MATH 271 – Summer 2016
Solutions to practice problems – Week 5
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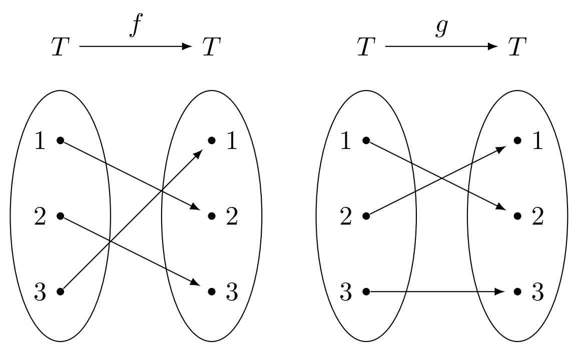
Part I (functions)

1. Let $T = \{1,2,3\}$, let $f\colon T \to T$ and $g\colon T \to T$ be defined by by $f = \{(1,2),(2,3),(3,1)\}$ and $g = \{(1,2),(2,1),(3,3)\}$. Draw the arrow diagrams for f and g. Determine each of the following functions as a collection of ordered pairs.
   (a) f^{-1}
   (b) g^{-1}
   (c) $f \circ g$
   (d) $g \circ f$

   Solution. From the definitions of f and g, we have that

   $$f(1) = 2,\ f(2) = 3,\ f(3) = 1 \quad \text{and} \quad g(1) = 2,\ g(2) = 1,\ g(3) = 3.$$

   Note that both f and g are onto and one-to-one. The arrow diagrams for f and g are



   Since f and g are one-to-one and onto, we can find their inverse functios. They are define by

   $$f^{-1}(1) = 3,\ f^{-1}(2) = 1,\ f^{-1}(3) = 2 \quad \text{and} \quad g^{-1}(1) = 2,\ g^{-1}(2) = 1,\ g(3)^{-1} = 3.$$

   Also, we can determine $f \circ g$ as

   $$(f \circ g)(1) = f(g(1)) = f(2) = 3, \quad (f \circ g)(2) = f(g(2)) = f(1) = 2, \quad (f \circ g)(3) = f(g(3)) = f(3) = 1$$

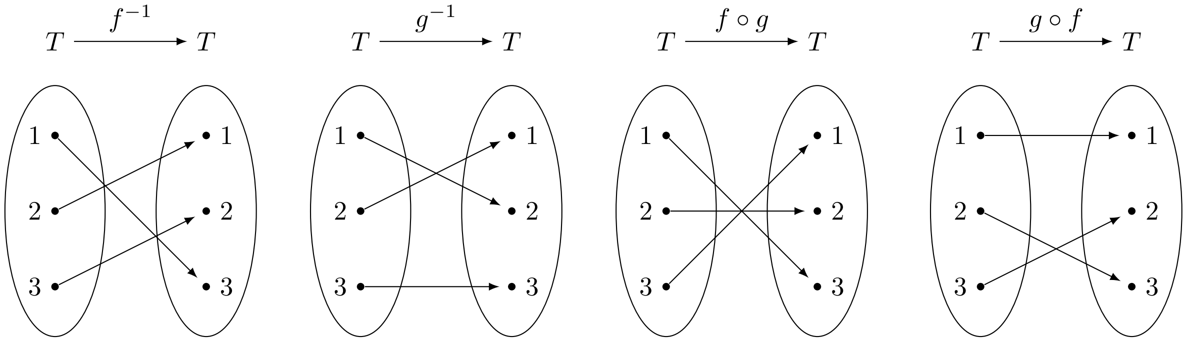
   and $g \circ f$ as

   $$(g \circ f)(1) = g(f(1)) = g(2) = 1, \quad (g \circ f)(2) = g(f(2)) = g(3) = 3, \quad (g \circ f)(3) = g(f(3)) = g(1) = 2.$$

