MATH 271 – Summer 2016
Extra practice problems – Week 6
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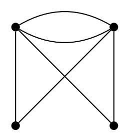
Here are some questions to help you study graph theory for the final.

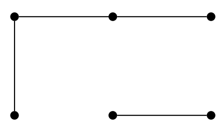
1. (a) Draw a graph with exactly 4 vertices and 6 edges.

   Solution. One example of such a graph is:



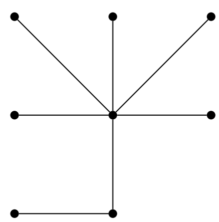
   (b) Draw a graph with exactly 6 vertices and 4 edges and exactly two connected components.

   Solution. One example of such a graph is:



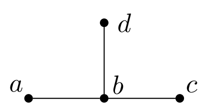
   (c) Draw a **simple** connected graph with exactly 8 vertices, one of which has degree 6.

   Solution. One example of such a graph is:



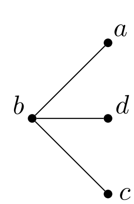
   (d) Does there exist a graph with exactly 8 vertices, so that three of the vertices have degree 3 and the remaining five vertices have degree 2? Explain.

   Solution. No. If such a graph existed, the sum of the degrees of all of its vertices would be: $3+3+3+2+2+2+2+2 = 19$, which is odd. However, the total sum of the degrees of the vertices of any graph must be even. Thus such a graph cannot exist.

2. Let G be the graph 

   (a) Is G bipartite?

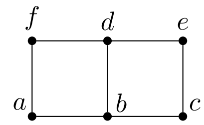
   Solution. Yes. We can redraw the graph as:



to see that the vertices can be partitioned as $\{b\}$ and $\{a, c, d\}$ so that no two vertices in the same subset are connected by an edge.

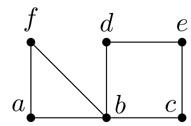
(b) Draw a simple graph H with exactly six vertices a, b, c, d, e, f and exactly seven edges and so that G is a subgraph of H.

Solution. One example of such a graph is:

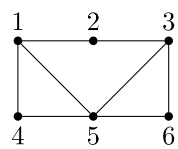


(c) Draw a simple graph F with exactly six vertices a, b, c, d, e, f so that G is a subgraph of F and F has an Euler circuit.

Solution. One example of such a graph is:



3. Consider the graph G given by



(a) Is G bipartite?

Solution. No. We will prove this as follows.

Proof (*that G is not bipartite*). Suppose that G were bipartite. Then the set of vertices can be partitioned into two disjoint subsets V_1 and V_2 such that the vertices in each subset are connected by edges only to vertices in the other subset and not to vertices in the same subset. Say that 1 is in V_1. Since vertices 4 and 5 are both connected to 1, both 4 and 5 must be in the same subset, say V_2. But this contradicts the statement that no two vertices in V_2 are connected, since 4 and 5 are connected by an edge. Hence G is not bipartite. □

(b) Does G have an Euler trail?

Solution. Yes. One trail would be the trail that starts at 1 and ends at 3 given by:

$$1 \to 4 \to 5 \to 6 \to 3 \to 5 \to 1 \to 2 \to 3.$$

This walk uses every edge exactly once and does not repeat any edges.

(c) Does G have an Euler circuit?

Solution. No, because G has at least one vertex with odd degree. Namely, vertices 1 and 3 have degree 3, which is odd. In order to have an Euler circuit, every vertex must have even degree.

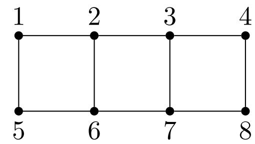
(d) Does G have a Hamiltonian circuit?

Solution. Yes. One circuit would be the circuit that starts at and ends at 1 given by:

$$1 \to 4 \to 5 \to 6 \to 3 \to 2 \to 1.$$

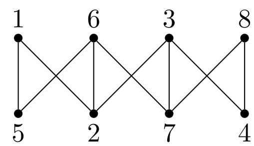
This walk uses every vertex exactly once and does not repeat any edges.

4. Consider the graph given by



(a) Is the graph bipartite?

Solution. Yes. One way to redraw the graph so that the bipartition is clear is:



None of the vertices in $\{1, 6, 3, 8\}$ are adjacent to each other and none of the vertices in $\{5, 2, 7, 4\}$ are adjacent to each other.

(b) Does this have an Euler trail?

Solution. No. There are more than two vertices that have odd degree. So this graph cannot have an Euler trail.

(c) Does the graph have an Euler circuit?

Solution. No. At least one vertex has odd degree, so there is no Euler circuit.

(d) Does the graph have a Hamiltonian circuit?

