MATH 271 – Summer 2016
Solutions to practice problems – Week 3

Part I

Determine which of the following statements are true and which are false. Prove the true statements. For the false statements, write the negation and prove that. Use the element method for all proofs.

1. $\forall A \subseteq \mathbb{Z}, \exists B \subseteq \mathbb{Z}$ so that $1 \in B - A$.
   This statement is false.
   Negation: $\exists A \subseteq \mathbb{Z}$ so that $\forall B \subseteq \mathbb{Z}$, $1 \notin B - A$.

   Proof (of negation). Let $A = \mathbb{Z}$. Let B be an arbitrary subset of $\mathbb{Z}$. Then $B - A = B - \mathbb{Z} = \emptyset$ and $1 \notin \emptyset$. Therefore $1 \notin B - A$. □

   Note: Any other set A that contains 1 will work as a counterexample.

2. $\forall A \subset \mathbb{Z}, \exists B \subseteq \mathbb{Z}$ so that $1 \notin B - A$.

   Proof. Let A be an arbitrary subset of $\mathbb{Z}$. Let $B = \emptyset$. Then $B - A = \emptyset - A = \emptyset$ and $1 \notin \emptyset$. Therefore $1 \notin B - A$. □

   Note: Any other set B that does not contain 1 will work.

3. For all sets A, B, and C, if $A \cup B = C$ then $C \in \mathcal{P}(A) \cup \mathcal{P}(B)$.
   This statement is false.
   Negation: There exists sets A, B, and C so that $A \cup B = C$ but $C \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

   Proof (of negation). Let Let $A = \{1\}$, $B = \{2\}$ and $C = \{1, 2\}$. Then $A \cup B = \{1, 2\} = C$. Note that $\mathcal{P}(A) = \{\emptyset, \{1\}\}$ and $\mathcal{P}(B) = \{\emptyset, \{2\}\}$, so $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{1\}, \{2\}\}$. But $\{1, 2\} \notin \{\emptyset, \{1\}, \{2\}\}$. Therefore $C \notin \mathcal{P}(A) \cup \mathcal{P}(B)$. □

4. For all sets A, B, and C, if $A \cap B = C$ then $C \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

   Proof. Suppose A, B, and C are arbitrary sets so that $A \cap B = C$. Then $C \subseteq A$ and $C \subseteq B$, which means $C \in \mathcal{P}(A)$ and $C \in \mathcal{P}(B)$. Therefore $C \in \mathcal{P}(A) \cap \mathcal{P}(B)$. □

5. For all sets A, B, and C, $(A \cup B) \cap C \subseteq A \cup (B \cap C)$.

   Proof. Let A, B, and C be arbitrary sets. Let $x \in (A \cup B) \cap C$. Then $x \in A \cup B$ and $x \in C$. Thus $x \in A$ or $x \in B$.

   Case 1: Suppose $x \in A$. Then $x \in A \cup (B \cap C)$.

   Case 2: Suppose $x \in B$. Then $x \in B \cap C$ since x is also in C. Hence $x \in A \cup (B \cap C)$.

   In either case, $x \in A \cup (B \cap C)$. □

6. For all sets A, B, and C, $A \cup (B \cap C) \subseteq (A \cup B) \cap C$.
   This statement is false.
   Negation: There exists sets A, B, and C so that $A \cup (B \cap C) \nsubseteq (A \cup B) \cap C$.

   Proof (of negation). Let $A = \{1, 2\}$, $B = \{2\}$ and $C = \{2\}$. Then $B \cap C = \{2\}$ and $A \cup (B \cap C) = \{1, 2\}$, so $1 \in A \cup (B \cap C)$. However, $A \cup B = \{1, 2\}$ and $(A \cup B) \cap C\{2\}$, so $1 \notin (A \cup B) \cap C$. □

7. For all sets A, B, and C, if $A \times B = A \times C$ then $B = C$.
   This statement is false.
   Negation: There exists sets A, B, and C so that $A \times B = A \times C$ but $B \neq C$.

   Proof (of negation). Let $A = \emptyset$, $B = \{1\}$ and $C = \{2\}$. Then $A \times B = \emptyset$ and $A \times C = \emptyset$, but $\{1\} \neq \{2\}$ so $B \neq C$. □

8. For all sets A, B, and C, if $A - B \subseteq C$ then $A - C \subseteq B$.

   Proof. Let A, B, and C be sets. Assume that $A - B \subseteq C$. (We want to show that $A - C \subseteq B$.) Let $x \in A - C$. This means that $x \in A$ and $x \notin C$. We will prove that $x \in B$ by contradiction. Suppose instead that $x \notin B$. Then $x \in A - B$ since $x \in A$ and $x \notin B$. This means that $x \in C$, since $x \in A - B$ and $A - B \subseteq C$. Thus $x \notin C$ and $x \in C$, a contradiction. So the assumption that $x \notin B$ is wrong, and thus $x \in B$. Therefore $A - C \subseteq B$. □

9. For all sets A, B, and C, if $A \cap B \subseteq C$ and $B \cap C \subseteq A$ then $C \cap A \subseteq B$.
   This statement is false.
   Negation: There exists sets A, B, and C so that $A \cap B \subseteq C$ and $B \cap C \subseteq A$ but $C \cap A \nsubseteq B$.

