C \cup A$ then $A \triangle B = C$."
Solution. This statement is false.
Negation: "There exist sets A, B, and C so that $A \subseteq B \cup C$ and $B \subseteq C \cup A$ but $A \triangle B \neq C$."
Proof (of negation). Let $A = \{1\}$, $B = \{1\}$, and $C = \{1, 2\}$. Then $B \cup C = \{1, 2\}$ and $C \cup A = \{1, 2\}$, so $A \subseteq B \cup C$ and $B \subseteq C \cup A$ since $\{1\} \subseteq \{1, 2\}$. However $A \triangle B = \emptyset$, since $B = A$ and we know that $A \triangle A = \emptyset$ from part (a). Thus $A \triangle B \neq C$ since $\emptyset \neq \{1, 2\}$. ☐

(c) "For all sets A, B, and C, if $A \triangle C = B \triangle C$ then $A = B$."
Solution. This statement is true.
Proof. Let A, B, and C be sets. Assume that $A \triangle C = B \triangle C$. We will show that $A = B$ by showing that $A \subseteq B$ and $B \subseteq A$.

- (To show that $A \subseteq B$.) Let $x \in A$. We will show that x is also in B. There are two cases: either $x \in C$ or $x \notin C$.
  - Case *i*) Suppose that $x \in C$. Then $x \notin A \triangle C$ since $x \in A$ and $x \in C$. So $x \notin B \triangle C$ since $A \triangle C = B \triangle C$. This means that $x \notin B \cup C$ or $x \in B \cap C$. But $x \in C$ means that $x \in B \cup C$. Thus it must be the case that $x \in B \cap C$, which means that $x \in B$ and $x \in C$.
  - Case *ii*) Suppose that $x \notin C$. Then $x \in A \triangle C$ since $x \in A$ and $x \notin C$. So $x \in B \triangle C$ since $A \triangle C = B \triangle C$. This means that $x \in B \cup C$ and $x \notin B \cap C$. But $x \notin C$ means that $x \notin B \cap C$. Thus it must be the case that $x \in B \cap C$, which means that $x \in B$ since $x \notin C$ and $x \in B \cup C$.

  In either case, $x \in B$. Therefore $A \subseteq B$.
- (To show that $B \subseteq A$.) Let $x \in B$. We will show that x is also in A. There are two cases: either $x \in C$ or $x \notin C$.
  - Case *i*) Suppose that $x \in C$. Then $x \notin B \triangle C$ since $x \in B$ and $x \in C$. So $x \notin A \triangle C$ since $B \triangle C = A \triangle C$. This means that $x \notin A \cup C$ or $x \in A \cap C$. But $x \in C$ means that $x \in A \cup C$. Thus it must be the case that $x \in A \cap C$, which means that $x \in A$ and $x \in C$.
  - Case *ii*) Suppose that $x \notin C$. Then $x \in B \triangle C$ since $x \in B$ and $x \notin C$. So $x \in A \triangle C$ since $B \triangle C = A \triangle C$. This means that $x \in A \cup C$ and $x \notin A \cap C$. But $x \notin C$ means that $x \notin A \cap C$. Thus it must be the case that $x \in A \cap C$, which means that $x \in A$ since $x \notin C$ and $x \in A \cup C$.

  In either case, $x \in A$. Therefore $B \subseteq A$.

Thus $A \subseteq B$ and $B \subseteq A$, which means that $A = B$. ☐

Problem 2. Consider the set $S = \{1,2,3,4,5,6,7,8,9,10\}$. For parts (a) through (e), you must give a brief explanation on how you get the answer. Simplify your answer to a number.

(a) How many nonempty subsets of S have the property that the product of their elements is even? (For example, $T = \{1,5,8\}$ is a nonempty subset of S, and the product of the elements of T is $1 \cdot 5 \cdot 8 = 40$.)

Solution. There are $2^{10} - 2^5 = 992$ subsets of S that have the property that the product of their elements is even.

Explanation. Subsets of S that have the property that the product of their elements is even must contain at least one even number. Let A be the set of subsets of S that contain no even numbers. Then the set we are interested in is $\mathcal{P}(S) - A$, since this is the set of subsets that contain at least one even number. Note that $|\mathcal{P}(S)| = 2^{10}$ and that $A = \mathcal{P}(\{1,3,5,7,9\})$, hence $|A| = 2^5$. Thus

$$\begin{aligned} |\mathcal{P}(S) - A| &= |\mathcal{P}(S)| - |A| \\ &= 2^{10} - 2^5, \end{aligned}$$

as desired. ☐

(b) How many subsets of S have exactly 5 elements?

Solution. There are $\binom{10}{5} = 252$ subsets of S that contain exactly 5 elements.

Explanation. To make a subset of S that contains 5 elements, we choose 5 elements from the 10 in S. ☐

(c) How many subsets of S have 3 as their smallest element?

