MATH 271 – Summer 2016

Practice problems – Week 4

1. Consider the set of 4-digit positive integers. How many of them...

   (a) ... are there total?
   Solution: There are $9 \cdot 10^3 = 9,000$. The recipe is as follows:
   1. Choose the first digit (which can't be 0) – 9 choices
   2. Choose the second digit – 10 choices
   3. Choose the third digit – 10 choices
   4. Choose the fourth digit – 10 choices

   (b) ... are odd?
   Solution: There are $9 \cdot 10^2 \cdot 5 = 4,500$. (Same as above, but the last digit must be odd – only 5 choices.)

   (c) ... have distinct digits?
   Solution: There are $9^2 \cdot 8 \cdot 7 = 4,536$. The recipe is as follows:
   1. Choose the first digit (which can't be 0) – 9 choices
   2. Choose the second digit (can't be the same as the first one) – 9 choices
   3. Choose the third digit (can't be the same as either the first or second)– 8 choices
   4. Choose the fourth digit (can't be the same as either the first, second, or third) – 7 choices

   (d) ... are odd and have distinct digits?
   Solution: There are $5 \cdot 8^2 \cdot 7 = 2,240$. The recipe is as follows:
   1. Choose the last digit (which must be odd) – 5 choices
   2. Choose the first digit (can't can't be zero and must be different from last) – 8 choices
   3. Choose the second digit (can't be the same as either the first or last)– 8 choices
   4. Choose the third digit (can't be the same as either the first, second, or last) – 7 choices

   (e) ... are even and have distinct digits?
   Solution: There are $1 \cdot 9 \cdot 8 \cdot 7 + 4 \cdot 8 \cdot 8 \cdot 7 = 2,296$. The trick is to note that there are two different ways to make these numbers: count the numbers that end in zero separately from the numbers that don't end in zero.

   - Let X be the set of four-digit numbers with distinct digits that end in zero. There are $1 \cdot 9 \cdot 8 \cdot 7$ of them. The recipe is:
     1. Choose the last digit (which must be zero) – 1 choice
     2. Choose the first digit (can't can't be zero) – 9 choices
     3. Choose the second digit (can't be zero or same as the first digit)– 8 choices
     4. Choose the third digit (can't be the same as either the first, second, or last) – 7 choices

     So $|X| = 1 \cdot 9 \cdot 8 \cdot 7$.
   - Let Y be the set of four-digit numbers with distinct digits and that don't end in zero. There are $4 \cdot 8 \cdot 8 \cdot 7$ of them. The recipe is:
     1. Choose the last digit (which must be even and can't be zero) – 4 choices
     2. Choose the first digit (can't can't be zero and can't be same as last digit) – 8 choices
     3. Choose the second digit (can't be the same as first or last digit)– 8 choices
     4. Choose the third digit (can't be the same as either the first, second, or last) – 7 choices

     So $|Y| = 4 \cdot 8 \cdot 8 \cdot 7$.

The set we are interested in is $X \cup Y$. Since $X \cap Y = \emptyset$, we see that $|X \cup Y| = |X| + |Y|$.

Alternate solution: Let S be the set of *even* four-digit integers with distinct digits. Let T be the set of *odd* four-digit integers with distinct digits. From part (d), we see that $|T| = 5 \cdot 8^2 \cdot 7$. From part (c), we see that $|T \cup S| = 9^2 \cdot 8 \cdot 7$. Note that $T \cap S = \emptyset$. Thus $|S| = |T \cup S| - |T|$. Hence $|S| = 9^2 \cdot 8 \cdot 7 - 5 \cdot 8^2 \cdot 7 = 2,296$.

(f) ... have their digits in strictly increasing order? (i.e. 1234)
Solution: There are $\binom{9}{4}$. Note that the first digit cannot be zero. So none of the digits can be zero. Recipe: Choose 4 distinct digits out of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and arrange them in increasing order.

