MATH 271 – Winter 2012
Final Exam – Solutions

1. (a) Use the Euclidean algorithm to find $\gcd(100, 57)$. Also use the algorithm to find integers x and y such that $\gcd(100, 57) = 100x + 57y$.

   Solution. Use the Euclidean algorithm to find that $\gcd(100, 57) = 1$ and $1 = 100 \cdot (4) + 57 \cdot (-7)$.

   (b) Use part (a) to find an inverse a for 57 modulo 100 so that $0 \le a \le 99$; that is, find an integer $a \in \{0, 1, \ldots, 99\}$ so that $57a \equiv 1 \pmod{100}$.

   Solution. From part (a), we see that -7 is an inverse of 57 modulo 100, because

$$57 \cdot (-7) = 1 + y \cdot 100 \equiv 1 \pmod{100}.$$

   Then 93 is another inverse of 57 modulo 100, because

$$57 \cdot (93) = 57 \cdot (-7) + 57 \cdot 100 \equiv 57 \cdot (-7) \equiv 1 \pmod{100}$$

   and thus $57 \cdot 93 \equiv 1 \pmod{100}$.

2. In this question, you may assume that every integer is either even or odd but not both. Otherwise use no facts about even or odd integers except for the definition. Let $\mathcal{S}$ be the statement

$$\text{for all integers } a \text{ and } b, \text{ if } a \text{ is odd and } b \mid a \text{ then } b \text{ is odd.}$$

   (a) Prove that $\mathcal{S}$ is true. Use contradiction.

   Solution. *Proof.* Let a and b be arbitrary integers. Assume that a is odd and $b \mid a$. Then there exists an integer k so that $a = bk$. (We want to show that b is odd.) Assume for the sake of getting a contradiction that b is even. Then $b = 2m$ for some integer m. However, $a = bk = (2m)k = 2(mk)$ and thus a is even, since mk is an integer. So a is both even and odd, which is a contradiction. Thus the assumption that b is even is wrong, so b is odd. □

   (b) Write out the *converse* of statement $\mathcal{S}$. Is it true or false? Explain.

   Solution. The converse is "For all integers a and b, if b is odd then a is odd and $b \mid a$." This statement is false. Its negation is: "There exists integers a and b such that b is odd, but a is even or $b \nmid a$".

   Proof (of the negation). Let $a = 1$ and $b = 1$. Then b is odd, but a is also odd, so a is not even. □

   (c) Write out the *contrapositive* of statement $\mathcal{S}$. Is it true or false? Explain.

   Solution. The contrapositive is "For all integers a and b, if b is even then a is even or $b \nmid a$." This statement is true, since the contrapositive is always logically equivalent to the original statement, which is true in this case.

3. Define the relation R on the set $\mathbb{Z}^+$ by:

$$\text{for all } a, b \in \mathbb{Z}^+, \quad a\,R\,b \text{ if and only if } \gcd(a, b) > 1.$$

   (a) Is R reflexive? Symmetric? Transitive? Give reasons.

   Solution. The relation R is symmetric, but neither reflexive nor transitive.

   Proof (that R is symmetric). Let $a, b \in \mathbb{Z}^+$ and suppose that $a\,R\,b$. Then $\gcd(a, b) > 1$, and thus $\gcd(b, a) > 1$ since $\gcd(a, b) = \gcd(b, a)$. Hence $b\,R\,a$. □

   Proof (that R is not reflexive). Let $a = 1$. Then $\gcd(a, a) = \gcd(1, 1) = 1$, which is not greater than 1, so $a \not\!R\, a$. □

Proof (*that R is not transitive*). Let $a = 2$, $b = 6$, and $c = 3$. Then $a \, R \, b$ since $\gcd(2, 6) = 2$ and $\gcd(6, 3) = 3$. But $\gcd(a, c) = \gcd(2, 3) = 1$, so $a \not R \, c$. □

(b) Find and simplify the *number* of integers $a \in \{1, 2, 3, \ldots, 100\}$ so that $a \, R \, 4$. Explain.

