MATH 271 – Winter 2014
Final Exam – Solutions

1. (a) Use the Euclidean algorithm to find $\gcd(102, 47)$. Also use the algorithm to find integers x and y such that $\gcd(102, 47) = 102x + 47y$.

   Solution. Use the Euclidean algorithm to find that $\gcd(102, 47) = 1$ and $1 = 102 \cdot (6) + 47 \cdot (-13)$.

   (b) Use part (a) to find an inverse a for 47 modulo 102 so that $0 \le a \le 101$; that is, find an integer $a \in \{0, 1, \ldots, 101\}$ so that $47a \equiv 1 \pmod{100}$.

   Solution. From part (a), we see that -13 is an inverse of 47 modulo 102,

$$47 \cdot (-13) = 1 + 102 \cdot (6) \equiv 1 \pmod{102}.$$

   Then 89 is another inverse of 47 modulo 102, because $89 = -13 + 102$ and

$$47 \cdot (89) = 47 \cdot (-13) + 47 \cdot (102) \equiv 47 \cdot (-13) \equiv 1 \pmod{102}.$$

2. For this problem, use not facts about $|$ ("divides into") other than its definition. Recall that $\mathbb{Z}$ denotes the set of all integers. Let $\mathcal{P}$ be the statement:

$$\text{“For all positive integers } a \text{ and } b, \text{ if } a \mid b \text{ then } (10a) \mid (2b).''$$

   (a) Is $\mathcal{P}$ true? Prove your answer.

   Solution. No, the statement is false.

   Proof (*that the statement is false, by counterexample*). Let $a = 1$ and $b = 1$. Then $a \mid b$ because $1 \mid 1$. But $10 \nmid 2$, so $(10a) \nmid (2b)$. □

   (b) Write out the *converse* of statement $\mathcal{P}$. Is the converse of $\mathcal{P}$ true? Explain.

   Solution. The converse is "For all integers a and b, if $(10a) \mid (2b)$ then $a \mid b$." This statement is true.

   Proof. Let a and b be arbitrary positive integers. Assume that $(10a) \mid (2b)$. Then there exists an integer k so that $2b = 10ak$. Dividing this equation by 2 gives us that $b = (5k)a$, where $5k$ is an integer. Thus $a \mid b$. □

   (c) Write out the *contrapositive* of $\mathcal{P}$. Is the contrapositive of $\mathcal{P}$ true? Explain.

   Solution. The contrapositive is "For all integers a and b, if $(10a) \nmid (2b)$ then $a \nmid b$." This statement is false, since the contrapositive is always logically equivalent to the original statement, which is false.

3. Of the following statements, one is true and one is false. Use the "element method" to prove the true statement. For the false statement, write out its negation and prove that.

   (a) For all sets A, B, and C, if $B \subseteq C$ then $A - C \subseteq A - B$.

   Solution. This statement is true.

   Proof. Let A, B, and C be arbitrary sets. Suppose that $B \subseteq C$. (We want to show that $A - C \subseteq A - B$.) Let $x \in A - C$. (We want to show that x is also in $A - B$.) Then $x \in A$ and $x \notin C$. (We will show that $x \notin B$.) Suppose that $x \in B$. Then $x \in C$, since $B \subseteq C$. But this is a contradiction, since we know that $x \notin C$. Then the assumption that $x \in B$ is wrong, hence $x \notin B$. Therefore $x \in A$ and $x \notin B$, which means that $x \in A - B$. □

(b) For all sets A, B, and C, if $A - C = A - B$ then $B = C$.

Solution. This statement is false. Its negation is: "There exist sets A, B, and C so that $A - C = A - B$ but $B \neq C$."

Proof (of the negation). Let $A = \emptyset$, $B = \emptyset$ and $C = \{1\}$. Then $A - B = \emptyset - \emptyset = \emptyset$ and $A - C = \emptyset - \{1\} = \emptyset$, so $A - B = A - C$. But $\emptyset \neq \{1\}$, thus $B \neq C$. □

4. Let f and g be functions from $\mathbb{Z}$ to $\mathbb{Z}$ defined by $f(x) = 2x$ and $g(x) = \left\lfloor \frac{x}{2} \right\rfloor$ for any $x \in \mathbb{Z}$.