   The functions as sets are

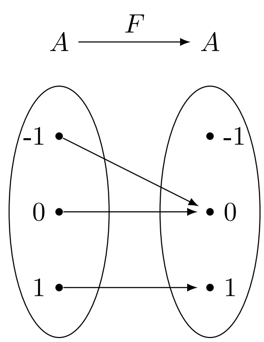
   $$\begin{aligned} f^{-1} &= \{(1,3),(2,1),(3,2)\}, \\ g^{-1} &= \{(1,2),(2,1),(3,2)\}, \\ f \circ g &= \{(1,3),(2,2),(3,1)\}, \\ \text{and } g \circ f &= \{(1,1),(2,3),(3,2)\}. \end{aligned}$$

The arrow diagrams for these functions are



2. Let $A = \{-1, 0, 1\}$ and let $F : A \to A$ be the function defined by $F(n) = \lceil \frac{n}{2} \rceil$ for all $n \in A$.
   (a) Is F one-to-one? Prove your answer.
   (b) Is F onto? Prove your answer.
   (c) Does there exist a function from A to A that is one-to-one but not onto? Prove your answer.
   (d) Does there exist a function from A to A that is onto but not one-to-one? Prove your answer.

   Solution. Note that $F(-1) = \lceil \frac{-1}{2} \rceil = 0$, $F(0) = \lceil 0 \rceil = 0$, and $F(1) = \lceil \frac{1}{2} \rceil = 1$. The arrow diagram for F is



   Then F is clearly not onto, since $F(x) \neq -1$ for all $x \in A$. Similarly, F is clearly not one-to-one, since $F(0) = F(-1) = 0$ and $0 \neq -1$.
   There cannot be a function from A to A that is one-to-one but not onto. The proof is as follows. Let G be a function from A to A and assume that G is one-to-one. Then the range of G has three elements, since A has three elements. The only subset of A with three elements is A itself, so G must be onto.
   There cannot be a function from A to A that is onto but not one-to-one. The proof is as follows. Let G be a function from A to A and assume that G is onto. Then the range of G is all of A. If G is not one-to-one, then the range of G must have fewer than three elements, since the domain has three elements. This is a contradiction, so G must be one-to-one.

3. Define the functions $h\colon \mathbb{Z} \to \mathbb{Z}$ and $g\colon \mathbb{Z} \to \mathbb{Z}$ by $h(n) = 3n$ and $g(n) = \lfloor \frac{n}{2} \rfloor$ for each $n \in \mathbb{Z}$. Prove or disprove each of the following statements.
   (a) h is one-to-one.

   Solution. h is one-to-one.

   Proof. Let x_1 and x_2 be arbitrary integers. Suppose that $h(x_1) = h(x_2)$. (We will show that x_1 must be equal to x_2.) Then $3x_1 = 3x_2$, and dividing both sides by 3 gives us $x_1 = x_2$. Therefore h is one-to-one. □

(b) g onto.

Solution. g is onto.

Proof. Let $y \in \mathbb{Z}$ be arbitrary. (We will show that there exists an $x \in \mathbb{Z}$ so that $g(x) = y$.) Pick $x = 2y$. Then $g(x) = g(2y) = \lfloor \frac{2y}{2} \rfloor = \lfloor y \rfloor = y$, since y is an integer. Thus $g(x) = y$, so g is onto. $\square$

(c) $h \circ g$ is onto.

Solution. $h \circ g$ is not onto.

Proof. Suppose that there exists an $x \in \mathbb{Z}$ so that $(h \circ g)(x) = 1$. Then

$$(h \circ g)(x) = h(g(x)) = h\left(\left\lfloor \frac{x}{2} \right\rfloor\right) = 3\left\lfloor \frac{x}{2} \right\rfloor = 1,$$

which means that $\lfloor \frac{x}{2} \rfloor = \frac{1}{3}$. But $\lfloor \frac{x}{2} \rfloor$ must be an integer by definition of floor, and $\frac{1}{3}$ is not an integer. This is a contradiction, so $h \circ g$ is not onto. $\square$

(d) $h \circ g$ is one-to-one.

Solution. $h \circ g$ is not one-to-one.

