# MATH 271 – Summer 2016
## Quiz 4 – solutions

For each of the following questions, give a brief explanation on how you get the answer.
You do not have to simplify your answer to a number.

**1.** (*4 points*) Consider all four-digit positive integers who digits are chosen from $\{1, 2, 3, 4, 5, 6\}$.

(a) How many of those numbers have at least one digit that is a 6?

**Solution.** The answer is $6^4 - 5^4$. The reasoning is as follows.

The total number of four-digit integers with digits chosen from $\{1, 2, 3, 4, 5, 6\}$ is $6^4$. All of the digits of these four integers can be any of the elements of $\{1, 2, 3, 4, 5, 6\}$. The recipe is as follows:

1. Choose the first digit. (6 choices)
2. Choose the second digit. (6 choices)
3. Choose the third digit. (6 choices)
4. Choose the fourth digit. (6 choices)

So the total number of four-digit integers with digits chosen from 1 to 6 is $6 \cdot 6 \cdot 6 \cdot 6 = 6^4$.

The total number of four-digit integers that have **no** 6's is $5^4$, since there are only 5 choices for each digit. Since we need to count the number of integers that have at least one 6, of which there are $6^4 - 5^4$.

(b) How many of those numbers have at least one digit that is a 6 **and** at least one digit that is a 5?

**Solution.** The answer is $6^4 - 5^4 - 5^4 + 4^4$. The reasoning is as follows.

Let $A$ be the set of four-digit integers with digits chosen from $\{1, 2, 3, 4, 5, 6\}$. Let $B$ be the subset of $A$ containing those integers that have no 6's and $C$ be the subset of $A$ containing integers that have no 5's. Then we are interested in counting the number of elements in $A - (B \cup C)$, since these are the integers with at least one 6 and one 5. Then

$$\begin{aligned} |A - (B \cup C)| &= |A| - |B \cup C| \\ &= |A| - (|B| + |C| - |B \cap C|) \\ &= |A| - |B| - |C| + |B \cap C|, \end{aligned}$$

where $|A| = 6^4$ and $|B| = |C| = 5^4$. Finally, the number of integers that have **no** 5's **and no** 6's is $|B \cap C| = 4^4$.

**2.** (*4 points*) Consider the letters of the word *CANADIAN*.

(a) How many ways can the letters be arranged?

**Solution.** There are $\binom{8}{3}\binom{5}{2}\binom{3}{1}\binom{2}{1}\binom{1}{1}$ ways. The reasoning is as follows.

There are 8 total letters, so there are 8 total spots to fill.

1. First choose three of the 8 spots to put the $A$'s. There are $\binom{8}{3}$ ways.
2. Choose two of the remaining 5 spots to put the $N$'s. There are $\binom{5}{2}$ ways.
3. Choose one of the remaining 3 spots to put the $C$. There are $\binom{3}{1}$ ways.
4. Choose one of the remaining 2 spots to put the $D$. There are $\binom{2}{1}$ ways.
5. Choose the one remaining spot to put the $I$. There are $\binom{1}{1}$ ways.

(b) How many of those arrangements do **not** have the two $N$'s next to each other?

**Solution.** There are $\binom{8}{3}\binom{5}{2}\binom{3}{1}\binom{2}{1}\binom{1}{1} - \binom{7}{1}\binom{6}{3}\binom{3}{1}\binom{2}{1}\binom{1}{1}$ ways. The reasoning is as follows.

We first count the number of ways that the letters can be arranged so that the two $N$'s **are** next to each other. We can think of '$NN$' as one letter, so there are now 7 spots for 7 'letters'.

1. First choose one of the 7 spots to put $NN$. There are $\binom{7}{1}$ ways.
2. Choose three of the remaining 6 spots to put the $A$'s. There are $\binom{6}{3}$ ways.
3. Choose one of the remaining 3 spots to put the $C$. There are $\binom{3}{1}$ ways.
4. Choose one of the remaining 2 spots to put the $D$. There are $\binom{2}{1}$ ways.
5. Choose the one remaining spot to put the $I$. There are $\binom{1}{1}$ ways.