Solution. The answer is 50. The reasoning is as follows. If a is a number such that $\gcd(a, 4) = 1$ then the only common divisor of a and b is 1. But the divisors of 4 are 1, 2, and 4. Hence any even number has a common divisor with 4, so $\gcd(a, 4) \geq 2$ for any even number a. If a is odd, then $\gcd(a, 4) = 1$. So we need to count all of the even numbers from 1 to 100. There are 50.

(c) Find the *number* of subsets $a \in \{1, 2, 3, \ldots, 100\}$ so that $a \, R \, 10$. Explain

Solution. The answer is 60. The reasoning is as follows. We need to count all of the integers from 1 to 100 that share at least one common divisor with 10 that is not 1. The divisors of 10 are 2, 5, and 10. So we must count all of the integers in the range that are divisible by either 2 or 5. The number of integers in the range that are divisible by 2 is 50. The number of integers that are divisible by 5 is 20. The number of integers that are divisible by both 2 and 5 are the ones divisible by 10. There are 10 of those. Hence, the number of integers a such that $a \, R \, 10$ is

$$50 + 20 - 10 = 60.$$

4. Let $\mathcal{S}$ be the statement: for all sets A, B, C, if $B \cap C = \emptyset$ then $B \subseteq A$ and $A \cap C = \emptyset$.

(a) **Disprove** $\mathcal{S}$ by writing out and proving its negation.

Solution. The negation is: "There exist sets A, B, and C so that $B \cap C = \emptyset$ but either $B \not\subseteq A$ or $A \cap C \neq C$."

Proof. Let $A = \emptyset$, $B = \{1\}$ and $C = \emptyset$. Then $B \cap C = \{1\} \cap \emptyset = \emptyset$ but $B \not\subseteq A$ since $\{1\} \not\subseteq \emptyset$. □

(b) Write out the *converse* of $\mathcal{S}$, and prove it using the element method and contradiction.

Solution. The converse is: "For all sets A, B, and C, if $B \subseteq A$ and $A \cap C = \emptyset$ then $B \cap C = \emptyset$."

Proof. Let A, B, and C be sets. Assume that $B \subseteq A$ and $A \cap C = \emptyset$. (We will show that $B \cap C = \emptyset$.) Assume that $B \cap C \neq \emptyset$. Then there exists an element $x \in B \cap C$. This means that $x \in B$ and $x \in C$. Then $x \in A$, since $x \in B$ and $B \subseteq A$. Thus $x \in A$ and $x \in C$, which means that $x \in A \cap C$ and thus $A \cap C \neq \emptyset$. However $A \cap C = \emptyset$, so this is a contradiction. This means that our assumption that $B \cap C \neq \emptyset$ was wrong. Therefore $B \cap C = \emptyset$. □

5. Let $\mathcal{X}$ be the set of all *nonempty* subsets of the set $\{1, 2, 3, \ldots, 10\}$. Define a relation $\mathscr{R}$ on $\mathcal{X}$ by:

for all $A, B \in \mathcal{X}$, $A \, \mathscr{R} \, B$ if and only if the smallest element of A is equal to the smallest element of B.

For example, $\{1, 2, 3\} \, \mathscr{R} \, \{1, 3, 5, 8\}$ because the smallest element of $\{1, 2, 3\}$ is 1 which is also the smallest element of $\{1, 3, 5, 8\}$.

(a) Prove that $\mathscr{R}$ is an equivalence relation on $\mathcal{X}$.

Solution. *Proof.* We prove that $\mathscr{R}$ is reflexive, symmetric, and transitive.