(g) ... have the property that the sum of their digits is even?
Solution: There are $9 \cdot 10 \cdot 10 \cdot 5 = 4,500$.

1. Choose the first digit (which can't be zero) – 9 choices
2. Choose the second digit – 10 choices
3. Choose the third digit – 10 choices
4. Choose the last digit (two cases) – 5 choices
   - Case i: If the sum of the first three digits is even, the last digit must be an even digit.
   - Case ii: If the sum of the first three digits is odd, the last digit must be an odd digit.

(h) ... are odd and have the property that the sum of their digits is even?
Solution: There are $5 \cdot 9 \cdot 10 \cdot 5 = 2,250$. The recipe is as follows.

1. Choose the last digit (which must be an odd digit) – 5 choices
2. Choose the first digit (which can't be zero) – 9 choices
3. Choose the second digit – 10 choices
4. Choose the third digit (two cases) – 5 choices
   - Case i: If the sum of the other three digits is even, the third digit must be an even digit.
   - Case ii: If the sum of the other three digits is odd, the third digit must be an odd digit.

(i) ... are odd and don't have the property that the sum of their digits is even?
Solution: There are $2,250$. The reason is as follows. Let A be the set of four-digit numbers that are odd. Let B be the set of odd four-digit numbers that have the property that the sum of their digits is even. The set we are interested in is $A - B$. Now $B \subseteq A$ and thus

$$\begin{aligned} |A - B| &= |A| - |B| \\ &= 4500 - 2250 \\ &= 2250, \end{aligned}$$

where $|A| = 4500$ from part (b) and $|B| = 2250$ from part (h).

2. Consider the set of integers $T = \{0, 1, 2, 3, \ldots, 99999\}$.

   (a) How many elements are in T?
   Solution. There are $10^5 = 100000$ elements in T. To make elements of T, we must fill each of the 5 spaces _ _ _ _ _ with one of the 10 digits in $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. We can ignore the leading zeros of any numbers made this way (i.e. $\underline{0}\,\underline{0}\,\underline{1}\,\underline{2}\,\underline{3}$ is just 123). Use the following recipe:

   1. Choose the first digit (10 choices)
   2. Choose the second digit (10 choices)
   3. Choose the third digit (10 choices)
   4. Choose the fourth digit (10 choices)
   5. Choose the fifth digit (10 choices)

   This makes all of the numbers we are interested in. So there are 10^5 elements in T.

(b) How many elements of T contain the digit 1 at least once?

Solution. Let A be the subset of integers of T that contain no 1's. Then $T - A$ is the set of integers in T that contain at least one 1. Since $A \subseteq T$, we have that $|T - A| = |T| - |A|$. To determine $|A|$, we use the following recipe:

1. Choose the first digit which can be anything but 1 (9 choices)
2. Choose the second digit which can be anything but 1 (9 choices)
3. Choose the third digit which can be anything but 1 (9 choices)
4. Choose the fourth digit which can be anything but 1 (9 choices)
5. Choose the fifth digit which can be anything but 1 (9 choices)

Thus $|A| = 9^5 = 59049$. Therefore we have $|T - A| = 10^5 - 9^5 = 40951$.

(c) How many integers in T contain each of the digits 1 and 2 at least once?

Solution. Let B be the subset of integers of T that contain no 2's. The set we are interested in is $T - (A \cup B)$ since $A \cup B$ is the set of integers in T that contain no 1's or contain no 2's. Note that $A \cup B \subseteq T$, so $|T - (A \cup B)| = |T| - |A \cup B|$ and $|A \cup B| = |A| + |B| - |A \cap B|$. To determine $|B|$, we use the following recipe:

1. Choose the first digit which can be anything but 2 (9 choices)
2. Choose the second digit which can be anything but 2 (9 choices)
3. Choose the third digit which can be anything but 2 (9 choices)
4. Choose the fourth digit which can be anything but 2 (9 choices)
5. Choose the fifth digit which can be anything but 2 (9 choices)

