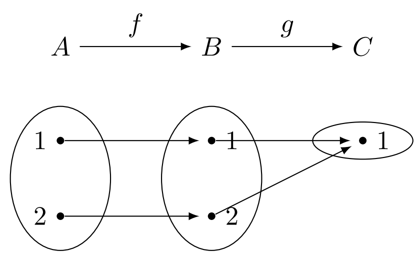
MATH 271 – Summer 2016
Assignment 4 – solutions

Problem 1.

Let A, B, and C be some sets and suppose that $f\colon A \to B$ and $g\colon B \to C$ are functions. Prove or disprove each of the following statements.

(a) If f is one-to-one then $g \circ f$ is one-to-one.

Solution. This statement is false. Consider the functions f and g defined by the following diagrams:



Then f is clearly one-to-one, but $g \circ f$ is not one-to-one.

(b) If both f and g are one-to-one then $g \circ f$ is one-to-one.

Solution. This statement is true.

Proof. Assume that f is one-to-one and g is one-to-one. (We will show that $g \circ f$ is one-to-one.) Let $a_1, a_2 \in A$ and assume that $(g \circ f)(a_1) = (g \circ f)(a_2)$. (We will show that $a_1 = a_2$.) Then

$$g\big(f(a_1)\big) = g\big(f(a_2)\big)$$

which means that $f(a_1) = f(a_2)$ since g is one-to-one. But f is also one-to-one, which means that $a_1 = a_2$. □

(c) If $g \circ f$ is one-to-one then f is one-to-one.

Solution. This statement is true.

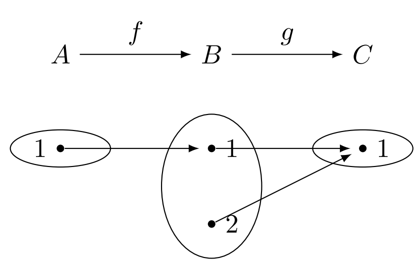
Proof. Assume that $g \circ f$ is one-to-one. (We will show that f is one-to-one.) Let $a_1, a_2 \in A$ ans assume that $f(a_1) = f(a_2)$. (We will show that $a_1 = a_2$.) Since $f(a_1) = f(a_2)$, applying g to both sides gives us $g\big(f(a_1)\big) = g\big(f(a_2)\big)$ or

$$(g \circ f)(a_1) = (g \circ f)(a_2).$$

But $g \circ f$ is one-to-one, so $a_1 = a_2$. □

(d) If $g \circ f$ is one-to-one then g is one-to-one.

Solution. This statement is false. Consider the functions defined by the following diagrams:



Then $g \circ f$ is clearly one-to-one, but g is not one-to-one.

(e) If $g \circ f$ is one-to-one and f is onto then g is one-to-one.

Solution. This statement is true.

Proof. Assume that $g \circ f$ is one-to-one and that f is onto. (We will show that g is one-to-one.) Let $b_1, b_2 \in B$ and assume that $g(b_1) = g(b_2)$. (We will show that $b_1 = b_2$.) Since f is onto, there exists an $a_1 \in A$ so that $f(a_1) = b_1$. Similarly, there exists an $a_2 \in A$ so that $f(a_2) = b_2$. Since $g(b_1) = g(b_2)$, this gives us

$$g\big(f(a_1)\big) = g\big(f(a_2)\big),$$

which means $(g \circ f)(a_1) = (g \circ f)(a_2)$. But $g \circ f$ is one-to-one, which means that $a_1 = a_2$. Hence $f(a_1) = f(a_2)$ and thus

$$b_1 = f(a_1) = f(a_2) = b_2,$$

so $b_1 = b_2$. □

Problem 2.

Consider the set $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and let $\mathcal{F}$ denote the set of all functions from S to S. Define a relation R on $\mathcal{F}$ by:

$$\text{for all } f, g \in \mathcal{F}, \quad f\,R\,g \text{ if and only if } \exists x \in S \text{ so that } f(x) = g(x).$$

Let $\alpha \in \mathcal{F}$ be the function defined by $\alpha(x) = 1$ for each $x \in S$. Let $h \in \mathcal{F}$ be the function defined by $h(x) = \lfloor \frac{x+3}{2} \rfloor$ for each $x \in S$.

