MATH 271 – Winter 2011
Final Exam – Solutions

1. (a) Use the Euclidean algorithm to find $\gcd(73, 50)$. Also use the algorithm to find integers x and y such that $\gcd(73, 50) = 73x + 50y$.

   Solution. Use the Euclidean algorithm to find that $\gcd(73, 50) = 1$ and that $1 = 73\cdot(-13)+50\cdot 19$.

   (b) Use part (a) to find an inverse a for 50 modulo 73 so that $0 \leq a \leq 72$; that is, find an integer $a \in \{0, 1, \ldots, 72\}$ so that $50a \equiv 1 \pmod{73}$.

   Solution. From part (a), we see that 19 is an inverse of 50 modulo 73, because

$$50 \cdot 19 = 1 + 13 \cdot 73 \equiv 1 \pmod{73}.$$

2. Let S be the statement:

$$\text{for all sets } A,\ B,\ C,\ \text{ if } A \subseteq B \text{ and } B \cap C = \emptyset \text{ then } A \cap C = \emptyset.$$

   (a) Prove that $\mathcal{S}$ is true. Use contradiction and the element method.

   Solution. *Proof.* Let A, B, and C be sets. Assume that $A \subseteq B$ and $B \cap C = \emptyset$. (We will show that $A \cap C = \emptyset$.) Assume that $A \cap C \neq \emptyset$. Then there exists an element $x \in A \cap C$. This means that $x \in A$ and $x \in C$. Then $x \in B$, since $x \in C$ and $C \subseteq B$. Thus $x \in A$ and $x \in B$, which means that $x \in A \cap B$ and thus $A \cap B \neq \emptyset$. However $A \cap B =$, so this is a contradiction. This means that our assumption that $A \cap C \neq \emptyset$ was wrong. Therefore $A \cap C = \emptyset$. □

   (b) Write out the *converse* of statement $\mathcal{S}$. Is it true or false? Explain.

   Solution. The converse is "For all sets A, B, and C, if $A \cap C = \emptyset$ then $A \subseteq B$ and $B \cap C = \emptyset$." This statement is false. Its negation is: "There exists sets A, B, and C so that $A \cap C = \emptyset$ but either $A \not\subseteq B$ or $B \cap C \neq \emptyset$.

   Proof (*of the negation*). Let $A = \{1\}$, $B = \emptyset$, and $C = \emptyset$. Then $A \cap C = \{1\} \cap \emptyset = \emptyset$, but $\{1\} \not\subseteq \emptyset$ so $A \not\subseteq B$. □

   (c) Write out the *contrapositive* of statement $\mathcal{S}$. Is it true or false? Explain.

   Solution. The contrapositive is "For all sets A,B,C, if $A \cap C \neq \emptyset$ then $A \not\subseteq B$ or $B \cap C \neq \emptyset$." This statement is true, since the contrapositive is always logically equivalent to the original statement, which is true in this case.

3. Let $X = \{1, 2, \ldots, 10\}$. Define the relation R on X by:

$$\text{for all } a, b \in X, \quad a\,R\,b \text{ if and only if } ab \text{ is even.}$$

   (a) Is R reflexive? Symmetric? Transitive? Give reasons.

   Solution. The relation R is symmetric, but neither reflexive nor transitive.

   Proof (*that R is symmetric*). Let $a, b \in X$ and suppose that $a\,R\,b$. Then ab is even, and thus ba is even since $ab = ba$. Hence $b\,R\,a$. □

   Proof (*that R is not reflexive*). Let $a = 1$. Then $aa = 1$, which is not even, so $a \not R\, a$. □

   Proof (*that R is not transitive*). Let $a = 1$, $b = 2$, and $c = 1$. Then $a\,R\,b$ since $ab = 2$, which is even, and $b\,R\,c$ since $bc = 2$, which is even. But $ac = 1$, which is not even, so $a \not R\, c$. □

   (b) Find and simplify the *number* of **two-element** subsets S of X that satisfy the following property: $\forall a \in S, a\,R\,1$. Explain.

Solution. The answer is $\binom{5}{2} = \frac{5!}{3!2!} = \frac{5\cdot 4}{2\cdot 1} = \frac{20}{2} = 10$. The reason is as follows. Suppose S is a set that has the desired property. Then S has two elements, say x and y. Then x and y must both be even, since $1\,R\,x$ and $1\,R\,y$ mean that $1 \cdot x = x$ and $1 \cdot y = y$ must be even. Therefore the only two-element sets with desired property are subsets of X where both elements are even. Since there are 5 even numbers in X, there are $\binom{5}{2}$ (i.e. 5 choose 2) ways to make subsets with the desired property.

