MATH 271 – Summer 2016
Practice problem solutions – Week 1

Part I

For each of the following statements, determine whether the statement is true or false. Prove the true statements. For the false statement, write out its negation and prove that. For the conditional statements, write out the converse and the contrapositive. Determine whether they are true or false and give reasoning.

1. $\forall n \in \mathbb{Z}$, $n^2 + 2n$ is even.
   This statement is False.
   Negation: $\exists n \in \mathbb{Z}$ such that $n^2 + 2n$ is odd.

   Proof (of the negation). Let $n = 1$. Then $n^2 + 2n = 1^2 + 2 \cdot 1 = 3$, which is odd. □

2. $\exists n \in \mathbb{Z}$ such that $n^3 + n$ is odd.
   This statement is False.
   Negation: $\forall n \in \mathbb{Z}$, $n^3 + n$ is even.

   Proof (of the negation). Let $n \in \mathbb{Z}$ be arbitrary. There are two cases: either n is even or n is odd. For both cases, we need to show that $n^3 + n$ is even.

   - Suppose n is even. Then there is an integer k such that $n = 2k$. Then
   $$\begin{aligned} n^3 + n &= n(n^2 + 1) \\ &= 2k(n^2 + 1) \end{aligned}$$
   where $k(n^2 + 1)$ is an integer, therefore $n^3 + n$ is even.
   - Suppose n is odd. Then there is an integer k such that $n = 2k + 1$. Then
   $$\begin{aligned} n^3 + n &= n(n^2 + 1) \\ &= n((2k + 1)^2 + 1) \\ &= n((4k^2 + 2k + 1) + 1) \\ &= 2n(2k^2 + k + 1) \end{aligned}$$
   where $n(2k^2 + k + 1)$ is an integer, therefore $n^3 + n$ is even.

   In both cases $n^2 + n$ is even. □

3. $\forall x \in \mathbb{R}$, $x^2 - x \geq 0$.
   This statement is False.
   Negation: $\exists x \in \mathbb{R}$ such that $x^2 - x \geq 0$.

   Proof (of the negation). Let $x = \frac{1}{2}$. Then $x^2 - x = \left(\frac{1}{2}\right)^2 + \frac{1}{2} = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4} \ngeq 0$. □

4. $\forall n \in \mathbb{Z}$, $n^2 - n \geq 0$.
   This statement is true.

Proof. Let $n \in \mathbb{Z}$. Since x is an integer, it must be that $x^2 - x$ is also an integer. Then

$$\begin{aligned} x^2 - x &= x^2 - x + \frac{1}{4} - \frac{1}{4} \\ &= x^2 - 2 \cdot \frac{1}{2}x + \left(\frac{1}{2}\right)^2 - \frac{1}{4} \\ &= \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} \\ &\geq -\frac{1}{4} \qquad (*) \end{aligned}$$

where $(*)$ is true, because the square of any number is greater than 0. Thus $x^2 - x$ is an integer that is larger than or equal to $-\frac{1}{4}$. But any *integer* that is larger than or equal to $-\frac{1}{4}$ must be greater than or equal to 0. □

5. $\forall x, y \in \mathbb{Z}$, if $x^2 + 2x = y^2 + 2y$ then $x = y$.
   This statement is false.
   Negation: $\exists x, y \in \mathbb{Z}$ so that $x^2 + 2x = y^2 + 2y$ but $x \neq y$.

   Proof (of the negation). Let $x = 0$ and $y = -2$. Then $x^2 + 2x = 0$ and

   $$y^2 + 2y = (-2)^2 + 2(-2) = 4 - 4 = 0.$$

   So $x^2 + x = y^2 + y$ but $x \neq y$. □

   Contrapositive: $\forall x, y \in \mathbb{Z}$, if $x \neq y$ then $x^2 + 2x \neq y^2 + 2y$.
   The contrapositive is false, because it is logically equivalent to the original statement, which is false.

   Converse: $\forall x, y \in \mathbb{Z}$, if $x = y$ then $x^2 + 2x = y^2 + 2y$. The converse is true.

   Proof (of the converse). Let x and y be integers. Suppose $x = y$. Then $x^2 + 2x = y^2 + 2y$ trivially by substitution. □

6. $\forall x, y \in \mathbb{Z}$, if $2x^2 + x = 2y^2 + y$ then $x = y$.
   This statement is true.

   Proof. Let x and y be integers. Suppose that $2x^2 + x = 2y^2 + y$. By rearranging, it follows that

   $$\begin{aligned} 0 &= 2(x^2 - y^2) + x - y \\ &= (x - y)(2(x + y) + 1). \end{aligned}$$

   We note that $2(x+y)+1$ is an odd integer. Therefore $2(x+y)+1 \neq 0$. By the Zero Product Property, it follows that $x - y = 0$. Thus $x = y$. □

   Contrapositive: $\forall x, y \in \mathbb{Z}$, if $x \neq y$ then $2x^2 + x \neq 2y^2 + y$.
   The contrapositive is true, because it is logically equivalent to the original statement, which is true.