Proof. Note that $(h \circ g)(0) = h(g(0)) = 3\lfloor \frac{0}{2} \rfloor = 0$ and $(h \circ g)(1) = h(g(1)) = 3\lfloor \frac{1}{2} \rfloor = 0$. So $(h \circ g)(0) = (h \circ g)(1)$ but $0 \neq 1$. $\square$

(e) $g \circ h$ is onto.

Solution. $g \circ h$ is not onto.

Proof. Suppose that there exists an $x \in \mathbb{Z}$ so that $(g \circ h)(x) = 2$. This means that $g(h(2)) = 2$, which is $g(3x) = \lfloor \frac{3x}{2} \rfloor = 2$. This implies that $2 \leq \frac{3x}{2} < 3$ by the definition of floor. This implies that $4 \leq 3x < 6$, or

$$\frac{4}{3} \leq x < 2.$$

But x is an integer. There are no integers that are greater or equal to $\frac{4}{3}$ and less than 2, so x is not an integer. This is a contradiction, since x is an integer. Hence, for all $x \in \mathbb{Z}$, $(g \circ h)(x) \neq 2$. Therefore $(g \circ h)$ is not onto. $\square$

(f) $g \circ h$ is one-to-one.

Solution. $g \circ h$ is one-to-one.

Proof. Let x_1 and x_2 be integers such that $(g \circ h)(x_1) = (g \circ h)(x_2)$. Then

$$\left\lfloor \frac{3x_1}{2} \right\rfloor = \left\lfloor \frac{3x_2}{2} \right\rfloor. \qquad (*)$$

Note that x_1 is either even or odd, so x_1 can be written as $x_1 = 2k_1 + r_1$ for some integers k_1 and r_1, where $r_1 = 0$ or $r_1 = 1$. Similarly, $x_2 = 2k_2 + r_2$ for some integers k_2 and r_2, where $r_2 = 0$ or $r_2 = 1$. Then $(*)$ becomes $\left\lfloor \frac{3(2k_1+r_1)}{2} \right\rfloor = \left\lfloor \frac{3(2k_2+r_2)}{2} \right\rfloor$, which reduces to $\lfloor 3k_1 + \frac{3r_1}{2} \rfloor = \lfloor 3k_2 + \frac{3r_2}{2} \rfloor$. This simplifies to

$$3k_1 + \left\lfloor \frac{3r_1}{2} \right\rfloor = 3k_2 + \left\lfloor \frac{3r_2}{2} \right\rfloor \qquad (**)$$

since $3k_1$ and $3k_2$ are integers. We examine the two cases: $r_1 = 0$ or $r_1 = 1$.

Case 1: Assume $r_1 = 0$. Then $\lfloor \frac{3r_1}{2} \rfloor = 0$ and $(**)$ reduces to $3k_1 = 3k_2 + \lfloor \frac{3r_2}{2} \rfloor$, and thus $3(k_1 - k_2) = \lfloor \frac{3r_2}{2} \rfloor$. Hence

$$k_1 - k_2 = \frac{1}{3} \left\lfloor \frac{3r_2}{2} \right\rfloor .$$

We show that $r_2 = 0$. Assume otherwise that $r_2 = 1$. Then $\frac{1}{3}\lfloor \frac{3r_2}{2} \rfloor = \frac{1}{3}\lfloor \frac{3}{2} \rfloor = \frac{1}{3}$, which is not an integer. This is a contradiction, so $r_2 \neq 1$. Hence $r_2 = 0$, which means that $k_1 - k_2 = 0$ and thus $k_1 = k_2$. Therefore $x_1 = x_2$ since

$$x_1 = 2k_1 + r_1 = 2k_1 + 0 = 2k_2 + 0 = 2k_2 + r_2 = x_2.$$

Case 2: Assume $r_1 = 1$. Then $\lfloor \frac{3r_1}{2} \rfloor = 1$ and $(**)$ reduces to $3k_1 + 1 = 3k_2 + \lfloor \frac{3r_2}{2} \rfloor$, and thus $3(k_1 - k_2) = \lfloor \frac{3r_2}{2} \rfloor - 1$. Hence

$$k_1 - k_2 = \frac{1}{3} \left(\left\lfloor \frac{3r_2}{2} \right\rfloor - 1 \right) .$$

