MATH 271 – Summer 2016
Midterm – solutions

1. (4 points) Use the Euclidean Algorithm to compute $\gcd(94, 49)$. Use this to find integers x and y so that $\gcd(94, 49) = 94x + 49y$.

   Solution: Note that

$$\begin{aligned} 94 &= 1 \cdot 49 + 45 \\ 49 &= 1 \cdot 45 + 4 \\ 45 &= 11 \cdot 4 + 1 \\ \text{and} \quad 4 &= 4 \cdot 1 + 0. \end{aligned}$$

   Using the Euclidean Algorithm, this implies that

$$\begin{aligned} \gcd(94, 49) &= \gcd(49, 45) \\ &= \gcd(45, 4) \\ &= \gcd(4, 1) \\ &= \gcd(1, 0) = 1. \end{aligned}$$

   Hence $\gcd(94, 49) = 1$. Now

$$\begin{aligned} 1 &= 45 - 11 \cdot 4 \\ &= 45 - 11 \cdot (49 - 45) && \text{since } 4 = 49 - 45 \\ &= -11 \cdot 49 + 12 \cdot 45 \\ &= -11 \cdot 49 + 12 \cdot (94 - 49) && \text{since } 45 = 94 - 49 \\ &= -11 \cdot 49 + 12 \cdot 94 - 12 \cdot 49 \\ &= 12 \cdot 94 - 23 \cdot 49, \end{aligned}$$

   so we can choose $x = 12$ and $y = -23$ so that $\gcd(94, 49) = 94x + 49y$.
   Alternatively, using the 'table method':

		x	y
R_1	94	1	0
R_2	49	0	1
$R_3 = R_1 - R_2$	45	1	-1
$R_4 = R_2 - R_3$	4	-1	2
$R_5 = R_3 - 11R_4$	1	12	-23

2. (10 points) For this problem, use no facts about $|$ ("divides") other than its definition.
   Let $\mathcal{S}$ be the statement: "For all integers a and b, if $a \mid b$ then $(6a) \mid (2b)$."

   (a) Prove that $\mathcal{S}$ is false by writing out its negation and proving that.

   Solution: The negation is: "There exist integers a and b so that $a \mid b$ but $(6a) \nmid (2b)$."

   Proof. Let $a = 1$ and $b = 1$. Then $a \mid b$ since $1 \mid 1$, but $(6a) \nmid (2b)$ since $6 \nmid 2$. □

   (b) Write out the converse of $\mathcal{S}$. Prove that the converse is true.

   Solution: The converse is: "For all integers a and b, if $(6a) \mid (2b)$ then $a \mid b$."

   Proof. Let a and b be integers. Assume that $(6a) \mid (2b)$. This means that there exists an integer m so that $2b = 6am$. Note that $6am = 2 \cdot 3am$ Dividing both sides by 2, we see that

$$b = 3am = a(3m),$$

where $3m$ is an integer. Hence a divides b. ☐

(c) Write out the contrapositive of $\mathcal{S}$. Is the contrapositive true? Explain.

Solution: The contrapositive is: "For all integers a and b, if $(6a) \nmid (2b)$ then $a \nmid b$." The contrapositive is false, because it is logically equivalent to the original statement which is false.

3. (8 points) Prove by induction that $\displaystyle\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$ for all integers $n \geq 1$.

Proof. We prove this by induction on n.

Base case ($n = 1$): Note that $\sum_{i=1}^{1} i(i+1) = 1(1+1) = 1 \cdot 2 = 2$ and $\frac{1(1+1)(1+2)}{3} = \frac{1 \cdot 2 \cdot 3}{3} = 2$. Thus $\sum_{i=1}^{1} i(i+1) = \frac{1(1+1)(1+2)}{3}$ since both sides are equal to 2.

Inductive step: Let $k \geq 1$ be an arbitrary integer and suppose that

$$\sum_{i=1}^{k} i(i+1) = \frac{k(k+1)(k+2)}{3}. \tag{IH}$$

(We will show that $\sum_{i=1}^{k+1} i(i+1) = \frac{(k+1)(k+1+1)(k+1+2)}{3}$.) Now

$$\begin{aligned}
\sum_{i=1}^{k+1} i(i+1) &= \sum_{i=1}^{k} i(i+1) + (k+1)(k+1+1) \\
&= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) && \text{by IH} \\
&= (k+1)(k+2)\left(\frac{k}{3} + 1\right) \\
&= \frac{(k+1)(k+2)(k+3)}{3} \\
&= \frac{(k+1)(k+1+1)(k+1+2)}{3},
\end{aligned}$$

which is what we wanted to show.

By the principle of mathematical induction, $\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$ for all integers $n \geq 1$. ☐

4. (10 points) Of the following statements, **one is true and one is false**. Prove the true statement. For the false statement, write its negation and prove that. Use the element method.

(a) For all sets A, B, and C, if $A \cap B = \emptyset$ and $C \subseteq B$ then $A \cap C = \emptyset$.

