MATH 271 – Summer 2016
Practice problems solutions– Week 2

Part I

For each pair of integers a and b, use the Euclidean Algorithm to compute $\gcd(a,b)$ and find integers x and y such that $\gcd(a,b) = ax + by$.

1. $a = 156$ and $b = 115$. We have

$$\begin{aligned} 156 &= 1 \cdot 115 + 41 \\ 115 &= 2 \cdot 41 + 33 \\ 41 &= 1 \cdot 33 + 8 \\ 33 &= 4 \cdot 8 + 1 \\ 8 &= 8 \cdot 1 + 0 \end{aligned}$$

This means that

$$\begin{aligned} \gcd(156,115) &= \gcd(115,41) \\ &= \gcd(41,33) \\ &= \gcd(33,8) \\ &= \gcd(8,1) \\ &= \gcd(1,0) = 1, \end{aligned}$$

so $\gcd(156,115) = 1$. To find integers x and y so that $1 = 156x + 115y$, note that

$$\begin{aligned} 1 &= 33 - 4 \cdot 8 \\ &= 33 - 4 \cdot (41 - 33) && (\text{since } 8 = 41 - 1 \cdot 33) \\ &= (-4) \cdot 41 + 5 \cdot 33 \\ &= (-4) \cdot 41 + 5 \cdot (115 - 2 \cdot 41) && (\text{since } 33 = 115 - 2 \cdot 41) \\ &= 5 \cdot 115 - 14 \cdot 41 \\ &= 5 \cdot 115 - 14 \cdot (156 - 1 \cdot 115) && (\text{since } 41 = 156 - 1 \cdot 115) \\ &= (-14) \cdot 156 + 19 \cdot 115. \end{aligned}$$

We can set $x = -14$ and $y = 19$ so that $\gcd(156,115) = 156x + 115y$.

Alternately, using the "table method":

		x	y
R_1	156	1	0
R_2	115	0	1
$R_3 = R_1 - R_2$	41	1	-1
$R_4 = R_2 - 2R_3$	33	-2	3
$R_5 = R_3 - R_4$	8	3	-4
$R_6 = R_4 - 4R_5$	1	-14	19

2. $a = 132$ and $b = 76$. We have

$$\begin{aligned} 132 &= 1 \cdot 76 + 56 \\ 76 &= 1 \cdot 56 + 20 \\ 56 &= 2 \cdot 20 + 16 \\ 20 &= 1 \cdot 16 + 4 \\ 16 &= 4 \cdot 4 + 0, \end{aligned}$$

so $\gcd(132, 76) = 4$. Using the "table method":

		x	y
R_1	132	1	0
R_2	76	0	1
$R_3 = R_1 - R_2$	56	1	-1
$R_4 = R_2 - R_3$	20	-1	2
$R_5 = R_3 - 2R_4$	16	3	-5
$R_6 = R_4 - R_5$	4	-4	7

We can set $x = -4$ and $y = 7$ so that $\gcd(132, 76) = 132x + 76y$.

3. $a = 2016$ and $b = 271$. We have

$$\begin{aligned}
2016 &= 7 \cdot 271 + 119 \\
271 &= 2 \cdot 119 + 33 \\
119 &= 3 \cdot 33 + 20 \\
33 &= 1 \cdot 20 + 13 \\
20 &= 1 \cdot 13 + 7 \\
13 &= 1 \cdot 7 + 6 \\
7 &= 1 \cdot 6 + 1 \\
6 &= 6 \cdot 1 + 0,
\end{aligned}$$

so $\gcd(2016, 271) = 1$. Using the "table method":

		x	y
R_1	2016	1	0
R_2	271	0	1
$R_3 = R_1 - 7R_2$	119	1	-7
$R_4 = R_2 - 2R_3$	33	-2	15
$R_5 = R_3 - 3R_4$	20	7	-52
$R_6 = R_4 - R_5$	13	-9	67
$R_7 = R_5 - R_6$	7	16	-119
$R_8 = R_6 - R_7$	6	-25	186
$R_9 = R_7 - R_8$	1	41	-305

We can set $x = 41$ and $y = -305$ so that $\gcd(20116, 271) = 2016x + 271y$.