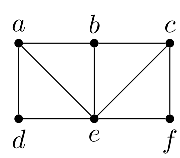
Solution. Yes. One circuit would be the circuit that starts at and ends at 1 given by:

$$1 \to 2 \to 3 \to 4 \to 8 \to 7 \to 2 \to 6 \to 5 \to 1.$$

This walk uses every vertex exactly once and does not repeat any edges.

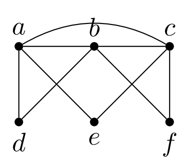
5. (a) Draw a **simple** graph with exactly six vertices and exactly nine edges.

Solution. One example of such a graph is:



(b) Draw a **simple** graph with exactly six vertices and exactly nine edges that is not bipartite but has an Euler circuit.

Solution. One example of such a graph is:



This graph is simple, because it has no loops and has no parallel edges. This graph is not bipartite since it is not possible to split up the the vertices a, b, and d into two separate sets so that none of the elements in each set are adjacent.

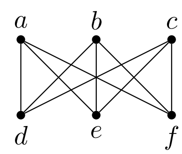
Finally, the graph has an Euler circuit because it is connected and each vertex has even degree. One example of an Euler circuit would be the closed walk that starts and ends at a given by:

$$a \to b \to d \to a \to e \to c \to b \to f \to c \to a.$$

This walk uses every edge exactly once, and starts and ends at the same point.

(c) Draw a **simple** graph with exactly six vertices and exactly nine edges that is bipartite but does not have an Euler circuit.

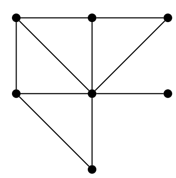
Solution. One example of such a graph is:



This graph is clearly bipartite since the upper points are only adjacent to the lower points and vice versa. It does not have an Euler trail, because it has at least one vertex with odd degree. Namely, all of the vertices have odd degree!

(a) Draw a **simple** graph G with exactly seven vertices and exactly ten edges, and so that some vertex of G has degree 6.

Solution. One example of such a graph is:

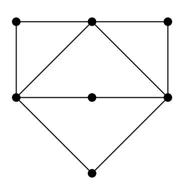


(b) Answer part (a) again, but so that your graph G does **not** have an Euler circuit. (Be sure to explain why you know that G does not have an Euler circuit.)

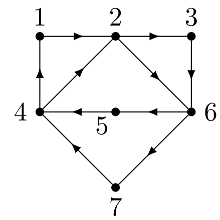
Solution. The graph given in the solution to part (a) does not have an Euler circuit, because at least one vertex has odd degree.

(c) Answer part (a) again, but so that your graph G **does** have an Euler circuit. (Be sure to explain why you know that G has an Euler circuit.)

Solution. One example of such a graph is:



This graph is simple, since it does not have any loops or parallel edges. This graph also has an Euler circuit. One example of an Euler circuit on this graph would be the circuit labeled here:

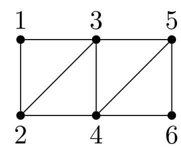


that starts and ends at vertex 1 and goes $1 \to 2 \to 3 \to 6 \to 5 \to 4 \to 2 \to 6 \to 7 \to 4 \to 1$. (Alternatively, we see that it must have an Euler circuit because it is connected and every vertex has even degree.)

6. Let G be the graph with vertices labeled $\{1, 2, 3, 4, 5, 6\}$, and for any two vertices i and j, there is an edge connecting vertex i and vertex j if and only if $1 \leq |i - j| \leq 2$.

   (a) Draw the graph G.

   Solution. The graph G looks like



   (b) Is G bipartite? Explain.

   Solution. No. There is no way to put the vertices 1, 2, and 3 into only two disjoint sets so that vertices in each subset are only connected to vertices in the other subset.

   (c) Does G have an Euler circuit? Explain.

   Solution. No. The vertices labeled 2 and 5 have odd degree, so there can not be an Euler circuit.

   (d) Does G have an Euler trail? Explain.

   Solution. Yes. One possible Euler trail starts at 2 and ends at 5 and looks like:

$$2 \to 1 \to 3 \to 2 \to 4 \to 6 \to 5 \to 4 \to 3 \to 5.$$

   This uses every edge exactly once.

   (e) Does G have a Hamiltonian circuit? Explain.

   Solution. Yes. One possible Hamiltonian circuit starts and ends at vertex 1 and looks like:

$$1 \to 2 \to 4 \to 6 \to 5 \to 3 \to 1.$$

   This goes through every vertex exactly once (with the exception of the start/end point).

For more practice, try the following problems from the book which have solutions in the back.

- Section 10.1: Problems 1, 3, 8, 15, 17, 18, 21, 24, 37ab.
- Section 10.2: Problems 1, 4, 8a, 9a, 12, 14, 19, 23.