# MATH 271 – Summer 2016
# Quiz 3 – Solutions

**1.** (*8 points*) Of the two following statements, one is true and one is false. Determine which statement is true and which is false. Prove the true statement and disprove the false statement. Use the element method.

(a) For all sets $A$, $B$, and $C$, if $A - B = C$ then $A = B \cup C$.
This statement is false.
**Negation**: There exist sets $A$, $B$, and $C$ so that $A - B = C$ but $A \neq B \cup C$.

*Proof (of negation).* Let $A = \{2\}$, $B = \{2, 3\}$ and $C = \emptyset$. Then $A - B = \{2\} - \{2, 3\} = \emptyset$ and $B \cup C = \{2, 3\} \cup \emptyset = \{2, 3\}$. But $\{2\} \neq \{2, 3\}$ and thus $A \neq B \cup C$. □

(b) For all sets $A$, $B$, and $C$, if $A - B \subseteq C$ then $A - C \subseteq B$.
This statement is true.

*Proof.* Let $A$, $B$, and $C$ be arbitrary sets. Suppose that $A - B \subseteq C$. (We want to show that $A - C \subseteq B$.) Let $x \in A - C$ be an arbitrary element. (We want to show that $x \in B$.) This means that $x \in A$ and $x \notin C$. Assume for the sake of getting a contradiction that $x \notin B$. Then $x \in A - B$ since $x \in A$ and $x \notin B$. Hence $x \in C$, since $x \in A - B$ and $A - B \subseteq C$. Thus $x \in C$ and $x \notin C$, which is a contradiction, so the assumption that $x \notin B$ is wrong. Therefore $x \in B$ and thus $A - C \subseteq B$. □

**2.** (*7 points*)
Consider the sequence $a_0, a_1, a_2, \ldots$ defined by $a_0 = 0$, $a_1 = 4$, and $a_n = 6a_{n-1} - 5a_{n-2}$ for all integers $n \geq 2$. Use strong mathematical induction to prove that $a_n = 5^n - 1$ for all integers $n \geq 0$.

*Proof.* We prove this by induction.

*Base cases*:

($n = 0$): Note that $a_0 = 0$ and $5^0 - 1 = 1 - 1 = 0$. Thus $a_0 = 5^0 - 1$.

($n = 1$): Note that $a_1 = 4$ and $5^1 - 1 = 5 - 1 = 4$. Thus $a_1 = 5^1 - 1$.

*Inductive step*: Let $k \geq 1$ be an arbitrary integer. Suppose that

$$a_i = 5^i - 1 \qquad \text{for all integers } i, \quad 0 \leq i \leq k. \tag{IH}$$

(We want to show that $a_{k+1} = 5^{k+1} - 1$.) Now

$$\begin{aligned}
a_{k+1} &= 6a_k - 5a_{k-1} && \text{by definition of the sequence} \\
&= 6(5^k - 1) - 5(5^{k-1} - 1) && \text{by IH (and because } 0 \leq k-1) \\
&= 6 \cdot 5^k - 6 - 5 \cdot 5^{k-1} + 5 \\
&= 6 \cdot 5^k - 5^k - 1 \\
&= (6 - 1) \cdot 5^k - 1 \\
&= 5 \cdot 5^k - 1 \\
&= 5^{k+1} - 1,
\end{aligned}$$

which is what we wanted to show.

Therefore, by induction, $a_n = 5^n - 1$ for all integers $n \geq 0$. □