- (Reflexive) Let $A \in \mathcal{X}$. Then A is a nonempty subset of $\{1, 2, \ldots, 10\}$. The smallest element of A is equal to itself. Thus $A \, \mathscr{R} \, A$. Hence $\mathscr{R}$ is reflexive.
- (Symmetric) Let $A, B \in \mathcal{X}$ and assume that $A \, \mathscr{R} \, B$. Let a be the smallest element of A and $b \in B$ be the smallest element of B. Then $a = b$ since $A \, \mathscr{R} \, B$. Thus $b = a$, which means that $B \, \mathscr{R} \, A$ and thus $\mathscr{R}$ is symmetric.

- (Transitive) Let $A, B, C \in \mathcal{X}$. Assume that $A \mathscr{R} B$ and $B \mathscr{R} C$. Let $a \in A$ be the smallest element of A, let $b \in B$ be the smallest element of B, and let $c \in C$ be the smallest element of C. Then $a = b$ and $b = c$, since $A \mathscr{R} B$ and $B \mathscr{R} C$. Thus $a = b = c$ and so $a = c$. Hence $A \mathscr{R} C$ and thus $\mathscr{R}$ is transitive.

Thus $\mathscr{R}$ is an equivalence relation because it is reflexive, symmetric, and transitive. ☐

(b) Find and simplify the number of equivalence classes of $\mathcal{R}$. Explain.

Solution. There are 10 different equivalence classes of $\mathscr{R}$ since there are 10 different possibilities for the smallest element of a subset of $\{1, 2, 3 \ldots, 10\}$.

(c) Find and simplify the number of elements in the equivalence class $[\{2, 6, 7\}]$. Explain.

Solution. The answer is $2^8 = 256$ elements. The reasoning is as follows. Note that 2 is the smallest element of $\{2, 6, 7\}$. A subset $A \in \mathcal{X}$ is related to $\{2, 6, 7\}$ under $\mathscr{R}$ if and only if the smallest element of A is 2. The number of subsets of $\{1, 2, 3, \ldots, 10\}$ that have 2 as the smallest element is 2^8. The recipe is as follows:

1. Make a subset of $\{3, 4, 5, 6, 7, 8, 9, 10\}$. There are 2^8 ways.
2. Add 2 to the subset.

(d) Find and simplify the number of *four-element* sets which are elements of the equivalence class $[\{2, 6, 7\}]$. Explain.

Solution. The answer is $\binom{8}{3} = \frac{8!}{5!3!} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 8 \cdot 7 = 56$. The reasoning is as follows. We must count the number of four-element sets that have 2 as there smallest element. These sets must contain 2, cannot contain 1, and must contain 3 more elements from $\{3, 4, 5, 6, 7, 8, 9, 10\}$. There are $\binom{8}{3}$ ways.

6. Let $\mathscr{F}$ denote the set of all functions from $\{1, 2, 3\}$ to $\{1, 2, 3\}$.

(a) One of statements (i) and (ii) is true and one is false. Prove the true statement. Write out the *negation* of the false statement and prove it.

(i) $\forall f \in \mathscr{F}, \exists g \in \mathscr{F}$ so that $g(f(1)) = 2$.

Solution. This statement is true.

Proof. Let $f \in \mathscr{F}$ be a function. Choose $g \in \mathscr{F}$ to be the function defined by $g(1) = 2$, $g(2) = 2$, and $g(3) = 2$. Then $g(f(1)) = 2$, since $f(1) \in \{1, 2, 3\}$. ☐

(ii) $\forall f \in \mathscr{F}, \exists g \in \mathscr{F}$ so that $f(g(1)) = 2$.

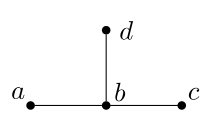
Solution. This statement is false. Its negation is: "$\exists f \in \mathcal{F}$ so that $\forall g \in \mathscr{F},\ f(g(1)) \neq 2$."