(c) Find the *number* of subsets S of X (of any size) that satisfy the following property: $\forall a \in S, \exists b \in S$ so that $a\,R\,b$. Explain

Solution. The answer is $2^{10} - 2^5 = 1024 - 32 = 992$. The reason is as follows. Note that any set that contains *only* odd elements do not satisfy the property, since the product of odd numbers is always odd. If a subset S has at least one even number $e \in S$, then for any $a \in A$ we can always choose $b = e$ so that ab is even. Therefore the sets S that satisfy the property are the sets that contain at least one even element. Let $A \subseteq \mathcal{P}(X)$ be the set of subsets of X that contain only odd elements. Then the set of objects that we want to count is $\mathcal{P}(X) - A$, since this set consists of all subsets of X that contain at least one even element. Note that $A = \mathcal{P}(\{1, 3, 5, 7, 9\})$, so $|A| = 2^5$, and $|\mathcal{P}(X)| = 2^{10}$. Finally,

$$\begin{aligned} |\mathcal{P}(X) - A| &= |\mathcal{P}(X)| - |A| \\ &= 2^{10} - 2^5. \end{aligned}$$

4. Let $\mathscr{F}$ denote the set of all functions from $\{1, 2, 3\}$ to $\{1, 2, 3, 4, 5\}$.

(a) Find and simplify the number of functions $f \in \mathscr{F}$ so that $f(1) = 4$.

Solution. The answer is $5^2 = 25$. The recipe is as follows:

1. Set $f(1)$ to be 4. (One way.)
2. Choose $f(2)$ to be any value from $\{1, 2, 3, 4, 5\}$. (5 ways)
3. Choose $f(3)$ to be any value from $\{1, 2, 3, 4, 5\}$. (5 ways)

So there are $1 \cdot 5 \cdot 5 = 5^2 = 25$ functions $f \in \mathscr{F}$ so that $f(1) = 4$.

(b) Find and simplify the number of *one-to-one* functions $f \in \mathscr{F}$ so that $f(1) \geq 4$.

Solution. The answer is $2 \cdot 4 \cdot 3 = 24$. The recipe is as follows:

1. Choose $f(1)$ to be either 4 or 5. (2 ways.)
2. Choose $f(2)$. It must be different from $f(1)$. (4 ways)
3. Choose $f(3)$. It must be different from $f(1)$ and $f(2)$ (3 ways)

(c) Find and simplify the number of functions $f \in \mathscr{F}$ so that $f(1) \neq f(2)$.

Solution. The answer is $5 \cdot 4 \cdot 5 = 100$. The recipe is as follows:

1. Choose $f(1)$ to be any value from $\{1, 2, 3, 4, 5\}$ (5 ways.)
2. Choose $f(2)$. It must be different from $f(1)$. (4 ways)
3. Choose $f(3)$ to be any value from $\{1, 2, 3, 4, 5\}$ (5 ways.)

5. (a) Give the definition of $a \equiv b \pmod{n}$ (i.e. "a is congruent to b modulo n"), for arbitrary integers a, b, n, where $n > 0$.

Solution. Given integers a, b, n, with $n > 0$, $a \equiv b \pmod{n}$ if and only if $n \mid (a - b)$.

(b) Prove that the relation $\equiv \pmod{n}$ ("congruence modulo n"), on the set $\mathbb{Z}$ of all integers, is **symmetric**. Use your definition from part (a). (Do not assume that the relation is an equivalence relation.)

Solution. *Proof.* Let a and b be arbitrary integers. Assume that $a \equiv b \pmod{n}$. Thus $n \mid (a-b)$. This means that there is an integer k so that $a - b = kn$. Now $b - a = (-k)n$, and $-k$ is an integer so $n \mid (b-a)$. Therefore $b \equiv a \pmod{n}$. □

(c) Now assume that "congruence modulo 7" is an equivalence relation on $\mathbb{Z}$. Find three elements of the equivalence class [3].

Solution. Some elements are: 3, 10, 17.

(d) Again consider the equivalence class [3] for the equivalence relation "congruence modulo 7" on $\mathbb{Z}$. Suppose that $S = \{1, 2, \ldots, N\}$, where N is a positive integer. Find all possible values of N so that $[3] \cap S$ contains exactly 10 elements.