   Converse: $\forall x, y \in \mathbb{Z}$, if $x = y$ then $2x^2 + x = 2y^2 + y$. The converse is true.

   Proof (of the converse). Let x and y be integers. Suppose $x = y$. Then $2x^2 + x = 2y^2 + y$ trivially by substitution. □

7. $\forall a, b, c \in \mathbb{Z}$, if $a|(b+c)$ and $a|(b-c)$ then $a|b$ and $a|c$.
This statement is false.
Negation: $\exists a, b, c \in \mathbb{Z}$ so that$a|(b+c)$ and $a|(b-c)$ but $a \nmid b$ and $a \nmid c$.

Proof (*of the negation*). Let $a = 2$, $b = 3$, and $c = 1$. Then $b + c = 4 = 2 \cdot 2$, which is divisible by 2, and $b - c = 2 = 2 \cdot 1$, which is divisible by 2. But $2 \nmid 3$ and $2 \nmid 1$. So $a|(b+c)$ and $a|(b-c)$ but $a \nmid b$ and $a \nmid c$. □

Contrapositive:$\forall a, b, c \in \mathbb{Z}$, if $a \nmid b$ and $a \nmid c$ then $a \nmid (b+c)$ or $a \nmid (b-c)$.
The contrapositive is false, because it is logically equivalent to the original statement, which is false.

Converse: $\forall a, b, c \in \mathbb{Z}$, if $a|b$ and $a|c$ then $a|(b+c)$ and $a|(b-c)$. The converse is true.

Proof (*of the converse*). Let a, b, and c be integers. Suppose $a|b$ and $a|c$. Then there are integers k and m such that $b = ak$ and $c = am$. Now

$$\begin{aligned} b + c &= ak + am \\ &= a(k+m), \end{aligned}$$

so $b + c$ is divisible by a since $k + m$ is an integer, and

$$\begin{aligned} b - c &= ak - am \\ &= a(k-m) \end{aligned}$$

so $b - c$ is divisible by a since $k - m$ is an integer. □

8. $\forall n \in \mathbb{Z}$, $\exists m \in \mathbb{Z}$ such that $n + m$ is even.
This statement is true.

Proof. Let n be an arbitrary integer. Let $m = n$. Then $n + m = 2n$, which is even. □

9. $\exists m \in \mathbb{Z}$ such that $\forall n \in \mathbb{Z}$, $n + m$ is even.
This statement is false.
Negation: $\forall m \in \mathbb{Z}$,$\exists n \in \mathbb{Z}$ such that $n + m$ is odd.

Proof. Let m be an arbitrary integer. Let $n = m + 1$. Then $n + m = 2m + 1$, which is odd. □

10. $\forall r \in \mathbb{Q}$, $\exists m \in \mathbb{Z}$ such that $rm \in \mathbb{Z}$.
This statement is true.

Proof. Let $r \in \mathbb{Q}$. Let $m = 0$. Then $rm = 0$, which is an integer. □

11. $\exists m \in \mathbb{Z}$ such that $\forall r \in \mathbb{Q}$, $rm \in \mathbb{Z}$.
This statement is true.

Proof. Let m be an arbitrary integer. Let $r = 0$, which is rational. Then $rm = 0$, which is an integer. □

12. For all positive integers n, there exists a positive integer m so that $3|(n+m)$.
This statement is true.

Proof. Let n be an integer. Let $m = 2n$, which is an integer. Then $n + m = 3m$, which is divisible by 3. □

13. There exists a positive integer m so that for all positive integers n, $3|(n+m)$.
This statement is false.
Negation: For all positive integers m, there exists a positive integer n so that $3 \nmid (n+m)$.

Proof (of the negation). Let m be a positive integer. Let $n = 2m+1$. Then $n+m = 3m+1$. The remainder from dividing $3m+1$ by 3 is 1, since m is an integer. Hence $3 \nmid (3m+1)$ and thus $3 \nmid (n+m)$. □

Part II

For each of the following statements, prove or disprove the statement. Note that you can use the fact that $\sqrt{2}$ is irrational. For all other irrational numbers, you must prove that they are in fact irrational.

1. $\forall x, y \in \mathbb{R}$, $\lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ This statement is false.
Negation: $\exists x, y \in \mathbb{R}$ so that $\lfloor x+y \rfloor \neq \lfloor x \rfloor + \lfloor y \rfloor$.