**Solution (Alternate).** There are $\binom{6}{3}\binom{3}{1}\binom{2}{1}\binom{1}{1}\binom{7}{2}$ ways. The reasoning is as follows.

First make all of the combinations of the six letters $AAACDI$. There are 6 spots for 6 letters.

1. First choose 3 of the 6 spots to put the $A$'s. There are $\binom{6}{3}$ ways.
2. Choose one of the remaining 3 spots to put the $C$. There are $\binom{3}{1}$ ways.
3. Choose one of the remaining 2 spots to put the $D$. There are $\binom{2}{1}$ ways.
4. There are now 7 'spots' (either in-between letters or at the beginning or end) to put the two $N$'s.

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They must go in two different spots, and there are 7 to choose from. There are $\binom{7}{2}$ ways.

**3.** (*7 points*) Let $f\colon \mathbb{Z} \to \mathbb{Z}$ and $g\colon \mathbb{Z} \to \mathbb{Z}$ be functions defined by $f(x) = \left\lceil \frac{x+1}{3} \right\rceil$ and $g(x) = 2x$ for each $x \in \mathbb{Z}$.

(a) Is $g$ one-to-one? Prove your answer.

**Solution.** $g$ is one-to-one.

*Proof (that $g$ is one-to-one).* Let $x_1$ and $x_2$ be arbitrary integers. Assume that $g(x_1) = g(x_2)$. Then $2x_1 = 2x_2$, and dividing both sides by 2 give us $x_1 = x_2$. Hence $g$ is one-to-one. □

(b) Is $f$ onto? Prove your answer.

**Solution.** $f$ is onto.

*Proof (that $f$ is onto).* Let $y$ be an arbitrary integer. (We will show that there exists an $x \in \mathbb{Z}$ so that $f(x) = y$.) Pick $x = 3y - 1$. Then $x$ is an integer and

$$f(x) = f(3y-1) = \left\lceil \frac{3y-1+1}{3} \right\rceil = \left\lceil \frac{3y}{3} \right\rceil = \lceil y \rceil = y$$

since $y$ is an integer. Therefore $f$ is onto. □

(c) Is $f \circ g$ one-to-one? Prove your answer.

**Solution.** $f \circ g$ is not one-to-one.

*Proof* (*that $f \circ g$ is not one-to-one*). Let $x_1 = 0$ and $x_2 = 1$. Then

$$(f \circ g)(x_1) = f(g(0)) = f(2 \cdot 0) = \left\lceil \frac{2 \cdot 0 + 1}{3} \right\rceil = \left\lceil \frac{1}{3} \right\rceil = 1$$

and

$$(f \circ g)(x_2) = f(g(1)) = f(2 \cdot 1) = \left\lceil \frac{2 \cdot 1 + 1}{3} \right\rceil = \left\lceil \frac{3}{3} \right\rceil = 1,$$

so $(f \circ g)(x_1) = (f \circ g)(x_2)$ but $x_1 \neq x_2$. Therefore $f \circ g$ is not one-to-one. □

(d) Is $g \circ f$ onto? Prove your answer.

**Solution.** $g \circ f$ is not onto.

*Proof* (*that $g \circ f$ is not onto*). Pick $y = 1$. (We will show that, for all $x \in \mathbb{Z}$, $(g \circ f)(x) \neq y$.) Suppose that there exists an $x \in \mathbb{Z}$ so that $(g \circ f)(x) = y$. This means that

$$\begin{aligned} 1 = y &= (g \circ f)(x) \\ &= 2 \left\lceil \frac{x+1}{3} \right\rceil. \end{aligned}$$

Hence $\frac{1}{2} = \left\lceil \frac{x+1}{3} \right\rceil$. But $\left\lceil \frac{x+1}{3} \right\rceil$ must be an integer, by the definition of ceiling, and $\frac{1}{2}$ is not an integer. This is a contradiction, so the assumption that there exists an $x \in \mathbb{Z}$ so that $(g \circ f)(x) = y$ is wrong. Hence $g \circ f$ is not onto. □