Thus $|B| = 9^5$. Note that $A \cap B$ is the set of integers in T that contain no 1's AND no contain no 2's. To determine $|A \cap B|$, we use the same recipe as above except that at each step there are only 8 choices (no 1's or 2's). Hence $|A \cap B| = 8^5$. Finlay, we have that

$$\begin{aligned} |T - (A \cup B)| &= |T| - |A \cup B| \\ &= |T| - (|A| + |B| - |A \cap B|) \\ &= |T| - |A| - |B| + |A \cap B| \\ &= 10^5 - 9^5 - 9^5 + 8^5 \\ &= 14670. \end{aligned}$$

(d) How many integers in T contain each of the digits 1, 2, and 3 at least once?

Solution. Let C be the subset of integers of T that contain no 3's. The set we are interested in is $T - (A \cup B \cup C)$ since $A \cup B \cup C$ is the set of integers in T that contain no 1's or contain no 2's or contain no 3's. Now $|T - (A \cup B \cup C)| = |T| - |A \cup B \cup C|$ and

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Similar to parts (b) and (c), note that $|C| = 9^5$ and that $|A \cap C| = |B \cap C| = 8^5$. Now $A \cap B \cap C$ is the set of integers in T that have no 1's and no 2's and no 3's. Analogously, it follows that $|A \cap B \cap C| = 7^5$. Thus

$$\begin{aligned} |T - (A \cup B \cup C)| &= |T| - |A \cup B \cup C| \\ &= |T| - (|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|) \\ &= |T| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C| \\ &= 10^5 - 9^5 - 9^5 - 9^5 + 8^5 + 8^5 + 8^5 - 7^5 \\ &= 4350. \end{aligned}$$

3. How many ways are there to arrange the 7 letters $AAABBBB$?

   Solution. There are $\binom{7}{3}\binom{4}{4}$ ways. The recipe is:

   1. Choose 3 of the 7 spots to put As. (there are $\binom{7}{3}$ ways)
   2. Put the 4 Bs in the remaining 4 spots (there are $\binom{4}{4} = 1$ ways)

4. How many ways are there to arrange the 12 letters of $AAABBBBCCCCC$?

   Solution. There are $\binom{12}{3}\binom{8}{4}\binom{5}{5}$ ways. The recipe is:

   1. Choose 3 of the 12 spots to put A's. (there are $\binom{12}{3}$ ways)
   2. Put the 4 Bs in 4 of the remaining 4 spots (there are $\binom{8}{4}$ ways)
   3. Put the 5 Cs in the remaining 5 spots (there are $\binom{5}{5}$ ways)

5. How many ways are there to arrange the 12 letters of $AAABBBBCCCCC$ without having two Cs together?

   Solution. There are $\binom{7}{3}\binom{4}{4}\binom{8}{5}$ ways. The recipe is:

   1. Arrange all of the As and Bs (there are $\binom{7}{3}\binom{4}{4}$ ways, from part (a))
   2. There are eight 'spots' where we can insert the Cs. For example, if in step 1 we made the string $ABBABBA$, we can add Cs in the indicated locations:

      $$_\uparrow A_\uparrow B_\uparrow B_\uparrow A_\uparrow B_\uparrow B_\uparrow A_\uparrow$$

      We need to choose 5 of those 8 spots to place Cs. (There are $\binom{8}{5}$ ways.) For example, we could choose the second, third, fifth, sixth and eight spots to make the string: $ACBCBACBCBAC$.

6. Consider the set $S = \{1, 2, 3, \ldots, 100\}$.

   (a) How many ways can two different integers be selected from the set $\{1, 2, 3, \ldots, 100\}$ such that the sum of their digits is even?

   Solution. Note that, if the sum of two integers is even, then they must be either both even or both odd. There are 50 odd integers and 50 even integers in S. There are $\binom{50}{2}$ ways to choose 2 odd integers and $\binom{50}{2}$ ways to choose 2 even integers. So the total number of ways to choose 2 different integers such that their sum is even is $\binom{50}{2} + \binom{50}{2} = 1225 + 1225 = 2450$.

