THE UNIVERSITY OF CALGARY
FACULTY OF SCIENCE
MATHEMATICS 271 (L01, L02)
FINAL EXAMINATION, WINTER 2013
TIME: 3 HOURS

NAME______________________________ID____________Section______

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Total (max. 80)	

SHOW ALL WORK. NO CALCULATORS PLEASE.

THE MARKS FOR EACH PROBLEM ARE GIVEN TO THE LEFT OF THE PROBLEM NUMBER. TOTAL MARKS [80]. THIS EXAM HAS 9 PAGES INCLUDING THIS ONE.

[8] 1. (a) Use the Euclidean algorithm to find $\gcd(77, 62)$. Also use the algorithm to find integers x and y such that $\gcd(77, 62) = 77x + 62y$.

(b) Use part (a) to find an inverse a for 62 modulo 77 so that $0 \leq a \leq 76$; that is, find an integer $a \in \{0, 1, \ldots, 76\}$ so that $62a \equiv 1 \pmod{77}$.

[12] 2. Let S be the statement:

for all positive integers a and b, if $a \mid b$ then $(5a) \mid (5b)$.

(a) Prove that S is true. Use the definition of "|" ("divides into").

(b) Write out the *converse* of statement S. Is it true or false? Give a proof or counterexample.

(c) Write out the *negation* of statement S. Is it true or false? Explain.

[12] 3. Let $\mathcal{S}$ be the power set $\mathscr{P}(\{1,2,\ldots,10\})$; that is, $\mathcal{S}$ is the set of all subsets of $\{1,2,\ldots,10\}$. Define the relation $\mathscr{R}$ on $\mathcal{S}$ by:

for all subsets A, B of $\{1,2,\ldots,10\}$, $A \mathscr{R} B$ if and only if $A \cup B$ has exactly 3 elements.

(a) Is $\mathscr{R}$ reflexive? Symmetric? Transitive? Give reasons.

(b) Find and simplify the *number* of subsets $A \subseteq \{1,2,3,\ldots,10\}$ so that $A \mathscr{R} \{1,2,7\}$. Explain.

(c) Find and simplify the *number* of subsets $A \subseteq \{1,2,3,\ldots,10\}$ so that $A \mathscr{R}\, \emptyset$. Explain.

[5] 4. (a) Write out the *contrapositive* of the following statement:

for all positive real numbers r, if r is irrational then $\sqrt{r}$ is irrational.

(b) Prove the statement in part (a) by using contradiction or the contrapositive. (Use no facts about rationals or irrationals except for the definitions.)

[5] 5. One of the following statements is true and one is false. Prove the true statement. Write out and prove the *negation* of the false statement.

(a) $\forall A \subseteq \mathbf{Z}\ \exists B \subseteq \mathbf{Z}$ so that $(1,2) \in A \times B$.

(b) $\forall A \subseteq \mathbf{Z}\ \exists B \subseteq \mathbf{Z}$ so that $(1,2) \notin A \times B$.

[12] 6. Define the relation R on the set $\mathbf{Z}^+$ of all positive integers by: for all $a, b \in \mathbf{Z}^+$, aRb if and only if the largest digit of a is equal to the largest digit of b. For example, $271\ R\ 770$ because the largest digit of 271 is 7 which is also the largest digit of 770.

(a) Prove that R is an equivalence relation on $\mathbf{Z}^+$.

(b) Find the *number* of equivalence classes of R. Explain.

(c) Find and simplify the *number* of positive integers between 100 and 1000 which are in the equivalence class [271]. Explain.

[12] 7. (a) Suppose that $f : \mathbf{Z} \to \mathbf{Z}$ is a one-to-one function. Define a function $g : \mathbf{Z} \to \mathbf{Z}$ by: for all $x \in \mathbf{Z}$, $g(x) = -f(x)$. Prove that g is also one-to-one.

(b) Suppose that $f : \mathbf{Z} \to \mathbf{Z}$ is an onto function. Define a function $g : \mathbf{Z} \to \mathbf{Z}$ by: for all $x \in \mathbf{Z}$, $g(x) = f(x) + 4$. Prove that g is also onto.

(c) Suppose that f and g are one-to-one functions from $\mathbf{Z}$ to $\mathbf{Z}$. Define the function $h : \mathbf{Z} \to \mathbf{Z}$ by $h(x) = f(x) + g(x)$ for all $x \in \mathbf{Z}$. Must h be one-to-one? Give a proof or counterexample.

[8] 8. (a) Draw a simple graph G with exactly seven vertices and exactly ten edges, and so that some vertex of G has degree 6.

(b) Answer part (a) again, but so that your graph G does **not** have an Euler circuit. (Be sure to explain why you know that G does not have an Euler circuit.)

(c) Answer part (a) again, but so that your graph G **does** have an Euler circuit. (Be sure to explain why you know that G has an Euler circuit.)

(d) Draw a **tree** T with exactly seven vertices and so that two of the vertices have degree 3.

[6] 9. Define the sequence $a_1, a_2, a_3, \ldots$ by: $a_1 = 1$, and $a_n = 7a_{n-1} + 4$ for all integers $n \geq 2$. Prove **by induction on** n that a_n is odd for all integers $n \geq 1$. (Use no facts about odd integers except the definition.)

[80]