(a) Find $f \circ g(1)$, $f \circ g(2)$, and $f \circ g(3)$.

Solution. We have $f \circ g(1) = f(g(1)) = 2\left\lfloor \frac{1}{2} \right\rfloor = 0$, $f \circ g(2) = f(g(2)) = 2\left\lfloor \frac{2}{2} \right\rfloor = 2$, and $f \circ g(2) = f(g(3)) = 2\left\lfloor \frac{3}{2} \right\rfloor = 2$.

(b) Is $f \circ g$ onto $\mathbb{Z}$? Explain.

Solution. No, $f \circ g$ is not onto.

Proof. Suppose that $f \circ g$ is onto. Then there exists an integer $x \in \mathbb{Z}$ so that $f \circ g(x) = 1$, since $1 \in \mathbb{Z}$. This means that $2\left\lfloor \frac{x}{2} \right\rfloor = 1$, and thus $\left\lfloor \frac{x}{2} \right\rfloor = \frac{1}{2}$. This is a contradiction, since $\left\lfloor \frac{x}{2} \right\rfloor$ must be an integer by definition of floor, but $\frac{1}{2}$ is not an integer. Hence there is no integer x so that $f \circ g(x) = 1$. Therefore $f \circ g$ is not onto. □

(c) Find $g \circ f(1)$, $g \circ f(2)$, and $g \circ f(3)$.

Solution. We have $g \circ f(1) = g(f(1)) = \left\lfloor \frac{2 \cdot 1}{2} \right\rfloor = 1$, $g \circ f(2) = g(f(2)) = \left\lfloor \frac{2 \cdot 2}{2} \right\rfloor = 2$, and $g \circ f(3) = g(f(3)) = \left\lfloor \frac{2 \cdot 3}{2} \right\rfloor = 3$.

(d) Is $g \circ f$ one-to-one? Explain.

Solution. Yes, $g \circ f$ is one-to-one.

Proof. Let x_1 and x_2 be arbitrary integers. Suppose that $(g \circ f)(x_1) = (g \circ f)(x_2)$. (We will show that $x_1 = x_2$.) This means that $\left\lfloor \frac{2x_1}{2} \right\rfloor = \left\lfloor \frac{2x_2}{2} \right\rfloor$, which simplifies to $\lfloor x_1 \rfloor = \lfloor x_2 \rfloor$. But $\lfloor x_1 \rfloor = x_1$ and $\lfloor x_2 \rfloor = x_2$ since x_1 and x_2 are integers. Therefore $x_1 = x_2$. Thus $g \circ f$ is one-to-one. □

5. Let $A = \{1, 2, 3, \ldots, 2014\} = \{x \mid 1 \leq x \leq 2014\}$. Let $\mathcal{P}$ be the set of **non-empty** subsets of A. Define the relation R on $\mathcal{P}$ by:

for any $X, Y \in \mathcal{P}$, $\quad X\,R\,Y$ if and only if the largest element of X equals the largest element of Y.

(a) Prove that R is an equivalence relation on $\mathcal{P}$.

Solution. *Proof.* We prove that R is reflexive, symmetric, and transitive.

- (Reflexive) Let $X \in \mathcal{P}$. Then X is a non-empty subset of A. The largest element of X is equal to itself. Thus $X\,R\,X$. Hence R is reflexive.
- (Symmetric) Let $X, Y \in \mathcal{P}$. Then X and Y are non-empty subsets of A. Assume that $X\,R\,Y$. Then the largest element of X is equal to the largest element of Y. Hence the largest element of Y is equal to the largest element of X. So $Y\,R\,X$. Therefore R is symmetric.
- (Transitive) Let $X, Y, Z \in \mathcal{P}$. Then X, Y, and Z are non-empty subsets of A. Assume that $X\,R\,Y$ and $Y\,R\,Z$. Then the largest element of X is equal to the largest element of Y, which is equal to the largest element of Z. Hence $X\,R\,Z$ and thus R is transitive.

Thus R is an equivalence relation because it is reflexive, symmetric, and transitive. □

(b) List all of the elements of $[\{3\}]$ (the equivalence class of $\{3\}$).