Part II

Use mathematical induction to prove the following statements.

1. For all integers $n \geq 1$, $\sum_{i=1}^{n} (i+1)2^i = n2^{n+1}$.

   Proof. We will prove this by induction on n.

   Base case ($n = 1$):

$$\sum_{i=1}^{1} (i+1)2^i = (1+1)2^1 = 2 \cdot 2 = 2^2 = 1 \cdot 2^{1+1},$$

   so the statement is true when $n = 1$.

   Induction step: Let $k \geq 1$ be an integer. Suppose that

$$\sum_{i=1}^{k} (i+1)2^i = k2^{k+1}. \tag{IH}$$

   (We want to show that $\sum_{i=1}^{k+1} (i+1)2^i = (k+1)2^{k+2}$.) Now

$$\begin{aligned}
\sum_{i=1}^{k+1} (i+1)2^i &= \sum_{i=1}^{k} (i+1)2^i + (k+1+1)2^{k+1} \\
&= k2^{k+1} + (k+1+1)2^{k+1} && \text{by IH} \\
&= (2k+2)2^{k+1} \\
&= (k+1)2 \cdot 2^{k+1} \\
&= (k+1)2^{k+2},
\end{aligned}$$

   so the statement is true for $n = k + 1$.

   By the principle of induction, $\sum_{i=1}^{n} (i+1)2^i = n2^{n+1}$ holds for all integers $n \geq 1$. □

2. For all integers $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$.

   Proof. We will prove this by induction on n.

   Base case ($n = 1$): We see that

$$\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1(1+1)} = \frac{1}{2} = \frac{1}{1+1},$$

   so the statement is true when $n = 1$.

   Induction step: Let $k \geq 1$ be an integer. Suppose that

$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}. \tag{IH}$$

(We want to show that $\sum\limits_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{k+2}$.) Now

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} && \text{by IH} \\
&= \frac{1}{(k+2)(k+1)} \left(k(k+2)+1\right) \\
&= \frac{k^2+2k+1}{(k+2)(k+1)} \\
&= \frac{(k+1)^2}{(k+2)(k+1)} \\
&= \frac{k+1}{k+2},
\end{aligned}$$

so the statement is true for $n = k+1$.

By the principle of induction, $\sum\limits_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ holds for all integers $n \geq 1$. □

3. For all integers $n \geq 1$, the sum of the first n positive odd integers is equal to n^2.
Solution. We first need to write the desired sum using summation notation. The sum of the odd integers can be written as

$$\begin{aligned}
1+3+5+7+\cdots(n^{th} \text{ odd}) &= \big(2-1\big) + \big(4-1\big) + \big(6-1\big) + \cdots + (2n-1) \\
&= \big(2\cdot 1-1\big) + \big(2\cdot 2-1\big) + \big(3\cdot 2-1\big) + \cdots + (2n-1) \\
&= \sum_{i=1}^{n}(2i-1).
\end{aligned}$$

What we need to prove is: For all integers $n \geq 1$, $\sum\limits_{i=1}^{n}(2i-1) = n^2$.

Proof. We will prove this by induction on n.

Base case ($n=1$):

$$\sum_{i=1}^{1}(2i-1) = 2\cdot 1 - 1 = 1 = 1^2.$$

Induction step: Let $k \geq 1$ be an integer. Suppose that

$$\sum_{i=1}^{k}(2i-1) = k^2. \tag{IH}$$

(We want to show that $\sum\limits_{i=1}^{k+1}(2i-1) = (k+1)^2$.) Now

$$\begin{aligned}
\sum_{i=1}^{k+1}(2i-1) &= \sum_{i=1}^{k}(2i-1) + (2(k+1)-1) \\
&= k^2 + 2k + 1 && \text{by IH} \\
&= (k+1)^2.
\end{aligned}$$

By the principle of induction, $\sum_{i=1}^{n}(2i-1) = n^2$ holds for all integers $n \geq 1$. $\square$

4. For all integers $n \geq 0$, $3^n + 1$ is even.

Proof. We will prove this by induction on n.