(a) Is R reflexive? Symmetric, Transitive? Prove your answers.

Solution. The relation R is reflexive and symmetric, but not transitive.

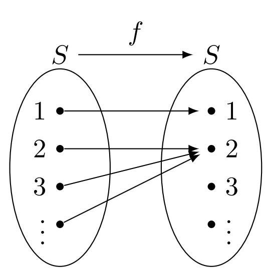
Proof (that R is reflexive). Let $f \in \mathcal{F}$ be arbitrary. Then $f(1) = f(1)$, so there is at least one $x \in S$ so that $f(x) = f(x)$. Hence $f\,R\,f$, so R is reflexive. □

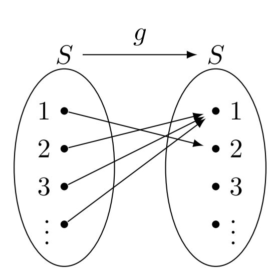
Proof (that R is symmetric). Let $f, g \in \mathcal{F}$ be arbitrary and assume that $f\,R\,g$. Then there is at least one $x \in S$ so that $f(x) = g(x)$. Hence $g(x) = f(x)$ for that x, which means that $g\,R\,f$, so R is symmetric. □

Proof (that R is not transitive). Let $f, g \in \mathcal{F}$ be the functions defined by

$$f(x) = \begin{cases} 1, & \text{if } x = 1, \\ 2, & \text{if } x \neq 1 \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} 2, & \text{if } x = 1, \\ 1, & \text{if } x \neq 1 \end{cases}$$

Function diagrams for these two functions are the following:





Then $f\,R\,\alpha$ since $f(1) = 1 = \alpha(1)$ and $\alpha\,R\,g$ since $g(2) = 1 = \alpha(2)$. But $f \not\!R\, g$ since $f(x) \neq g(x)$ for all $x \in S$. Thus R is not transitive. □

(b) Prove or disprove the statement: "$\exists f \in \mathcal{F}$ so that $\forall g \in \mathcal{F}$, $f\, R\, g$".

Solution. This statement is false. Its negation is: "$\forall f \in \mathcal{F}$, $\exists g \in \{$ so that $f \not R\, g$."

Proof (*of the negation*). Let $f \in \mathcal{F}$ be arbitrary. Pick g to be the function defined by

$$g(x) = \begin{cases} 2, & \text{if } f(x) = 1, \\ 1, & \text{if } f(x) \neq 1 \end{cases}$$

It is clear that $g(x) \neq f(x)$ for all $x \in S$. Indeed, let x be an element of S. Then $f(x) = 1$ or $f(x) \neq 1$. If $f(x) = 1$, then $g(x) = 2 \neq 1$. If $f(x) \neq 1$, then $g(x) = 1 \neq f(x)$. In either case, $g(x) \neq f(x)$ for all $x \in S$. This proves the statement. □

(c) How many functions $f \in \mathcal{F}$ are there so that $f\, R\, \alpha$? Explain.

Solution. There are $10^{10} - 9^{10}$. The reasoning is as follows. Consider the set

$$A = \{f \in \mathcal{F} \,|\, f\, R\, \alpha\}.$$

We want to count $|A|$. It is easier to count the complement $A^c = \mathcal{F} - A$, which is the set of functions that are *not* related to α. That is,

$$A^c = \{f \in \mathcal{F} \,|\, f \not R\, \alpha\}.$$

Note that $|A| = |\mathcal{F}| - |A^c|$. The recipe for counting the functions that are not related to α is as follows:

1. Choose a value for $f(1)$. It cannot be 1, otherwise $f(1) = \alpha(1)$. There are 9 other choices.
2. Choose a value for $f(2)$. It cannot be 1, otherwise $f(2) = \alpha(2)$. There are 9 other choices.

⋮

10. Choose a value for $f(10)$. It cannot be 1, otherwise $f(10) = \alpha(10)$. There are 9 other choices.

Hence $|A^c| = 9^{10}$. There are 10^{10} total functions from S to S, so

$$|A| = |\mathcal{F}| - |A^c| = 10^{10} - 9^{10}.$$

(d) How many functions $f \in \mathcal{F}$ are there so that $f\, R\, h$? Explain.