Solution. The equivalence class of $\{3\}$ under the equivalence relation R is

$$[\{3\}] = \Big\{\{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}\Big\}.$$

(c) How many equivalence classes does R have? Explain.

Solution. There are 2014 equivalence classes of R. The reasoning is as follows. For any subset X of A, the largest element of X must be one of $\{1, 2, \ldots, 2014\}$. Furthermore, for each of the 2014 elements x of $\{1, 2, \ldots, 2014\}$, there is a set $\{x\} \subseteq A$ whose largest element is x.

(d) How many elements does the equivalence class $[\{271\}]$ have? Explain.

Solution. There are $2^{270} - 1$ elements of $[\{271\}]$. The reasoning is as follows. A set X is related to $\{271\}$ if and only if the largest element of X is 271, since the largest element of $\{271\}$ is 271. To make a subset $X \subseteq A$ that is related to $\{271\}$, we make a non-empty subset of $\{1, 2, 3, \ldots, 270\}$ and include 271 in it. There are 2^{270} subsets of $\{1, 2, 3, \ldots, 270\}$, one of which is empty.

6. Let $A = \{1, 2, 3, \ldots, 2014\} = \{x \mid 1 \le x \le 2014\}$. Define the relation R on A by:

for any $x, y \in A$, $\quad x \, R \, y$ if and only if there exists a prime p so that $p \mid x$ and $p \mid y$.

(a) Is R reflexive? symmetric? transitive? Explain.

Solution. The relation R is symmetric, but neither reflexive nor transitive.

Proof (*that R is not reflexive*). Note that $1 \in A$, but there are no primes p so that $p \mid 1$. So $1 \not{R} 1$. Hence R is not reflexive. □

Proof (*that R is symmetric*). Let $x, y \in A$ and suppose that $x \, R \, y$. Then there exists a prime p so that $p \mid x$ and $p \mid y$. Hence $p \mid y$ and $p \mid x$, and thus $y \, R \, x$. Hence R is symmetric. □

Proof (*that R is not transitive*). Let $x = 3$, $y = 6$, and $z = 2$. Then $x \, R \, y$ since $3 \mid 3$ and $3 \mid 6$, and 3 is prime. Also $y \, R \, z$ since $2 \mid 6$ and $2 \mid 2$, and 2 is prime. But there are no primes that divide both 2 and 3, so $2 \nmid 3$. Hence R is not transitive. □

(b) Find three elements a, b, c of A so that $271 \, R \, a$, $271 \, R \, b$ and $271 \, R \, c$.

Solution. Three examples of elements a, b, and c are $a = 542 = 271 \cdot 2$, $b = 813 = 271 \cdot 3$ and $c = 1084 = 271 \cdot 4$.

(c) How many elements x of A are there so that $271 \, R \, x$? Explain.

Solution. There are 7 elements. The reasoning is as follows. Since 271 is prime, the only prime that divides 271 is 271. Hence, any element $x \in A$ so that $271 \, R \, x$ must be divisible by 271. There are 7 multiples of 271 that are less than or equal to 2014. They are:

$$271, \; 271 \cdot 2 = 542, \; 271 \cdot 3 = 813, \; 271 \cdot 4 = 1084, \; 271 \cdot 5 = 1355, \; 271 \cdot 6 = 1626, \text{ and } 271 \cdot 7 = 1897.$$

(Note that $271 \cdot 8 = 2168$, which is greater than 2014.)

7. Only one of the following statements is true. Prove the true statement. For the other two statements, write out their negations and prove them. You can use the fact that $\sqrt{2}$ is irrational. For irrational numbers other than $\sqrt{2}$, you must explain why they are irrational.

(a) For all non-zero real numbers a and b, if a is rational and b is irrational then ab is irrational.

Solution. This statement is true.