Base case ($n = 0$): We have

$$3^0 + 1 = 1 + 1 = 2$$

which is even.

Induction step: Let $k \geq 0$ be an integer. Suppose that

$$3^k + 1 \text{ is even.} \tag{IH}$$

(We want to show that $3^{k+1} + 1$ is even.) By IH, there exists an integer m so that $3^k + 1 = 2m$. Thus $3^k = 2m - 1$. Now

$$\begin{aligned} 3^{k+1} + 1 &= 3 \cdot 3^k + 1 \\ &= 3 \cdot (2m - 1) + 1 && \text{by IH} \\ &= 6m - 3 + 1 \\ &= 2(3m - 1), \end{aligned}$$

where $3m - 1$ is an integer. Therefore $3^{k+1} + 1$ is even.

By the principle of induction, $n \geq 0$, $3^n + 1$ is even for all integers $n \geq 0$. $\square$

5. For all integers $n \geq 1$, $5^n - 4n - 1$ is divisible by 16.

Proof. We will prove this by induction on n.

Base case ($n = 1$): We have

$$5^1 - 4 \cdot 1 - 1 = 5 - 4 - 1 = 0$$

which is divisible by 16.

Induction step: Let $k \geq 1$ be an integer. Suppose that

$$5^k - 4k - 1 \text{ is divisible by 16.} \tag{IH}$$

(We want to show that $5^{k+1} - 4(k+1) - 1$ is divisible by 16.) By IH, there exists an integer m so that $5^k - 4k - 1 = 16m$. Thus $5^k = 16m + 4k + 1$. Now

$$\begin{aligned} 5^{k+1} - 4(k+1) - 1 &= 5 \cdot 5^k - 4(k+1) - 1 \\ &= 5(16m + 4k + 1) - 4k - 4 - 1 && \text{by IH} \\ &= 16 \cdot 5m + 20k + 5 - 4k - 5 \\ &= 16 \cdot 5m + 16k \\ &= 16(5m + k), \end{aligned}$$

where $5m + k$ is an integer. Therefore $5^{k+1} - 4(k+1) - 1$ is divisible by 16.

By the principle of induction, $5^n - 4n - 1$ is divisible by 16 for all integers $n \geq 1$. $\square$

6. For all integers $n \geq 4$, $n! > 2^n$.

Proof. We will prove this by induction on n.

Base case ($n = 4$): We have $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$ and $2^4 = 16$. Then $4! > 2^4$ since $24 > 16$.

Induction step: Let $k \geq 4$ be an integer. Suppose that

$$k! > 2^k. \tag{IH}$$

(We want to show that $(k+1)! > 2^{k+1}$.) Now

$$\begin{aligned}
(k+1)! &= k!(k+1) \\
&> 2^k(k+1) && \text{by IH} \\
&> 2^k \cdot (2) && \text{because } k+1 > 2 \\
&= 2^{k+1}.
\end{aligned}$$

By the principle of induction, $n! > 2^n$ for all integers $n \geq 4$. □

7. For all integers $n \geq 2$, $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$.

Proof. We will prove this by induction on n.

Base case ($n = 2$): Note that $1 < \sqrt{2}$. Now

$$\begin{aligned}
\sqrt{2} = \frac{2}{\sqrt{2}} &= \frac{1}{\sqrt{2}}(1+1) \\
&< \frac{1}{\sqrt{2}}\left(\sqrt{2}+1\right) \\
&= 1 + \frac{1}{\sqrt{2}} \\
&= \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}},
\end{aligned}$$

so $\frac{1}{\sqrt{2}} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}}$.

Induction step: Let $k \geq 4$ be an integer. Suppose that

$$\sqrt{k} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}}. \tag{IH}$$

(We want to show that $\sqrt{k+1} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$.)