   Proof (of negation). Let $A = \{1\}$, $B\{2\}$, and $C = \{1\}$. Then $A \cap B = \emptyset$ and $B \cap C = \emptyset$ and $\emptyset \subseteq C$ and $\emptyset \subseteq A$. Thus $A \cap B \subseteq C$ and $B \cap C \subseteq A$. However, $C \cap A = \{1\}$ and $\{1\} \nsubseteq \{2\}$. Therefore $C \cap A \nsubseteq B$. □

10. For all sets A, B, and C, if $A - (B \cap C) = \emptyset$ then $A - C = \emptyset$.

    Proof. Let A, B, and C be sets. Suppose that $A - (B \cap C) = \emptyset$. (We want to show that $A - C = \emptyset$.) Assume for the sake of getting a contradiction that $A - C \neq \emptyset$. Then there exists an element $x \in A - C$. This means that $x \in A$ and $x \notin C$. Then $x \notin B \cap C$ since $x \notin C$. Thus $x \in A$ and $x \notin B \cap C$, which means that $x \in A - (B \cap C)$. But $A - (B \cap C) = \emptyset$, so $x \notin A - (B \cap C)$. This is a contradiction, so the assumption that $A - C \neq \emptyset$ is wrong. Therefore $A - C = \emptyset$. □

11. For all sets A, B, and C, if $A - C = \emptyset$ then $A - (B \cap C) = \emptyset$.
    This statement is false.
    Negation: There exists sets A, B, and C so that $A - C = \emptyset$ but $A - (B \cap C) \neq \emptyset$.

    Proof (of negation). Let $A = \{1\}$, $B = \{2\}$, and $C = \{1\}$. Then $A - C = \emptyset$ and $B \cap C = \emptyset$, but $A - (B \cap C) = \{1\}$ and $\{1\} \neq \emptyset$. Therefore $A - (B \cap C) \neq \emptyset$. □

Part II

1. Suppose A and B are arbitrary subsets of $\mathbb{Z}$ such that $(2, 3) \in A \times B$ and $(3, 4) \in A \times B$, but $(1, 3) \notin A \times B$.

   (a) Find another element in $A \times B$ that is not $(2, 3)$ or $(3, 4)$. Explain.
   Solution. $(2, 4) \in A \times B$ and $(3, 3) \in A \times B$.

   Proof. Note that $(2, 3) \in A \times B$ means that $2 \in A$ and $3 \in B$. Similarly, $(3, 4) \in A \times B$ means that $3 \in A$ and $4 \in B$. Thus $(2, 4) \in A \times B$ and $(3, 3) \in A \times B$. □

   (b) Find another element that is not in $A \times B$. Explain.
   Solution. $(1, 7) \notin A \times B$.

   Proof. Note that $(1, 3) \notin A \times B$ means that $1 \notin A$ or $3 \notin B$. However, we know from part (a) that $3 \in B$, so it must be the case that $1 \notin A$. Thus $(1, 7) \notin A \times B$, since $1 \notin A$. □

   Note: Any number other than 7 will also work.

2. Suppose A and B are arbitrary subsets of $\mathbb{Z}$ such that $A \cap B = \{1\}$.

   (a) Find an element of $A \times B$. Explain why it is an element of $A \times B$.
   Solution. $(1, 1) \in A \times B$.

   Proof. Note that $1 \in A \cap B$ so $1 \in A$ and $1 \in B$. Thus $(1, 1) \in A \times B$, by definition of the product of sets. □

(b) Find an element of the complement $(A \times B)^c$. (Here, assume that the universal set is $\mathbb{Z} \times \mathbb{Z}$.) Explain.
Solution. $(2,2) \notin A \times B$.

Proof. Suppose instead that $(2,2) \in A \times B$. Then $2 \in A$ and $2 \in B$ which means that $2 \in A \cap B$. But $A \cap B = \{1\}$ and $2 \notin \{1\}$, and thus $2 \notin A \cap B$. This is a contradiction so the supposition that $(2,2) \in A \times B$ is wrong. Therefore $(2,2) \notin A \times B$. □

Note: Any number other than 2 will also work.

Part III

1. Consider the set of 4-digit positive integers. How many of them...

(a) ... are there total?
Solution: There are $9 \cdot 10^3 = 9,000$. The recipe is as follows:

1. Choose the first digit (which can't be 0) – 9 choices
2. Choose the second digit – 10 choices
3. Choose the third digit – 10 choices
4. Choose the fourth digit – 10 choices

(b) ... are odd?
Solution: There are $9 \cdot 10^2 \cdot 5 = 4,500$. (Same as above, but the last digit must be odd – only 5 choices.)