Solution. There are $10^{10} - 9^{10}$. The reasoning is as follows. Consider the set

$$H = \{f \in \mathcal{F} \,|\, f\, R\, h\}.$$

We want to count $|H|$. It is easier to count the complement $H^c = \mathcal{F} - H$, which is the set of functions that are *not* related to h. That is,

$$H^c = \{f \in \mathcal{F} \,|\, f \not R\, h\}.$$

Note that $|H| = |\mathcal{F}| - |H^c|$. The recipe for counting the functions that are not related to h is as follows:

1. Choose a value for $f(1)$. It cannot be 2 (since $h(1) = 2$), otherwise $f(1) = h(1)$. There are 9 other choices.
2. Choose a value for $f(2)$. It cannot be 2 (since $h(2) = 2$), otherwise $f(2) = h(2)$. There are 9 other choices.
3. Choose a value for $f(3)$. It cannot be 3 (since $h(3) = 3$), otherwise $f(3) = h(3)$. There are 9 other choices.

⋮

10. Choose a value for $f(10)$. It cannot be 6 (since $h(10) = 6$), otherwise $f(10) = h(10)$. There are 9 other choices.

Hence $|H^c| = 9^{10}$. There are 10^{10} total functions from S to S, so

$$|H| = |\mathcal{F}| - |H^c| = 10^{10} - 9^{10}.$$

(e) How many functions $f \in \mathcal{F}$ are there so that $f \not R \alpha$ or $f \not R h$? Explain.

Solution. The answer is $2 \cdot 9^{10} - 8^{10}$. The reasoning is as follows.

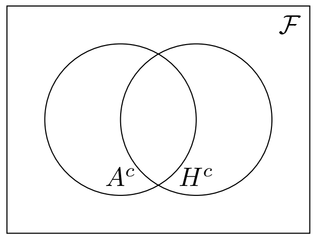
The functions that we want to count are the functions $f \in \mathcal{F}$ such that

$$f \not R \alpha \quad \text{or} \quad f \not R h.$$

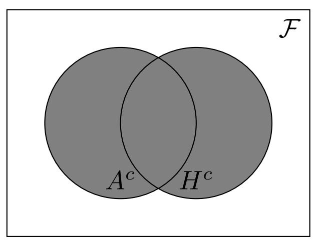
Note that $f \not R \alpha \Leftrightarrow f \notin A \Leftrightarrow f \in A^c$. Similarly, note that $f \not R h \Leftrightarrow f \notin H \Leftrightarrow f \in H^c$. This means that we want to count the functions f so that

$$f \in A^c \quad \text{or} \quad f \in H^c.$$

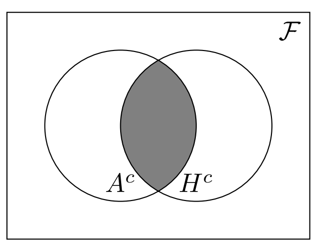
That is, we want to count the functions f in $A^c \cup H^c$. Consider the following picture:



We want to count the union of A^c and H^c. That is, we want to count the shaeded region of the following diagram:



The number of elements in this set is $|A^c \cup H^c| = |A^c| + |H^c| - |A^c \cap H^c|$, where $A^c \cap H^c$ is the set that is the shaded region of the following diagram:



Note that $f \in A^c \cap H^c$ means that $f \not R \alpha$ **and** $f \not R h$. We can count the nubmer of functions that are not related to both α and h by the following recipe:

1. Choose a value for $f(1)$. It cannot be 1, otherwise $f(1) = \alpha(1) = 1$, and it cannot be 2 (since $h(1) = 2$), otherwise $f(1) = h(1)$. There are 8 other choices.
2. Choose a value for $f(2)$. It cannot be 1, otherwise $f(2) = \alpha(2) = 1$, and it cannot be 2 (since $h(2) = 2$), otherwise $f(2) = h(2)$. There are 8 other choices.

3. Choose a value for $f(3)$. It cannot be 1, otherwise $f(3) = \alpha(3) = 1$, and it cannot be 3 (since $h(3) = 3$), otherwise $f(3) = h(3)$. There are 8 other choices.