Note that $k^2 < k(k+1)$. Hence $k < \sqrt{k(k+1)}$ and thus $k+1 < \sqrt{k(k+1)} + 1$. Therefore

$$\sqrt{k+1} < \frac{\sqrt{k(k+1)}+1}{\sqrt{k+1}}. \tag{$*$}$$

Now

$$\begin{aligned}
\sqrt{k+1} &< \frac{\sqrt{k(k+1)}+1}{\sqrt{k+1}} && \text{by } (*) \\
&= \frac{\sqrt{k(k+1)}}{\sqrt{k+1}} + \frac{1}{\sqrt{k+1}} \\
&= \sqrt{k} + \frac{1}{\sqrt{k+1}} \\
&< \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}, && \text{by IH}
\end{aligned}$$

which is what we wanted to show.

By the principle of induction, $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$ for all integers $n \geq 2$. ☐

8. For any real number $x > -1$ and all positive integers n, $(1+x)^n \geq 1 + nx$.

Proof. Let $x > -1$ be an arbitrary real number. We will prove that $(1+x)^n \geq 1 + nx$ for all integers $n \geq 1$ by induction on n.

Base case ($n = 1$):

$$(1+x)^1 = 1 + x = 1 + 1 \cdot x \geq 1 + 1 \cdot x.$$

Induction step: Let $k \geq 1$ be an integer. Suppose that

$$(1+x)^k \geq 1 + kx. \qquad \text{(IH)}$$

(We want to show that $(1+x)^{k+1} \geq 1 + (k+1)x$.) Now

$$\begin{aligned}
(1+x)^{k+1} &= (1+x)^k(1+x) \\
&\geq (1+kx)(1+x) && \text{by IH and because } (1+x) > 0 \\
&= 1 + x + kx + kx^2 \\
&= 1 + (k+1)x + kx^2 \\
&\geq 1 + (k+1)x, && \text{because } kx^2 \geq 0,
\end{aligned}$$

which is what we wanted to show.

By the principle of induction, $(1+x)^n \geq 1 + nx$ for all integers $n \geq 1$. ☐

Part II

Use *strong* mathematical induction to prove the following statements.

1. The sequence $a_1, a_2, a_3, \ldots$ is defined by letting $a_1 = 3$, $a_2 = 5$ and $a_k = 3a_{k-1} - 2a_{k-2}$ for all integers $k \geq 3$. Prove that $a_n = 2^n + 1$ for all integers $n \geq 1$.

Proof. We will prove the statement by strong induction on n.

Base cases

($n = 1$): $2^1 + 1 = 2 + 1 = 3$ and $a_1 = 3$.
($n = 2$): $2^2 + 1 = 4 + 1 = 5$ and $a_2 = 5$.

Induction step: Let $k \geq 2$ be an integer. Suppose that

$$a_i = 2^i + 1 \quad \text{for each integer } i, \quad 1 \leq i \leq k. \qquad \text{(IH)}$$

(We want to show that $a_{k+1} = 2^{k+1} + 1$.) Now

$$\begin{aligned}
a_{k+1} &= 3a_k - 2a_{k-1} \\
&= 3(2^k + 1) - 2(2^{k-1} + 1) && \text{by IH} \\
&= 3 \cdot 2 \cdot 2^{k-1} + 3 - 2 \cdot 2^{k-1} - 2 \\
&= (6-4)2^{k-1} + 1 \\
&= 2^2 \cdot 2^{k-1} + 1 \\
&= 2^{k+1} + 1.
\end{aligned}$$

By the principle of induction, $a_n = 2^n + 1$ for all integers $n \geq 1$. $\square$

2. Consider the sequence defined by $t_1 = t_2 = t_3 = 1$ and $t_k = t_{k-1} + t_{k-2} + t_{k-3}$ for all $k \geq 4$. Prove that $t_n < 2^n$ for all $n \in \mathbb{Z}^+$.

Proof. We will prove the statement by strong induction on n.

Base cases

($n = 1$): Note that $a_1 = 1$, and $2^1 = 2$, and $1 < 2$. Hence $a_1 < 2^1$.
($n = 2$): Note that $a_2 = 1$, and $2^2 = 4$, and $1 < 4$. Hence $a_2 < 2^2$.
($n = 3$): Note that $a_3 = 1$, and $2^3 = 8$, and $1 < 8$. Hence $a_3 < 2^3$.