Solution. The first ten positive elements of [3] are 3, 10, 17, 24, 31, 38, 45, 52, 59, and 66. Then eleventh element is 73. For S to contain exactly 10 elements, it must contain all of these elements, so N must be at least 66. However, if N is greater than 72, then S also contains 73 which means that $|[3] \cap S| \geq 11$. Therefore N can be any number from $\{66, 67, 68, 69, 70, 71, 72\}$.

6. $\mathbb{Q}$ is the set of rational numbers. Two of the following statements are true and one is false. Prove the true statements. Write our the *negation* of the false statement and prove it.

(a) $\forall q \in \mathbb{Q}, \exists n \in \mathbb{Z}$ so that $q + n = 271$.

Solution. This statement is false. Its negation is: "There exists a $q \in \mathbb{Q}$ so that for all $n \in \mathbb{Z}$, $q + n \neq 271$."

Proof (of the negation). Let $q = \frac{1}{2}$. Let n be an arbitrary integer. (We will show that $q+n \neq 271$.) Suppose that $q + n = 271$. Then $n = 271 - q - 271 - \frac{1}{2}$. But $271 - \frac{1}{2} = \frac{541}{2}$, which is not an integer. This is a contradiction, since $n \in \mathbb{Z}$. Hence the assumption that $q + n = 271$ was wrong. Therefore $q + n \neq 271$. □

(b) $\forall n \in \mathbb{Z}, \exists q \in \mathbb{Q}$ so that $q + n = 271$.

Solution. This statement is true.

Proof. Let n be an arbitrary integer. Choose $q = 271 - n$. Then q is a rational number and $q + n = 271 - n + n = 271$. □

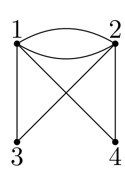
(c) $\exists n \in \mathbb{Z}$ so that $271 - n$ is even.

Solution. This statement is true.

Proof. Choose $n = 271$. Then n is an integer and $271 - n = 0$, which is even. □

7. (a) Draw a graph with exactly 4 vertices and 6 edges, and give its adjacency matrix.

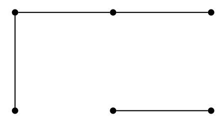
Solution. One example of such a graph and its adjacency matrix is:



$$\begin{bmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

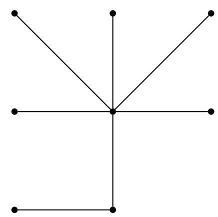
(b) Draw a graph with exactly 6 vertices and 4 edges and exactly two connected components.

Solution. One example of such a graph is:



(c) Draw a **tree** with exactly 8 verticies, one of which has degree 6.

Solution. One example of such a graph is:



(d) Does there exist a graph with exactly 8 vertices, so that three of the vertices have degree 3 and the remaining five vertices have degree 2? Explain.

Solution. No. If such a graph existed, the sum of the degrees of all of its vertices would be: $3+3+3+2+2+2+2+2=19$, which is odd. However, the total sum of the degrees of the vertices of any graph must be even. Thus such a graph can not exist.

8. Prove by induction on n that $6 \mid (7^n + 11)$ for all integers $n \geq 0$.

Solution. Let $P(n)$ be the statement "$6 \mid (7^n + 11)$" for each $n \in \mathbb{Z}$.

Proof. We will prove that $P(n)$ is true for all integers $n \geq 0$ by induction on n.

Base case $(n = 0)$: We have

$$7^0 + 11 \cdot 1 + 11 = 12$$

which is divisible by 6, so $P(0)$ is true.

Induction step: Let $k \geq 1$ be an integer. Suppose that

$$6 \mid (7^k + 11). \tag{IH}$$

(We want to show that $6 \mid (7^{k+1} + 11)$ is divisible by 6.) By IH, there exists an integer m so that $6 \mid 7^k + 11 = 6m$. Then

$$7^k = 6m - 11. \tag{$*$}$$

Now

$$\begin{aligned} 7^{k+1} + 11 &= 5 \cdot 7 \cdot 7^k + 11 \\ &= 7 \cdot (6m - 11) + 11 && \text{by } (*) \\ &= 6 \cdot 7m - 77 + 11 \\ &= 6 \cdot 7m - 66 \\ &= 6 \cdot (7m - 11), \end{aligned}$$

where $7m - 11$ is an integer. Therefore $7^{k+1} + 11$ is divisible by 6.

By the principle of induction, $6 \mid (7^n + 11)$ is divisible by 6 for all integers $n \geq 0$. □