We show that $r_1 = 1$. Assume otherwise that $r_1 = 0$. Then $\frac{1}{3}\left(\lfloor \frac{3r_1}{2} \rfloor - 1\right) = \frac{1}{3}(0 - 1) = -\frac{1}{3}$, which is not an integer. This is a contradiction, so $r_1 \neq 0$. Hence $r_1 = 1$, which means that $k_1 - k_2 = 0$ and thus $k_1 = k_2$. Therefore $x_1 = x_2$ since

$$x_1 = 2k_1 + r_1 = 2k_1 + 1 = 2k_2 + 1 = 2k_2 + r_2 = x_2.$$

In either case, we showed that $x_1 = x_2$. Hence $g \circ h$ is one-to-one. ☐

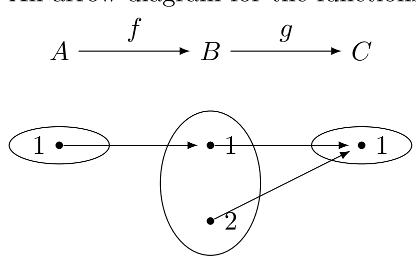
4. Let A, B, and C be some sets and suppose that $f\colon A \to B$ and $g\colon B \to C$ are functions. Prove or disprove each of the following statements.

   (a) If $g \circ f$ is onto then f is onto.

   Solution. This statement is false. The is "There exist sets A, B, and C and functions $f\colon A \to B$ and $g\colon B \to C$ so that $g \circ f$ is onto but f is not onto."

   Proof. Let $A = \{1\}$, $B = \{1, 2\}$, and $C = \{1\}$ and define the functions $f\colon A \to B$ and $g\colon B \to C$ by $f(1) = 1$ and $g(1) = g(2) = 1$. Then $g \circ f$ is onto since $(g \circ f)(1) = 1$ and 1 is the only element in C. ☐

   An arrow diagram for the functions in the proof above is given below.



   (b) If $g \circ f$ is onto then g is onto.

   Solution. This statement is true.

   Proof. Suppose $g \circ f$ is onto. (We will show that g is onto.) Let $c \in C$. (We will show that there exists a $b \in B$ so that $g(b) = c$). Since $g \circ f\colon A \to C$ is onto, there exists an $a \in A$ so that $(g \circ f)(a) = c$. Pick $b = f(a)$. Then $b \in B$ and $g(b) = g(f(a)) = (g \circ f)(a) = c$. Thus g is onto. ☐

   (c) If $g \circ f$ is onto and g is one-to-one then f is onto.

Solution. This statement is true.

Proof. Suppose $g \circ f$ is onto and g is one-to-one. (We will show that f is onto.) Let $b \in B$. (We will show that there exists an $a \in A$ so that $f(a) = b$). Let $c = g(b)$ then $c \in C$. Since $g \circ f$ is onto, there exists an $a \in A$ so that $(g \circ f)(a) = c$. Thus $c = g(b) = (g \circ f)(a) = g(f(a))$. In particular,

$$g(b) = g(f(a)).$$

But g is one-to-one, which means that $b = f(a)$. Therefore f is onto. □

5. Find two functions $f\colon \mathbb{Z} \to \mathbb{Z}$ and $g\colon \mathbb{Z} \to \mathbb{Z}$ so that $f \circ g = I_{\mathbb{Z}}$ but f and g are not invertible.

Solution. Let f be defined by $f(x) = \lfloor \frac{x}{4} \rfloor$ and g be defined by $g(x) = 4x + 1$ for each $x \in \mathbb{Z}$. Then f and g are not invertible. Indeed, f is not one-to-one, since $f(0) = f(1) = 0$ and g is not onto since $g(x) \neq 0$ for all $x \in \mathbb{Z}$. However, we will show that $f \circ g = I_{\mathbb{Z}}$. (That is, we will show that, for all $x \in \mathbb{Z}$, $(f \circ g)(x) = x$.)