Solution. There are $2^7 = 128$ subsets of S that have 3 as their smallest element.

Explanation. A subset of S that has 3 as its smallest element must contain 3, and it cannot contain 1 or 2. To make a subset of S that 3 as its smallest element, we first make any subset of the 7 remaining elements $\{4,5,6,7,8,9,10\}$ and we then include 3 in it. There are 2^7 possibilities. ☐

(d) How many subsets of S have 3 as their smallest element and have exactly 5 elements?

Solution. There are $\binom{7}{4} = 35$ subsets of S with exactly 5 elements that have 3 as their smallest element.

Explanation. A subset of S that has 3 as its smallest element must contain 3, and it cannot contain 1 or 2. To make a subset of S with exactly 5 elements that 3 as its smallest element, we choose 4 of the 7 remaining elements $\{4,5,6,7,8,9,10\}$ and we include 3 in it. There are $\binom{7}{4}$ possibilities. ☐

(e) How many subsets of S have 6 as their smallest element and have exactly 5 elements?

Solution. There are $\binom{4}{4} = 1$ subsets of S with exactly 5 elements that have 6 as their smallest element.

Explanation. A subset of S that has 6 as its smallest element must contain 6, and it cannot contain any of the numbers from 1 to 5. To make a subset of S with exactly 5 elements that 6 as its smallest element, we choose 4 of the 4 remaining elements $\{7,8,9,10\}$ and we include 6 in it. There are $\binom{4}{4}$ possibilities. (That is, there is only one possibility since the only set that fits the criteria is the set $\{6,7,8,9,10\}$.) ☐

(f) Use the method of combinatorial proof to prove the following identity:

$$\binom{10}{5} = \binom{9}{4} + \binom{8}{4} + \binom{7}{4} + \binom{6}{4} + \binom{5}{4} + \binom{4}{4}.$$

Use complete sentences. (Hint: Find two different ways to count the number of subsets of S that have exactly 5 elements. Use parts (b), (d), and (e).)

Proof. We count the number of subsets of S that contain exactly 5 elements in two different ways. For the first way, we choose 5 elements from the 10, as in part (b). So there are

$$\binom{10}{5} \tag{$*$}$$

total subsets of S with exactly 5 elements.

For the second way of counting the number of subsets of S with exactly 5 elements, we split up the counting into six different types of subsets. For any subset of S with exactly 5 elements, there are 6 possibilities for the smallest element of that subset.

- We first count the number of subsets of S that have exactly 5 elements and have 1 as their smallest element. There are $\binom{9}{4}$ of these, since such subsets must include 1 and we then choose 4 of the 9 remaining elements.
- Similarly, the number of subsets of S with 5 elements and have 2 as their smallest element is $\binom{8}{4}$.
- The number of subsets of S with exactly 5 elements and have 3 as their smallest element is $\binom{7}{4}$.
- The number of subsets of S with exactly 5 elements and have 4 as their smallest element is $\binom{6}{4}$.
- The number of subsets of S with exactly 5 elements and have 5 as their smallest element is $\binom{5}{4}$.
- The number of subsets of S with exactly 5 elements and have 6 as their smallest element is $\binom{4}{4}$.

This accounts for all of the possible subsets of S with exactly 5 elements. So there are

$$\binom{9}{4} + \binom{8}{4} + \binom{7}{4} + \binom{6}{4} + \binom{5}{4} + \binom{4}{4} \tag{$**$}$$

total subsets of S that contain exactly 5 elements. Since the number of subsets must be the same no matter how we count them, the two numbers in $(*)$ and $(**)$ must be the same. This proves the identity. □

Problem 3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = 2\lfloor x \rfloor - x$ for each $x \in \mathbb{R}$. Let $g\colon \mathbb{R} \to \mathbb{R}$ be the function defined by $g(x) = \frac{x}{x^2+1}$ for all $x \in \mathbb{R}$.

(a) Prove that f is one-to-one.

Proof. Let $x_1, x_2 \in \mathbb{R}$ and suppose that $f(x_1) = f(x_2)$. That is,

$$2\lfloor x_1 \rfloor - x_1 = 2\lfloor x_2 \rfloor - x_2. \tag{1}$$

(We want to show that $x_1 = x_2$.) From the definition of floor, we see that

$$0 \le x_1 - \lfloor x_1 \rfloor < 1 \qquad \text{and} \qquad 0 \le x_1 - \lfloor x_2 \rfloor < 1.$$