   (b) How many ways can two different integers be selected from the set $\{1, 2, 3, \ldots, 100\}$ such that the sum of their digits is odd?

   Solution. Note that, if the sum of two integers is odd, then one must be even and one must be odd. To pick two integers that satisfy this, we must pick one even integer in S and one odd integer in S. There are $\binom{50}{1}\binom{50}{1} = 50 \cdot 50 = 2500$ ways.

   Alternatively, note that there are $\binom{100}{2}$ ways to pick different integers from the set S. Since the sum of $\binom{50}{2} + \binom{50}{2} = 2450$ of those picks are even (from part a), we know that there must be $\binom{100}{2} - \binom{50}{2} - \binom{50}{2} = 4950 - 2450 = 2500$ ways.

7. Consider the set $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d, e, f\}$.

   (a) How many functions are there from X to Y?

   Solution. There are 6^4 such functions. Recipe: For each $x \in X$, we need to choose a value for $f(x)$. There are six options for each of the four different $x \in X$, so there are 6^4 functions. The explicit recipe is:

   1. Choose a value for $f(1)$ (There are 6 choices, since there are six different things in Y.)
   2. Choose a value for $f(2)$ (6 choices)

3. Choose a value for $f(3)$ (6 choices)
4. Choose a value for $f(4)$ (6 choices)

So there are $6 \cdot 6 \cdot 6 \cdot 6 = 6^4 = 1296$ different ways to make a function from X to Y.

(b) How many functions are there from Y to X?

Solution. There are $4^6 = 4096$ such functions. Same reasoning as part (a), but there are four ways to choose $f(y)$ for each of the six different $y \in Y$.
(Note: In general, if X and Y are finite sets, the number of functions from X to Y is $|Y|^{|X|}$)

(c) How many one-to-one functions there from X to Y?

Solution. There are $6 \cdot 5 \cdot 4 \cdot 3$ such functions. The recipe is as follows:

1. Choose $F(1)$ (6 choices)
2. Choose $F(2)$, which must be different from $F(1)$. (5 choices)
3. Choose $F(3)$, which must be different from $F(1)$ and $F(2)$. (4 choices)
4. Choose $F(4)$, which must be different from $F(1)$ and $F(2)$ and $F(3)$. (3 choices)

(d) How many one-to-one functions are there from Y to X?

Solution. There are none, because $|X| < |Y|$. (Think about why this is.)

(e) How many functions F are there from X to Y so that $f(1) = a$?

Solution. There are $1 \cdot 6^3$ such functions. The recipe is as follows:

1. Set $F(1) = a$ (1 choice)
2. Choose $F(2)$ (6 choices)
3. Choose $F(3)$ (6 choices)
4. Choose $F(4)$ (6 choices)

(f) How many one-to-one functions F are there from X to Y so that $F(1)$ is not a vowel?

Solution. There are $4 \cdot 5 \cdot 4 \cdot 3$ such functions. The recipe is as follows:

1. Choose $F(1)$ from $\{b, c, d, f\}$ (4 choices)
2. Choose $F(2)$ from Y, which must be different from $F(1)$. (5 choices)
3. Choose $F(3)$ from Y, which must be different from $F(1)$ and $F(2)$. (4 choices)
4. Choose $F(4)$ from Y, which must be different from $F(1)$ and $F(2)$ and $F(3)$. (3 choices)

In addition to the problems given below, you can try these problems in the book to practice your counting. Solutions for these problems can be found in the back of the textbook.

- Section 9.2: 11, 12(a), 13(a), 14(a,b,d), 16(a,b,c,d), 32(a,b), 38(a), 39(a,c).
- Section 9.3: 3, 4, 6, 8(a), 11, 16(a,b), 23(a), 37.
- Section 9.5: 6, 9(b), 11(a,c,f), 13(a,d), 15, 17(a), 19.