Proof (that $f \circ g = I_{\mathbb{Z}}$). Let $x \in \mathbb{Z}$ be arbitrary. Then $(f \circ g)(x) = \lfloor \frac{4x+1}{4} \rfloor = \lfloor x + \frac{1}{4} \rfloor = x + \lfloor \frac{1}{4} \rfloor = x$, since x is an integer and $\lfloor \frac{1}{4} \rfloor = 0$. Hence $(g \circ f)(x) = x = I_{\mathbb{Z}}(x)$, so $f \circ g = I_{\mathbb{Z}}$. □

6. Let $t\colon \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ be the function defined by $t(x, y) = x + y\sqrt{2}$ for all $(x, y) \in \mathbb{Q} \times \mathbb{Q}$. Is t one-to-one? Is t onto? Prove your answers.

Solution. The function is one-to-one but not onto.

Proof (that t is one-to-one). Let (x_1, y_1) and (x_2, y_2) be in $\mathbb{Q} \times \mathbb{Q}$. Assume that $t(x_1, y_1) = t(x_2, y_2)$. This means that $x_1 + y_1\sqrt{2} = x_2 + y_2\sqrt{2}$, which becomes

$$x_2 - x_1 = (y_1 - y_2)\sqrt{2}. \tag{1}$$

(We will show that $x_1 = x_2$ and $y_1 = y_2$.) Suppose instead that that $y_1 \neq y_2$. (We will derive a contradiction.) Then $y_1 - y_2 \neq 0$, so we can divide by $y_1 - y_2$, and this implies that

$$\sqrt{2} = \frac{x_2 - x_1}{y_1 - y_2}.$$

But the ratio of two rational numbers is another rational number. This means that $\sqrt{2}$ is rational, which is a contradiction. Therefore $y_1 = y_2$ and thus $y_1 - y_2 = 0$. Thus $x_2 - x_1 = 0$ from (1), so $x_1 = x_2$. □

We will use the fact that $\sqrt{3}$ is irrational. We will also use a few facts about rational and irrational numbers that we have proved in the course.

Proof (that t is not onto). Suppose there exist rational numbers x and y so that $t(x, y) = \sqrt{3}$. Then $x + y\sqrt{2} = \sqrt{3}$. This implies that $(x + y\sqrt{2})^2 = (\sqrt{3})^2$, or

$$x^2 + 2xy\sqrt{2} + 2y^2 = 3$$

which reduces to $xy\sqrt{2} = \frac{3-x^2-y^2}{2}$. But $\frac{3-x^2-y^2}{2}$ is rational and xy rational, and the only way the product of a rational and an irrational number can be rational is if the rational number is zero. Hence $xy = 0$ so either $x = 0$ or $y = 0$. We will show that "$x = 0$ or $y = 0$" leads to a contradiction.

Suppose that $y = 0$, then $x = \sqrt{3}$. But $\sqrt{3}$ is irrational and x is rational, which is a contradiction. So y cannot be zero. Suppose instead that $x = 0$, then $3 - y^2 = 0$, which means that $y = \sqrt{3}$. But $\sqrt{3}$ is irrational and y is rational, a contradiction.

Hence the assumption that "there exist rational numbers x and y so that $t(x, y) = \sqrt{3}$" is wrong. Therefore t is not onto. □

7. Let $H\colon (\mathbb{R}-\{1\}) \to (\mathbb{R}-\{1\})$ be the function defined by $H(x) = \frac{x+1}{x-1}$ for each $x \in \mathbb{R}-\{1\}$.

   (a) Show that H is one-to-one.