Set $r_1 = x_1 - \lfloor x_1 \rfloor$ and $r_2 = x_2 - \lfloor x_2 \rfloor$ so that $x_1 = \lfloor x_1 \rfloor + r_1$ and $x_2 = \lfloor x_2 \rfloor + r_2$, where r_1 and r_2 are real numbers so that $0 \le r_1 < 1$ and $0 \le r_2 < 1$. From this we know that

$$-1 < r_1 - r_2 < 1. \tag{2}$$

Now (1) becomes $2\lfloor x_1 \rfloor - (\lfloor x_1 \rfloor + r_1) = 2\lfloor x_2 \rfloor - (\lfloor x_2 \rfloor + r_2)$, which simplifies to

$$r_1 - r_2 = \lfloor x_1 \rfloor - \lfloor x_2 \rfloor. \tag{3}$$

From (2) and (3), we see that $r_1 - r_2$ is an integer (since $\lfloor x_1 \rfloor - \lfloor x_2 \rfloor$ is an integer) and $r_1 - r_2$ is strictly greater than -1 and strictly less than 1. Therefore $r_1 - r_2 = 0$, since the only integer between -1 and 1 is zero. Thus $r_1 = r_2$. Hence $\lfloor x_1 \rfloor - \lfloor x_2 \rfloor = 0$ and thus $\lfloor x_1 \rfloor = \lfloor x_2 \rfloor$. Therefore $x_1 = \lfloor x_1 \rfloor + r_1 = \lfloor x_2 \rfloor + r_2 = x_2$. This means that f is one-to-one. □

(b) Prove that f is onto.

Proof. Let $y \in \mathbb{R}$ be arbitrary. Choose $x = 2\lceil y \rceil - y$. We will show that $f(x) = y$.

We first show that, for all numbers $z \in \mathbb{R}$, $\lfloor -z \rfloor = -\lceil z \rceil$.

Let z be an arbitrary real number. By definition of floor, $\lfloor -z \rfloor$ is the unique integer such that

$$\lfloor -z \rfloor \leq -z < \lfloor -z \rfloor + 1. \tag{4}$$

Multiplying (4) by -1, we see that this becomes $-\lfloor -z \rfloor \geq z > -\lfloor -z \rfloor - 1$. Flipping this around, we see that

$$-\lfloor -z \rfloor - 1 < z \leq -\lfloor -z \rfloor. \tag{5}$$

Where $-\lfloor -z \rfloor$ is an integer. But, from the definition of ceiling, $\lceil z \rceil$ is the unique integer such that

$$\lceil z \rceil - 1 < z \leq \lceil z \rceil. \tag{6}$$

Examining (5) and (6) we see that $-\lfloor -z \rfloor = \lceil z \rceil$, since they must be the same integer (from the definition of ceiling). Hence $\lfloor -z \rfloor = -\lceil z \rceil$.

Now $\lfloor 2\lceil y \rceil - y \rfloor = 2\lceil y \rceil + \lfloor -y \rfloor$, since $2\lceil y \rceil$ is an integer. Thus

$$\begin{aligned}
f(x) &= 2\lfloor x \rfloor - x \\
&= 2\lfloor 2\lceil y \rceil - y \rfloor - (2\lceil y \rceil - y) \\
&= 2(2\lceil y \rceil + \lfloor -y \rfloor) - 2\lceil y \rceil + y \\
&= 2(2\lceil y \rceil - \lceil y \rceil) - 2\lceil y \rceil + y && \text{since } \lfloor -y \rfloor = -\lceil y \rceil \\
&= 2\lceil y \rceil - 2\lceil y \rceil + y \\
&= y.
\end{aligned}$$

Thus f is onto. □

(c) Is g one-to-one? Prove your answer.

Solution. g is not one-to-one.

Proof (that g is not one-to-one). Let $x_1 = \frac{1}{4}$ and $x_2 = 4$. It is clear that $x_1 \neq x_2$. However

$$\begin{aligned}
g(x_1) &= \frac{\frac{1}{4}}{\left(\frac{1}{4}\right)^2 + 1} \\
&= \frac{4}{16 + 1} \\
&= \frac{4}{17}
\end{aligned}$$

and

$$\begin{aligned}
g(x_2) &= \frac{4}{4^2 + 1} \\
&= \frac{4}{16 + 1} \\
&= \frac{4}{17}.
\end{aligned}$$

Thus $g(x_1) = g(x_2)$ but $x_1 \neq x_2$ which means that g is not one-to-one. □

(d) Is g onto? Prove your answer.

Solution. g is not onto.

Proof (*that g is not onto*). Let $y = 1$. We will show that, for all $x \in \mathbb{R}$, $g(x) \neq y$. Suppose that there is an $x \in \mathbb{R}$ so that $g(x) = 1$. Then $\frac{x}{x^2+1} = 1$ which means that $x^2 - x + 1 = 0$. Using the quadratic equation, we see that

$$x = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{-3}}{2}.$$

But this is not a real number, since $\sqrt{-3} \notin \mathbb{R}$. This is a contradiciton. Hence there is no $x \in \mathbb{R}$ so that $g(x) = 1$. Thus g is not onto. □

Note: For any $x \in \mathbb{R}$ with $x \neq 0$, it is true that $g(x) = g(\frac{1}{x})$.