MATH 271 – Summer 2016
Practice problems – Week 5
University of Calgary
Mark Girard

Part I (functions)

1. Let $T = \{1, 2, 3\}$, let $f\colon T \to T$ and $g\colon T \to T$ be defined by by $f = \{(1,2),(2,3),(3,1)\}$ and $g = \{(1,2),(2,1),(3,3)\}$. Draw the arrow diagrams for f and g. Determine each of the following functions as a collection of ordered pairs.
   (a) f^{-1}
   (b) g^{-1}
   (c) $f \circ g$
   (d) $g \circ f$

2. Let $A = \{-1, 0, 1\}$ and let $F : A \to A$ be the function defined by $F(n) = \lceil \frac{n}{2} \rceil$ for all $n \in A$.
   (a) Is F one-to-one? Prove your answer.
   (b) Is F onto? Prove your answer.
   (c) Does there exist a function from A to A that is one-to-one but not onto? Prove your answer.
   (d) Does there exist a function from A to A that is onto but not one-to-one? Prove your answer.

3. Define the functions $h\colon \mathbb{Z} \to \mathbb{Z}$ and $g\colon \mathbb{Z} \to \mathbb{Z}$ by $h(n) = 3n$ and $g(n) = \lfloor \frac{n}{2} \rfloor$ for each $n \in \mathbb{Z}$. Prove or disprove each of the following statements.
   (a) h is one-to-one.
   (b) g is onto.
   (c) $h \circ g$ is onto.
   (d) $h \circ g$ is one-to-one.
   (e) $g \circ h$ is onto.
   (f) $g \circ h$ is one-to-one.

4. Let A, B, and C be some sets and suppose that $f\colon A \to B$ and $g\colon B \to C$ are functions. Prove or disprove each of the following statements.
   (a) If $g \circ f$ is onto then f is onto.
   (b) If $g \circ f$ is onto then g is onto.
   (c) If $g \circ f$ is onto and g is one-to-one then f is onto.

5. Find two functions $f\colon \mathbb{Z} \to \mathbb{Z}$ and $g\colon \mathbb{Z} \to \mathbb{Z}$ so that $f \circ g = I_{\mathbb{Z}}$ but f and g are not invertible.

6. Let $t\colon \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ be the function defined by $t(x, y) = x + y\sqrt{2}$ for all $(x, y) \in \mathbb{Q} \times \mathbb{Q}$. Is t one-to-one? Is t onto? Prove your answers.

7. Let $H\colon (\mathbb{R} - \{1\}) \to (\mathbb{R} - \{1\})$ be the function defined by $H(x) = \frac{x+1}{x-1}$ for each $x \in \mathbb{R} - \{1\}$.
   (a) Show that H is one-to-one.
   (b) Show that H is onto.
   (c) Find a formula for $H^{-1}(x)$ such that $H^{-1} \circ H = H \circ H^{-1} = I_{\mathbb{R}-\{1\}}$.

Part II (relations)

1. Let $B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. For each of the following relations, draw the directed graph. For each of the relations, determine whether it is reflexive, symmetric, or transitive.

   (a) Define the relation Q on A by $a\,Q\,b \Leftrightarrow a \mid b$.

   (b) Define the relation R on A by $a\,R\,b \Leftrightarrow 3 \mid (a - b)$.

   (c) Define the relation S on A by $a\,S\,b \Leftrightarrow 5 \mid (a^2 - b^2)$.

   (d) Define the relation T on A by $a\,T\,b \Leftrightarrow 1 \leq |a - b| \leq 3$.

   (e) Define the relation V on A by $a\,V\,b \Leftrightarrow \lfloor \frac{a}{3} \rfloor \leq \lfloor \frac{b}{3} \rfloor$.

2. Let $A = \{1, 2, 3, 4\}$. For each of the following questions, describe your relations as a subset of $A \times A$ (for example $R = \{(1, 2), (2, 1)\}$) and draw its directed graph.

   (a) Find a relation R on A that is reflexive, but is neither symmetric nor transitive.

   (b) Find a relation R on A that is transitive, but is neither reflexive nor symmetric.

   (c) Find a relation R on A that is symmetric, but is neither reflexive nor transitive.

   (d) Find a relation R on A that is reflexive and symmetric, but not transitive.

   (e) Find a relation R on A that is reflexive and transitive, but not symmetric.

   (f) Find a relation R on A that is symmetric and transitive, but not reflexive.

3. Let R be the relation on $\mathbb{Z}^+ \times \mathbb{Z}^+$ defined by

$$\forall (a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \quad \text{and } \forall (c, d) \in \mathbb{Z}^+ \times \mathbb{Z}^+, \qquad (a, b)\,R\,(c, d) \text{ if and only if } a + b < c + d.$$

   (a) Is R reflexive? Symmetric? Transitive? Prove your answers.

   (b) Is it true that, for all $(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, there exists $(c, d) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ so that $(a, b)\,R\,(c, d)$? Prove your answer.

   (c) Is it true that, for all $(c, d) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, there exists $(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ so that $(a, b)\,R\,(c, d)$? Prove your answer.

   (d) Is it true that there exists $(c, d) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ so that for all $(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, $(a, b)\,R\,(c, d)$? Prove your answer.

   (e) How many elements (a, b) in $\mathbb{Z}^+ \times \mathbb{Z}^+$ are there so that $(a, b)\,R\,(3, 3)$?