Induction step: Let $k \geq 3$ be an integer. Suppose that

$$a_i < 2^i \quad \text{for each integer } i, \quad 1 \leq i \leq k. \tag{IH}$$

(We want to show that $a_{k+1} < 2^{k+1}$.) Now

$$\begin{aligned}
a_{k+1} &= a_k + a_{k-1} + a_{k-2} \\
&< 2^k + 2^{k-1} + 2^{k-2} && \text{by IH} \\
&= 2^{k+1}\left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}\right) \\
&= 2^{k+1}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) \\
&= 2^{k+1}\left(\frac{4+2+1}{8}\right) \\
&= 2^{k+1} \cdot \frac{7}{8} \\
&< 2^{k+1} && \text{because } \tfrac{7}{8} < 1.
\end{aligned}$$

By the principle of induction, $a_n < 2^n$ for all integers $n \geq 1$. $\square$

3. Let a_n be the sequence defined by $a_1 = 1$, $a_2 = 8$, $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$. Prove that $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for all $n \in \mathbb{Z}^+$.

Proof. We will prove the statement by strong induction on n.

Base cases

($n = 1$): Note that $3 \cdot 2^{1-1} + 2(-1)^1 = 3 \cdot 1 - 2 = 1$ and $a_1 = 1$. Thus $a_1 = 3 \cdot 2^{1-1} + 2(-1)^1$.
($n = 2$): Note that $3 \cdot 2^{2-1} + 2(-1)^2 = 3 \cdot 2 + 2 = 8$ and $a_2 = 8$. Thus $a_2 = 3 \cdot 2^{2-1} + 2(-1)^2$.

Induction step: Let $k \geq 2$ be an integer. Suppose that

$$a_i = 3 \cdot 2^{i-1} + 2(-1)^i \quad \text{for each integer } i, \quad 1 \leq i \leq k. \tag{IH}$$

(We want to show that $a_{k+1} = 3 \cdot 2^k + 2(-1)^{k+1}$.) Now

$$\begin{aligned}
a_{k+1} &= a_k + 2a_{k-1} \\
&= \left(3 \cdot 2^{k-1} + 2(-1)^k\right) + 2\left(3 \cdot 2^{(k-1)-1} + 2(-1)^{k-1}\right) && \text{by IH} \\
&= 3 \cdot 2^{k-1} + 2(-1)^k + 3 \cdot 2 \cdot 2^{k-2} + 4(-1)^{k-1} \\
&= 6 \cdot 2^{k-1} - 2(-1)^{k+1} + 4(-1)^{k+1} \\
&= 3 \cdot 2^k + 2(-1)^{k+1}.
\end{aligned}$$

By the principle of induction, $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for all integers $n \geq 1$. □

4. The sequence $s_1, s_2, s_3, \ldots$ is defined by: $s_1 = 1$, and for all integers $k \geq 2$, $s_k = 2 \cdot s_{\lfloor \frac{k}{2} \rfloor}$. Prove by induction that $s_n \leq n$ for all integers $n \geq 1$.

Proof. We will prove the statement by strong induction on n.

Base case (Note that only one base case is needed!):

($n = 1$): We see that $s_1 = 1$ and $1 \leq 1$. Thus $s_1 \leq 1$.

Induction step: Let $k \geq 1$ be an integer. Suppose that

$$s_i \leq i \quad \text{for each integer } i, \quad 1 \leq i \leq k. \tag{IH}$$

(We want to show that $s_{k+1} \leq k+1$.) Now $1 \leq \left\lfloor \frac{k+1}{2} \right\rfloor \leq k$, since $k \geq 1$. Thus $s_{\lfloor \frac{k+1}{2} \rfloor} \leq \left\lfloor \frac{k+1}{2} \right\rfloor$ (by IH) and

$$\begin{aligned} s_{k+1} &= 2 \cdot s_{\lfloor \frac{k+1}{2} \rfloor} \\ &\leq 2 \left\lfloor \frac{k+1}{2} \right\rfloor && \text{by IH} \\ &\leq 2 \cdot \frac{k+1}{2} && \text{by definition of floor} \\ &= k+1, \end{aligned}$$

so $s_{k+1} \leq k+1$.

By the principle of induction, $s_n \leq n$ for all integers $n \geq 1$. □