   Proof. Let $x_1, x_2 \in \mathbb{R}-\{1\}$. Suppose that $H(x_1) = H(x_2)$. Then $\frac{x_1+1}{x_1-1} = \frac{x_2+1}{x_2-1}$ and thus $(x_1+1)(x_2-1) = (x_2+1)(x_1-1)$. Multiplying out gives us

   $$x_1x_2 - x_1 + x_1 - 1 = x_1x_2 + x_1 - x_1 - 1$$

   which simplifies to $x_1 = x_2$. Thus H is one-to-one. □

   (b) Show that H is onto.

   Proof. Let $y \in \mathbb{R}-\{1\}$. Then $y \in \mathbb{R}$ and $y \neq 1$. Pick $x = \frac{y+1}{y-1}$, which is allowed since $y-1 \neq 0$. Now

   $$\begin{aligned} H(x) &= \frac{\frac{y+1}{y-1}+1}{\frac{y+1}{y-1}-1} \\ &= \frac{y+1+y-1}{y+1-(y-1)} \\ &= \frac{2y}{2} \\ &= y. \end{aligned}$$

   Thus H is onto. □

   (c) Find a formula for $H^{-1}(x)$ such that $H^{-1}\circ H = H \circ H^{-1} = I_{\mathbb{R}-\{1\}}$.

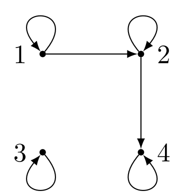
   Solution. From part (d), we see that a formula for H^{-1} is $H^{-1}(x) = \frac{x+1}{x-1}$.

Part II (relations)

1. Let $A = \{1,2,3,4\}$. For each of the following questions, describe your relations as a subset of $A \times A$ (for example $R = \{(1,2),(2,1)\}$) and draw its directed graph.

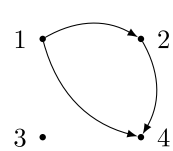
   (a) Find a relation R on A that is reflexive, but is neither symmetric nor transitive.

   Solution. Consider the relation $R = \{(1,1),(1,2),(2,2),(2,4),(3,3),(4,4)\}$ with directed graph:

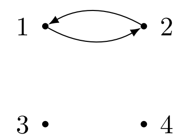
   (b) Find a relation R on A that is transitive, but is neither reflexive nor symmetric.

   Solution. Consider the relation $R = \{(1,2),(1,4),(2,4)\}$ with directed graph:

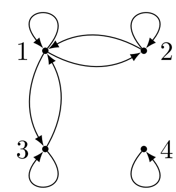
   (c) Find a relation R on A that is symmetric, but is neither reflexive nor transitive.

Solution. Consider the relation $R = \{(1,2),(2,1)\}$ with directed graph:



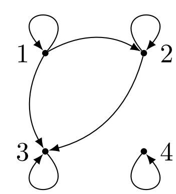
(d) Find a relation R on A that is reflexive and symmetric, but not transitive.

Solution. Consider the relation $R = \{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1),(3,3),(4,4)\}$ with directed graph:



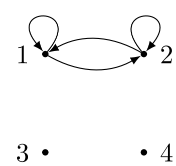
(e) Find a relation R on A that is reflexive and transitive, but not symmetric.

Solution. Consider the relation $R = \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3),(4,4)\}$ with directed graph:

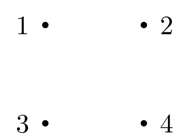


(f) Find a relation R on A that is symmetric and transitive, but not reflexive.

Solution. Consider the relation $R = \{(1,1),(1,2),(2,1),(2,2)\}$ with directed graph:



Alternatively, consider the empty relation $R = \emptyset$ with directed graph:

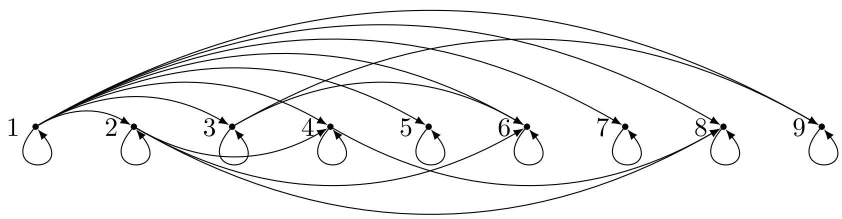


This relation is vacuously symmetric and transitive (there are no arrows to check), and clearly not reflexive.