Proof (of the negation). Let $f \in \mathscr{F}$ be the function defined by $f(x) = 1$ for all $x \in \{1, 2, 3\}$. Let $g \in \mathcal{F}$ be arbitrary. Then $g(1) \in \{1, 2, 3\}$ but $f(g(1)) = 1$, which is not equal to 2. ☐

(b) Let $f \in \mathscr{F}$ be defined by: $f(1) = 2$, $f(2) = 3$, $f(3) = 2$. Find and simplify the *number* of functions $g \in \mathscr{F}$ so that $f(g(f(1))) = 2$.

Solution. The answer is $2 \cdot 3 \cdot 3 = 18$. Suppose $g \in \mathscr{F}$ is a function so that $f(g(f(1))) = 2$. Note that $f(1) = 2$, so this means that $f(g(2)) = 2$. Then $g(2) \neq 2$, otherwise $f(g(2)) = 3$. Hence, a recipe for making functions g that satisfy the desired property is as follows:

1. Choose $g(2)$. It can't be 2, so there are 2 choices (1 or 3).
2. Choose $g(1)$. There are 3 choices.
3. Choose $g(3)$. There are 3 choices.

7. Let G be the graph 

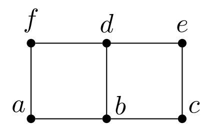
(a) Give the adjacency matrix of G.

Solution. The adjacency matrix of G is:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

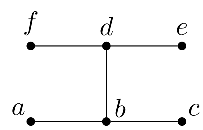
(b) Draw a simple graph H with exactly six vertices a, b, c, d, e, f and exactly seven edges and so that G is a subgraph of H.

Solution. One example of such a graph is:



(c) Draw a **tree** T with exactly 6 vertices a, b, c, d, e, f, each with degree either 1 or 3 and so that G is a subgraph of T.

Solution. One example of such a graph is:



8. Prove by induction on n that

$$1 \cdot 1 + 2 \cdot 3 + 3 \cdot 5 + \cdots + n(2n-1) > \frac{n^3}{2}$$

for all integers $n \geq 1$.

Solution. Note that the sum can be written as $\sum_{i=1}^{n} i(2i-1)$. For each $n \geq 1$, let $P(n)$ be the statement "$\sum_{i=1}^{n} i(2i-1) > \frac{n^3}{2}$".

Proof. We will prove that $P(n)$ is true for all integers $n \geq 1$ by induction on n.

Base case $(n = 1)$: We have $\sum_{i=1}^{n} i(2i-1) = 1 \cdot 1 = 1$ and $1 > \frac{1}{2} = \frac{1^3}{2}$, so $P(1)$ is true.

Induction step: Let $k \geq 1$ be an integer. Suppose that

$$\sum_{i=1}^{k} i(2i-1) > \frac{n^3}{2} \qquad \text{(IH)}$$

(We want to show that $\sum_{i=1}^{k+1} i(2i-1) > \frac{(k+1)^3}{2}$.) Now

$$\begin{aligned}
\sum_{i=1}^{k+1} i(2i-1) &= \sum_{i=1}^{k} i(2i-1) + (k+1)(2(k+1)-1) \\
&= \sum_{i=1}^{k} i(2i-1) + (k+1)(2k+1) \\
&> \frac{k^3}{2} + 2k^3 + 3k + 1 && \text{by IH} \\
&= \frac{k^3}{2} + \frac{4k^3 + 6k + 2}{2} \\
&= \frac{k^3}{2} + \frac{3k^3 + 3k + 1}{2} + \frac{k^3 + 3k + 1}{2} \\
&= \frac{k^3 + 3k^2 + 3k + 1}{2} + \frac{k^3 + 3k + 1}{2} \\
&> \frac{k^3 + 3k^2 + 3k + 1}{2} && \text{where } k^3 + 3k + 1 > 0 \text{ since } k \geq 1 \\
&= \frac{(k+1)^3}{2},
\end{aligned}$$

which is what we wanted to show.

By the principle of induction, $\sum_{i=1}^{n} i(2i-1) > \frac{n^3}{2}$ for all integers $n \geq 1$. □