Proof (of the negation). Let $x = \frac{1}{2}$ and $y = \frac{1}{2}$. Then $\lfloor x+y \rfloor = \lfloor \frac{1}{2}+\frac{1}{2} \rfloor = 1$ but $\lfloor x \rfloor + \lfloor y \rfloor = \lfloor \frac{1}{2} \rfloor + \lfloor \frac{1}{2} \rfloor = 0$. Thus $\lfloor x+y \rfloor \neq \lfloor x \rfloor + \lfloor y \rfloor$ since $0 \neq 1$. □

2. $\exists$ a positive real number a so that $\forall$ real numbers x, if $x - \lfloor x \rfloor < a$ then $\lfloor 3x \rfloor = 3\lfloor x \rfloor$.
This statement is true.

Proof. Let $a = \frac{1}{3}$. Let x be an arbitrary real number. Suppose that $x - \lfloor x \rfloor < a = \frac{1}{3}$. Let $n = \lfloor x \rfloor$. By definition of the floor, $n \leq x < n+1$. By supposition,

$$x - n < \frac{1}{3}, \qquad (*)$$

since $n = \lfloor x \rfloor$. Thus $x < n + \frac{1}{3}$, by adding n to both sides of the inequality in $(*)$. Hence

$$n \leq x < n + \frac{1}{3}$$

by combining the inequalities. Multiplying every part of the last inequality by 3, we have

$$3n \leq 3x < 3n+1,$$

where $3n$ is an integer, since n is an integer. By definition of floor, $\lfloor 3x \rfloor = 3n$. Therefore $\lfloor 3x \rfloor = 3\lfloor x \rfloor$, which is what we needed to prove. □

3. $2 - \sqrt{2}$ is irrational.

Proof. Suppose, for the sake of contradiction, that $2-\sqrt{2}$ is rational. Then there exist integers a and b such that $2 - \sqrt{2} = \frac{a}{b}$ and $b \neq 0$. Then

$$\begin{aligned}\sqrt{2} &= -\frac{a}{b} + 2 \\ &= \frac{2b-a}{b}\end{aligned}$$

where $2b-a$ is an integer and b is a nonzero integer. This is a contradiction to the irrationality of $\sqrt{2}$. Therefore $2-\sqrt{2}$ is irrational. □

4. $3\sqrt{2}$ is irrational.

Proof. Suppose, for the sake of contradiction, that $3\sqrt{2}$ is rational. Then there exist integers a and b such that $3\sqrt{2} = \frac{a}{b}$ and $b \neq 0$. Then

$$\begin{aligned}\sqrt{2} &= \frac{1}{3}\frac{a}{b} \\ &= \frac{a}{3b}\end{aligned}$$

where a is an integer and $3b$ is a nonzero integer. This is a contradiction to the irrationality of $\sqrt{2}$. Therefore $3\sqrt{2}$ is irrational. $\square$

5. $\forall x, y \in \mathbb{R}$, if x and y are irrational then $x + y$ is irrational. This statement is false.
   Negation: $\exists x, y \in \mathbb{R}$ so that x and y are irrational but $x + y$ is rational.

   Proof. Let $x = \sqrt{2}$ and $y = 2 - \sqrt{2}$. Then x is irrational and y is irrational (which we know from the problem above). But $x + y = 2$, which is rational. $\square$

6. $\forall x, y \in \mathbb{R}$, if x and y are irrational then xy is irrational. This statement is false.
   Negation: $\exists x, y \in \mathbb{R}$ so that x and y are irrational but xy is rational.

   Proof. Let $x = \sqrt{2}$ and $y = 3\sqrt{2}$. Then x is irrational and y is irrational (which we know from the previous problem). But $xy = 6$, which is rational. $\square$

7. $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ so that $x + y$ is rational.

   Proof. Let $x \in \mathbb{R}$ be arbitrary. Let $y = -x$. Then $x + y = x - x = 0$, which is rational. $\square$

8. $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ so that $x + y$ is irrational.

   Proof. Let $x \in \mathbb{R}$ be arbitrary. Let $y = \sqrt{2} - x$. Then $x + y = \sqrt{2}$, which is irrational. $\square$

9. $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ so that xy is irrational. This is false.
   Negation: $\exists x \in \mathbb{R}$ so that $\forall y \in \mathbb{R}$, xy is rational.

   Proof. Let $x = 0$. Let y be an arbitrary real number. Then $xy = 0$, which is rational. $\square$

10. $\forall x \in \mathbb{R}$ such that $x \neq 0$, $\exists y \in \mathbb{R}$ so that xy is irrational.

    Proof. Let x be a nonzero real number. Let $y = \frac{1}{x}\sqrt{2}$. Then $xy = \sqrt{2}$ is irrational. $\square$