Solution: This statement is true.

Proof. Let A, B, and C be sets. Suppose that $A \cap B = \emptyset$ and $C \subseteq B$. (We want to show that $A \cap C = \emptyset$.) Suppose instead that $A \cap C \neq \emptyset$. Then there exists an element $x \in A \cap C$. This means that $x \in A$ and $x \in C$. Thus $x \in B$, since $x \in C$ and $C \subseteq B$. Hence $x \in A \cap B$, since $x \in A$ and $x \in B$. But $A \cap B = \emptyset$, so $x \in \emptyset$, which is a contradiction. Thus the assumption that $A \cap C \neq \emptyset$ must be wrong. Therefore $A \cap C = \emptyset$. ☐

(b) For all sets A, B, and C, $A - (B \cup C) = (A - B) \cup C$.

Solution: This statement is false.
Its negation is: "There exist sets A, B, and C so that $A - (B \cup C) \neq (A - B) \cup C$."

Proof. Let $A = \emptyset$, $B = \emptyset$, and $C = \{4\}$. Then $A - (B \cup C) = \emptyset - (B \cup C) = \emptyset$, but $(A - B) \cup C = (\emptyset - \emptyset) \cup \{4\} = \emptyset \cup \{4\} = \{4\}$. Hence $A - (B \cup C) \neq (A - B) \cup C$. □

5. (8 points) In this problem, use no properties of rational and irrational numbers other than the definition of rational.

   (a) Use proof by contradiction to prove the following statement.
   "For all real numbers x, if x is irrational then $\frac{1}{x}$ is irrational."

   Proof. Let $x \in \mathbb{R}$. Suppose that x is irrational. Assume (for the sake of getting a contradiction) that $\frac{1}{x}$ is rational. Then there exist integers a and b such that $\frac{1}{x} = \frac{a}{b}$ and $b \neq 0$. Note that $b = ax$, and thus $a \neq 0$ since $b \neq 0$. Now

   $$x = \frac{b}{a}$$

   where a and b are integers and $a \neq 0$. This means that x is rational and x is irrational, a contradiction. So the assumption that $\frac{1}{x}$ is rational must be wrong. Therefore $\frac{1}{x}$ is irrational. □

   (b) Prove the following statement. You may use the result from part (a).
   "For all irrational numbers x, there exists an irrational number y so that xy is rational."

   Proof. Let x be an irrational real number. Choose $y = \frac{1}{x}$. From part (a), we know that y is also irrational. Now $xy = x\frac{1}{x} = 1$, which is rational. □

6. (10 points) Consider the following sets of integers:

   $$A = \{x \in \mathbb{Z} \mid 10 \leq x \leq 100 \text{ and } x \text{ is even}\}$$
   $$\text{and} \quad B = \{x \in \mathbb{Z} \mid 10 \leq x \leq 100 \text{ and the sum of the digits of } x \text{ is even}\}.$$

   (Example: The sum of the digits of 12 is $1 + 2 = 3$.)
   No explanation is needed for your answers for (a), (b), or (c).

   (a) Give three elements of $A - B$.

   Solution: Some elements of $A - B$ include: 10, 12, 14.

   (b) Give three elements of $B - A$.

   Solution: Some elements of $B - A$ include: 11, 13, 15.

   (c) Give three elements of $\mathcal{P}(A \cap B)$.

   Solution: Some elements of $\mathcal{P}(A \cap B)$ include: $\emptyset$, $\{20\}$, $\{20, 40\}$.

   (d) Give one element of $(A \times B) - (B \times A)$. Explain your answer.

   Solution: One element of $(A \times B) - (B \times A)$ would be $(10, 11)$. Note that $(10, 11) \in A \times B$, since $10 \in A$ and $11 \in B$. Also note that $(10, 11) \notin B \times A$, since $10 \notin B$. Thus $(10, 11) \in (A \times B) - (B \times A)$.

   (e) How many elements does $A - B$ have? Explain your answer.

   Solution: There are 26 elements in $A - B$. The reasoning is as follows. Note that $|A - B| = |A| - |A \cap B|$. Also note that $|A| = 46$. Now $A \cap B$ consists of all $x \in \mathbb{Z}$ with $10 \leq x \leq 100$ such that x is even and the sum of the digits of x is even. This is exactly the set of two-digit numbers that have both digits even. There are 20 elements in $A \cap B$. Here is a recipe:

   1. Choose the one's digit. It must be even, so it can be one of $\{0, 2, 4, 6, 8\}$. There are 5 choices.
   2. Choose the ten's digit. It must be even, but it can't be zero, so it can be one of $\{2, 4, 6, 8\}$. There are 4 choices.

   Hence $|A \cap B| = 5 \cdot 4 = 20$. Therefore

   $$\begin{aligned} |A - B| &= |A| - |A \cap B| \\ &= 46 - 20 \\ &= 26. \end{aligned}$$