2. Let $B = \{1,2,3,4,5,6,7,8,9\}$. For each of the following relations, draw the directed graph. For each of the relations, determine whether it is reflexive, symmetric, or transitive.

(a) Define the relation Q on A by $a\,Q\,b \Leftrightarrow a \mid b$.

Solution. Here is the directed graph.



This relation is reflexive and transitive, but not symmetric.

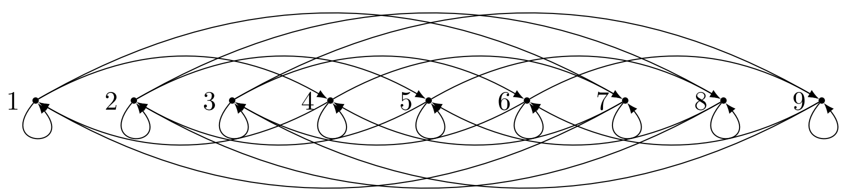
Proof. We prove that Q is reflexive and transitive, but not symmetric.

- Let $a \in B$. Then $a\,Q\,a$ since $a \mid a$ and $a \neq 0$. Hence Q is reflexive.
- Note that $1 \mid 2$ but $2 \mid 1$, so $1\,Q\,2$ but $2 \not{Q}\, 1$. Hence Q is not symmetric.
- Let $a, b, c \in S$ and assume that $a\,Q\,b$ and $b\,Q\,c$. This means that there are integers m and k so that $b = ka$ and $c = mb$. Thus $c = m(ka) = (mk)a$, so $a \mid c$ hence $a\,Q\,c$. There fore Q is transitive.

□

(b) Define the relation R on A by $a\,R\,b \Leftrightarrow 3 \mid (a - b)$.

Solution. Here is the directed graph.



This relation is reflexive, symmetric, and transitive.

Proof. We prove that R is reflexive, symmetric, and transitive.

- Let $a \in B$. Then $a\,R\,a$ since $a - a = 0$ and $3 \mid 0$. Hence R is reflexive.
- Let $a, b \in B$ and assume that $a\,R\,b$. Then $3 \mid (a - b)$ which means that there is an integer k so that $3k = a - b$. Hence $(-k)3 = b - a$, so $3 \mid (b - a)$ and thus $b\,R\,a$. Hence R is symmetric.
- Let $a, b, c \in S$ and assume that $a\,R\,b$ and $b\,R\,c$. Thus $3 \mid (a - b)$ and $3 \mid (b - c)$ This means that there are integers m and k so that $a - b = 3k$ and $c - b = 3m$. Thus

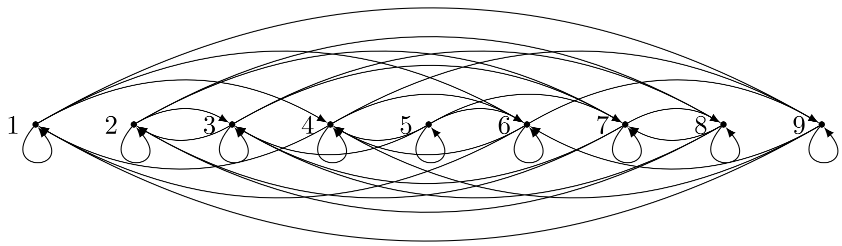
$$\begin{aligned} a - c &= a - b + b - c \\ &= 3k - 3m \\ &= 3(k - m) \end{aligned}$$

so $3 \mid (a - c)$ since $k - m$ is an integer. Hence $a\,R\,c$ so R is transitive.

□

(c) Define the relation S on A by $a\,S\,b \Leftrightarrow 5 \mid (a^2 - b^2)$.

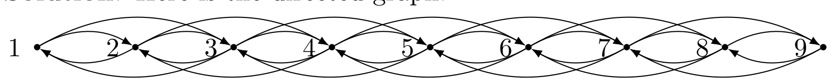
Solution. Here is the directed graph.