$\vdots$

10. Choose a value for $f(10)$. It cannot be 1, otherwise $f(10) = \alpha(10) = 1$, and it cannot be 6 (since $h(10) = 6$), otherwise $f(10) = h(10)$. There are 8 other choices.

Hence $|A^c \cap H^c| = 8^{10}$. Now

$$|A^c \cup H^c| = |A^c| + |H^c| - |A^c \cap H^c| = 9^{10} + 9^{10} - 8^{10}.$$

Problem 3.

Consider the set $\mathbb{Z}^+$ of all positive integers. Let S be the relation on $\mathbb{Z}^+ \times \mathbb{Z}^+$ defined by

$$\text{for all } (a,b) \text{ and } (c,d) \text{ in } \mathbb{Z}^+ \times \mathbb{Z}^+ \qquad (a,b)\, S\, (c,d) \text{ if and only if } a + 2b = c + 2d$$

(a) Prove that S is an equivalence relation on $\mathbb{Z}^+ \times \mathbb{Z}^+$.

Solution. *Proof.* We prove that S is reflexive, symmetric, and transitive.

- (*Reflexive*) Let $(a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ be an arbitrary pair of positive integers. Then $a + 2b = a + 2b$, so $(a,b)\, S\, (a,b)$. Hence S is reflexive.
- (*Symmetric*) Let $(a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ and $(c,d) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ be arbitrary pairs of positive integers. Assume that $(a,b)\, S\, (c,d)$. Then $a + 2b = c + 2d$ which means that $c + 2d = a + 2b$. Hence $(c,d)\, S\, (a,b)$ and thus S is symmetric.
- (*Transitive*) Let $(a,b), (c,d), (e,f) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ be arbitrary pairs of positive integers. Assume that $(a,b)\, S\, (b,c)$ and $(b,d)\, S\, (e,f)$. Then $a + 2b = c + 2d$ and $c + 2d = e + 2f$ and thus $a + 2b = d + 2f$ since "=" is transitive. Hence $(a,b)\, S\, (e,f)$ and thus S is transitive.

Thus S is an equivalence relation because it is reflexive, symmetric, and transitive. □

(b) List all elements of $[(3,3)]$ and all elements of $[(4,4)]$.

Solution. Note that $3 + 2 \cdot 3 = 9$. Then $(a,b)\, S\, (3,3)$ if and only if $a + 2b = 9$. The elements of $[(3,3)]$ are

$$[(3,3)] = \{(1,4),\ (3,3),\ (5,2),\ (7,1)\}.$$

Note that $4 + 2 \cdot 4 = 12$. Then $(a,b)\, S\, (4,4)$ if and only if $a + 2b = 12$. The elements of $[(4,4)]$ are

$$[(4,4)] = \{(2,5),\ (4,4),\ (6,3),\ (8,2),\ (10,1)\}.$$

(c) Is there an equivalence class of S that has exactly 271 elements? Explain.

Solution. Yes. Consider the equivalence class of $(1, 271)$. The elements are

$$\begin{aligned}[(1,271)] &= \{(1,271),\ (3,270),\ (5,269),\ \ldots,\ (541,1)\} \\ &= \{(543 - 2k, k) \mid k = 1, 2, \ldots, 271\},\end{aligned}$$

which has 271 elements.

(d) How many equivalence classes of S are there that contain at most 271 elements?

Solution. There are $2 \times 271 = 542$ equivalence classes that contain at most 271 elements. Note that for each $n \in \mathbb{Z}^+$, there are two exactly equivalence classes that contain exactly n elements. Indeed, we see that

$$[(1,1)] = \{(1,1)\} \qquad \text{and} \qquad [(2,1)] = \{(2,1)\}$$

are the only equivalence classes that contain exactly 1 element. For each n, the equivalence classes

$$[(1,n)] = \{(1,n),(3,n-1),(5,n-2),\ldots,(2n-1,1)\}$$

and

$$[(2,n)] = \{(2,n),(4,n-1),(6,n-2),\ldots,(2n,1)\}$$

are the only classes that contain exactly n elements. For each $n = 1,2,3\ldots,271$ there are exactly 2 equivalence classes containig n elements. So there are a total of $2 \cdot 271$ classes that have at most 271 elements.