Proof. Let a and b be arbitrary non-zero real numbers. Assume that a is rational and b is irrational. Since a is rational, there exist integers x and y so that $a = \frac{x}{y}$ and $y \neq 0$. Furthermore, $x \neq 0$ since $a \neq 0$. (We will show that ab is irrational.) Suppose that ab is rational. Then there exist integers m and n so that $ab\frac{m}{n}$ and $n \neq 0$. Then $\frac{m}{n} = ab = \frac{x}{y}b$. Since $x \neq 0$ we can divide by x and multiply by y to both sides of the equation. This gives us

$$b = \frac{my}{nx}.$$

where my and nx are integers and $nx \neq 0$ since $x \neq 0$ and $n \neq 0$. Hence b is rational. This is a contradiction, since b is irrational. Thus the assumption that ab is rational is wrong. Therefore ab is irrational. □

(b) For all real numbers a and b, if both a and b are irrational then $a + b$ is irrational.

Solution. This statement is false. Its negation is "There exist real numbers a and b so that both a and b are irrational but $a + b$ is not irrational."

Proof (of the negation). Let $a = \sqrt{2}$ and let $b = -\sqrt{2}$. Then a is irrational. We first show that b is irrational. Suppose instead that b is rational. Then there exist integers m and n so that $-\sqrt{2} = \frac{m}{n}$ and $n \neq 0$. Then $\sqrt{2} = -\frac{m}{n} = \frac{-m}{n}$ where $-m$ is an integer and $n \neq 0$. Thus $\sqrt{2}$ is rational. This is a contradiction, so the assumption that b is rational is false. Hence b is irrational. Now $a + b = \sqrt{2} - \sqrt{2} = 0$, which is rational. □

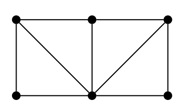
(c) For all real numbers a and b, if both a and b are irrational then ab is irrational.

Solution. This statement is false. Its negation is "There exist real numbers a and b so that both a and b are irrational but ab is rational."

Proof (of the negation). Let $a = \sqrt{2}$ and $b = \sqrt{2}$. Then both a and b are irrational, since $\sqrt{2}$ is irrational. But $ab = \sqrt{2}\sqrt{2} = 2$, which is rational. □

8. (a) Draw a simple graph with exactly six vertices and exactly nine edges.

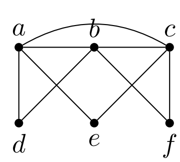
Solution. One example of such a graph is:



This graph is simple, because it has no loops and has no parallel edges.

(b) Draw a simple graph with exactly six vertices and exactly nine edges that is not bipartite but has an Euler circuit.

Solution. One example of such a graph is:



This graph is simple, because it has no loops and has no parallel edges. This graph is not bipartite since it is not possible to split up the the vertices a, b, and d into two separate sets so that none of the elements in each set are adjacent.

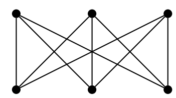
Finally, the graph has an Euler circuit because it is connected and each vertex has even degree. One example of an Euler circuit would be the closed walk that starts and ends at a given by:

$$a \to b \to d \to a \to e \to c \to b \to f \to c \to a.$$

This walk uses every edge exactly once, and starts and ends at the same point.

(c) Draw a simple graph with exactly six vertices and exactly nine edges that is bipartite but does not have an Euler circuit.

Solution. One example of such a graph is:



This graph is clearly bipartite since the upper points are only adjacent to the lower points and vice versa. It does not have an Euler trail, because it has at least one vertex with odd degree. Namely, all of the vertices have odd degree!

9. Prove by induction on n that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for all integers $n \geq 1$.

Solution. Let $P(n)$ be the statement: '$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$'.

Proof. We will prove that $P(n)$ is true for all integers $n \geq 1$ by induction on n.

Base case ($n = 1$): We have $\sum_{i=1}^{n} i = 1$, and $\frac{1(1+1)}{2} = 1$. So $P(1)$ is true.

Induction step: Let $k \geq 1$ be an integer. Suppose that

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}. \qquad \text{(IH)}$$

(We want to show that $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$.) Now

$$\begin{aligned}
\sum_{i=1}^{k+1} i &= \sum_{i=1}^{k} i + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) && \text{by IH} \\
&= (k+1)\left(\frac{k}{2} + 1\right) \\
&= \frac{k+1}{2}(k+2) \\
&= \frac{(k+1)(k+2)}{2},
\end{aligned}$$

which is what we wanted to show.

By the principle of induction, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for all integers $n \geq 1$. □