This relation is reflexive, symmetric, and transitive.

(d) Define the relation T on A by $a\,T\,b \Leftrightarrow 1 \leq |a-b| \leq 3$. (Proof is similar to part (b).)

Solution. Here is the directed graph.



This relation is symmetric, but not reflexive and not transitive.

Proof. We prove that R is symmetric and not reflexive and not transitive.

- Note that $1 \in B$ and $|1-1| = 0 < 1$. Hence $1 \not{T}\, 1$ so T is not reflexive.
- Let $a, b \in B$ and assume that $a\,T\,b$. Then $1 \leq |a-b| \leq 3$. Since $|a-b| = |b-a|$, this means that $1 \leq |b-a| \leq 3$ and thus $b\,T\,a$. Hence T is symmetric.
- Note that $1\,T\,3$ and $3\,T\,5$ since $1 \leq |1-3| \leq 3$ and $1 \leq |3-5| \leq 3$. However $|1-5| = 4$ and thus $1 \not{T}\, 5$. Hence T is not transitive.

□

3. Let R be the relation on $\mathbb{Z}^+ \times \mathbb{Z}^+$ defined by

$$\forall (a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \quad \text{and } \forall (c,d) \in \mathbb{Z}^+ \times \mathbb{Z}^+, \qquad (a,b)\,R\,(c,d) \text{ if and only if } a+b < c+d.$$

(a) Is R reflexive? Symmetric? Transitive? Prove your answers.

Solution. The relation R is transitive, but neither reflexive nor symmetric.

Proof (that R is transitive). Let (a,b), (c,d), and $(e,f) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. Assume that $(a,b)\,R\,(c,d)$ and $(c,d)\,R\,(e,f)$. Then $a+b < c+d$ and $c+d < e+f$ and thus $1+b < e+f$. Hence $(a,b)\,R\,(e,f)$. Therefore R is transitive. □

Proof (that R is not reflexive). Note that $(1,1) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, but $(1,1) \not{R}\, (1,1)$ since $1+1 \not< 1+1$. □

Proof (that R is not symmetric). Note that $(1,1)\,R\,(2,2)$ since $1+1 < 2+2$. However $(2,2) \not{R}\, (1,1)$ since $2+2 \not< 1+1$. Thus R is not symmetric. □

(b) Is it true that, for all $(a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, there exists $(c,d) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ so that $(a,b)\,R\,(c,d)$? Prove your answer.

Solution. This statement is true.

Proof. Let $(a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. Then $(a,b)\,R\,(a,b+1)$ since $a+b < a+b+1$ and $(a,b+1) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. □

(c) Is it true that, for all $(c,d) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, there exists $(a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ so that $(a,b)\,R\,(c,d)$? Prove your answer.

Solution. This statement is false. It's negation is "There exists $(c,d) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ so that for all $(a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, $(a,b) \not R\, (c,d)$"

Proof (of the negation). Pick $(c,d) = (1,1)$. Let $(a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. Then $a \leq 1$ and $b \leq 1$ and thus $a \not< 1$ and $b \not< 1$. Hence $a + b \not< 1 + 1 = c + d$ and thus $(a,b) \not R\, (c,d)$. □

(d) Is it true that there exists $(c,d) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ so that for all $(a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, $(a,b)\, R\, (c,d)$? Prove your answer.

Solution. This statement is false. It's negation is "For all $(c,d) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, there exists $(a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ so that $(a,b) \not R\, (c,d)$"

Proof (of the negation). Let $(c,d) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. Then $c + d \not< c + d$ and thus $(c,d) \not R\, (c,d)$, so we can pick $(a,b) = (c,d)$. □

(e) How many elements (a,b) in $\mathbb{Z}^+ \times \mathbb{Z}^+$ are there so that $(a,b)\, R\, (3,3)$?

Solution. There are 10 possible pairs since there are 10 pairs $(a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ so that $a + b < 6$. They are:

$$\{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (3,1), (3,2), (4,1)\}$$