(c) ... have distinct digits?
Solution: There are $9^2 \cdot 8 \cdot 7 = 4,536$. The recipe is as follows:

1. Choose the first digit (which can't be 0) – 9 choices
2. Choose the second digit (can't be the same as the first one) – 9 choices
3. Choose the third digit (can't be the same as either the first or second)– 8 choices
4. Choose the fourth digit (can't be the same as either the first, second, or third) – 7 choices

(d) ... are odd and have distinct digits?
Solution: There are $5 \cdot 8^2 \cdot 7 = 2,240$. The recipe is as follows:

1. Choose the last digit (which must be odd) – 5 choices
2. Choose the first digit (can't can't be zero and must be different from last) – 8 choices
3. Choose the second digit (can't be the same as either the first or last)– 8 choices
4. Choose the third digit (can't be the same as either the first, second, or last) – 7 choices

(e) ... are even and have distinct digits?
Solution: There are $1 \cdot 9 \cdot 8 \cdot 7 + 4 \cdot 8 \cdot 8 \cdot 7 = 2,296$. The trick is to note that there are two different ways to make these numbers: count the numbers that end in zero separately from the numbers that don't end in zero.

- Let X be the set of four-digit numbers with distinct digits that end in zero. There are $1 \cdot 9 \cdot 8 \cdot 7$ of them. The recipe is:
  1. Choose the last digit (which must be zero) – 1 choice
  2. Choose the first digit (can't can't be zero) – 9 choices
  3. Choose the second digit (can't be zero or same as the first digit)– 8 choices
  4. Choose the third digit (can't be the same as either the first, second, or last) – 7 choices

  So $|X| = 1 \cdot 9 \cdot 8 \cdot 7$.
- Let Y be the set of four-digit numbers with distinct digits and that don't end in zero. There are $4 \cdot 8 \cdot 8 \cdot 7$ of them. The recipe is:

1. Choose the last digit (which must be even and can't be zero) – 4 choices
2. Choose the first digit (can't can't be zero and can't be same as last digit) – 8 choices
3. Choose the second digit (can't be the same as first or last digit)– 8 choices
4. Choose the third digit (can't be the same as either the first, second, or last) – 7 choices

So $|Y| = 4 \cdot 8 \cdot 8 \cdot 7$.

The set we are interested in is $X \cup Y$. Since $X \cap Y = \emptyset$, we see that $|X \cup Y| = |X| + |Y|$.

Alternate solution: Let S be the set of *even* four-digit integers with distinct digits. Let T be the set of *odd* four-digit integers with distinct digits. From part (d), we see that $|T| = 5 \cdot 8^2 \cdot 7$. From part (c), we see that $|T \cup S| = 9^2 \cdot 8 \cdot 7$. Note that $T \cap S = \emptyset$. Thus $|S| = |T \cup S| - |T|$. Hence $|S| = 9^2 \cdot 8 \cdot 7 - 5 \cdot 8^2 \cdot 7 = 2,296$.

(f) ... have their digits in strictly increasing order? (i.e. 1234)
Solution: There are $\binom{9}{4}$. Note that the first digit cannot be zero. So none of the digits can be zero. Recipe: Choose 4 distinct digits out of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and arrange them in increasing order.

(g) ... have the property that the sum of their digits is even?
Solution: There are $9 \cdot 10 \cdot 10 \cdot 5 = 4,500$.

1. Choose the first digit (which can't be zero) – 9 choices
2. Choose the second digit – 10 choices
3. Choose the third digit – 10 choices
4. Choose the last digit (two cases) – 5 choices
   - Case i: If the sum of the first three digits is even, the last digit must be an even digit.
   - Case ii: If the sum of the first three digits is odd, the last digit must be an odd digit.

(h) ... are odd and have the property that the sum of their digits is even?
Solution: There are $5 \cdot 9 \cdot 10 \cdot 5 = 2,250$. The recipe is as follows.

1. Choose the last digit (which must be an odd digit) – 5 choices
2. Choose the first digit (which can't be zero) – 9 choices
3. Choose the second digit – 10 choices
4. Choose the third digit (two cases) – 5 choices
   - Case i: If the sum of the other three digits is even, the third digit must be an even digit.
   - Case ii: If the sum of the other three digits is odd, the third digit must be an odd digit.

(i) ... are odd and don't have the property that the sum of their digits is even?
Solution: There are 2,250. The reason is as follows. Let A be the set of four-digit numbers that are odd. Let B be the set of odd four-digit numbers that have the property that the sum of their digits is even. The set we are interested in is $A - B$. Now $B \subseteq A$ and thus

$$\begin{aligned} |A - B| &= |A| - |B| \\ &= 4500 - 2250 \\ &= 2250, \end{aligned}$$

where $|A| = 4500$ from part (b) and $|B| = 2250$ from part (h).