MATH 271 – Summer 2016
Quiz 2 – Solutions

1. (*4 points*) Use the method of proof by contradiction to prove the following statement: "For all real numbers x, if x^2 is irrational then x is irrational."

Proof. Let x be an arbitrary real number. Assume that x^2 is irrational. We will show that x is irrational by contradiction. Suppose that x is rational. Then there exist integers a and b such that $x = \frac{a}{b}$ and $b \neq 0$. Now

$$\begin{aligned} x^2 &= \left(\frac{a}{b}\right)^2 \\ &= \frac{a^2}{b^2} \end{aligned}$$

where a^2 and b^2 are integers and $b^2 \neq 0$ since $b \neq 0$. Thus x^2 is rational, but x^2 was assumed to be irrational. This is a contradiction, so the assumption that x is rational must be wrong. Therefore x is irrational. $\square$

2. (*4 points*) Use the Euclidean Algorithm to compute $\gcd(181, 123)$. Find integers x and y such that $\gcd(181, 123) = 181x + 123y$.

Solution: Note that

$$\begin{aligned} 181 &= 1 \cdot 123 + 58 \\ 123 &= 2 \cdot 58 + 7 \\ 58 &= 8 \cdot 7 + 2 \\ 7 &= 3 \cdot 2 + 1 \\ 2 &= 2 \cdot 1 + 0. \end{aligned} \qquad (*)$$

This implies that

$$\begin{aligned} \gcd(181, 123) &= \gcd(123, 58) \\ &= \gcd(58, 7) \\ &= \gcd(7, 2) \\ &= \gcd(2, 1) \\ &= \gcd(1, 0) = 1. \end{aligned}$$

Substituting in values from $(*)$, we find

$$\begin{aligned} 1 &= 7 - 3 \cdot 2 \\ &= 7 - 3 \cdot (58 - 8 \cdot 7) \\ &= (-3) \cdot 58 + 25 \cdot 7 \\ &= (-3) \cdot 58 + 25 \cdot (123 - 2 \cdot 58) \\ &= 25 \cdot 123 - 53 \cdot 58 \\ &= 25 \cdot 123 - 53 \cdot (181 - 123) \\ &= (-53) \cdot 181 + 78 \cdot 123. \end{aligned}$$

We can set $x = -53$ and $y = 78$ so that $\gcd(181, 123) = 181x + 123y$.
Alternatively, using the "table method":

		x	y
R_1	181	1	0
R_2	123	0	1
$R_3 = R_1 - R_2$	58	1	-1
$R_4 = R_2 - 2R_3$	7	-2	3
$R_5 = R_3 - 8R_4$	2	17	-25
$R_6 = R_4 - 3R_5$	1	-53	78

3. (*7 points*) Use mathematical induction to prove that $5^n - 4n - 1$ is divisible by 16 for all integers $n \geq 1$.

Proof. We prove this by induction.

Base case ($n = 1$): $5^1 - 4 \cdot 1 - 1 = 5 - 4 - 1 = 0$ and 0 is divisible by 16. So 16 divides $5^1 - 4 \cdot 1 - 1$.

Inductive step: Let $k \geq 1$ be an integer. Assume that $5^k - 4k - 1$ is divisible by 16 (IH). Then there is an integer m such that

$$5^k - 4k - 1 = 16m \tag{$*$}$$

by IH. (We want to show that $5^{k+1} - 4(k+1) - 1$ is divisible by 16.) Now

$$\begin{aligned}
5^{k+1} - 4(k+1) - 1 &= 5 \cdot 5^k - 4(k+1) - 1 \\
&= 5 \cdot (\underbrace{5^k - 4k - 1}_{=16m} + 4k + 1) - 4(k+1) - 1 \\
&= 5 \cdot 16m - 20k + 5 - 4k - 4 - 1 \qquad \text{because } 5^k - 4k - 1 = 16m \text{ from } (*) \\
&= 16 \cdot 5m - 16k \\
&= 16(5m - k).
\end{aligned}$$

Therefore $5^{k+1} - 4(k+1) - 1$ is divisible y 16 since $5m - k$ is an integer.

By induction, $5^n - 4n - 1$ is divisible by 16 for all integers $n \geq 1$ ☐