THE UNIVERSITY OF CALGARY
FACULTY OF SCIENCE
MATHEMATICS 271 (L01, L02)
FINAL EXAMINATION, WINTER 2011
TIME: 3 HOURS

NAME______________________________ID____________Section______

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Total (max. 80)	

SHOW ALL WORK. NO CALCULATORS PLEASE.

THE MARKS FOR EACH PROBLEM ARE GIVEN TO THE LEFT OF THE PROBLEM NUMBER. TOTAL MARKS [80]. THIS EXAM HAS 9 PAGES INCLUDING THIS ONE.

[8] 1. (a) Use the Euclidean algorithm to find $\gcd(73, 50)$. Also use the algorithm to find integers x and y such that $\gcd(73, 50) = 73x + 50y$.

(b) Use part (a) to find an inverse a for 50 modulo 73 so that $0 \leq a \leq 72$; that is, find an integer $a \in \{0, 1, \ldots, 72\}$ so that $50a \equiv 1 \pmod{73}$.

[11] 2. Let S be the statement:

$$\text{for all sets } A, B, C, \text{ if } A \subseteq B \text{ and } B \cap C = \emptyset \text{ then } A \cap C = \emptyset.$$

(a) Prove that S is true. Use contradiction and the element method.

(b) Write out the *converse* of statement S. Is it true or false? Explain.

(c) Write out the *contrapositive* of statement S. Is it true or false? Explain.

[13] 3. Let $X = \{1, 2, \ldots, 10\}$. Define the relation R on X by:

for all $a, b \in X$, aRb if and only if ab is even.

(a) Is R reflexive? Symmetric? Transitive? Give reasons.

(b) Find and simplify the *number* of **two-element** subsets S of X that satisfy the following property: $\forall a \in S$, $aR1$. Explain.

(c) Find the *number* of subsets S of X (of any size) that satisfy the following property: $\forall a \in S\ \exists b \in S$ so that aRb. Explain.

[9] 4. Let $\mathscr{F}$ denote the set of all functions from $\{1, 2, 3\}$ to $\{1, 2, 3, 4, 5\}$.
(a) Find and simplify the number of functions $f \in \mathscr{F}$ so that $f(1) = 4$.

(b) Find and simplify the number of *one-to-one* functions $f \in \mathscr{F}$ so that $f(1) \geq 4$.

(c) Find and simplify the number of functions $f \in \mathscr{F}$ so that $f(1) \neq f(2)$.

[11] 5. (a) Give the definition of $a \equiv b \pmod{n}$ ("a is congruent to b modulo n"), for arbitrary integers a, b, n where $n > 0$.

(b) Prove that the relation $\equiv \pmod{n}$ ("congruence modulo n"), on the set $\mathbb{Z}$ of all integers, is **symmetric**. Use your definition from part (a). (Do not assume that the relation is an equivalence relation.)

(c) Now assume that "congruence modulo 7" is an equivalence relation on $\mathbb{Z}$. Find three elements in the equivalence class [3].

(d) Again consider the equivalence class [3] for the equivalence relation "congruence modulo 7" on $\mathbb{Z}$. Suppose that $S = \{1, 2, \ldots, N\}$, where N is a positive integer. Find all possible values of N so that $[3] \cap S$ contains exactly 10 elements.

[10] 6. $\mathbb{Q}$ is the set of all rational numbers. Two of the following statements are true and one is false. Prove the true statements. Write out the *negation* of the false statement and prove it.

(a) $\forall q \in \mathbb{Q}\ \exists n \in \mathbb{Z}$ so that $q + n = 271$.

(b) $\forall n \in \mathbb{Z}\ \exists q \in \mathbb{Q}$ so that $q + n = 271$.

(c) $\exists n \in \mathbb{Z}$ so that $271 - n$ is even.

[11] 7. (a) Draw a graph with exactly 4 vertices and 6 edges, and give its adjacency matrix.

(b) Draw a graph with exactly 6 vertices and 4 edges and exactly two connected components.

(c) Draw a **tree** with exactly 8 vertices, one of which has degree 6.

(d) Does there exist a graph with exactly 8 vertices, so that three of the vertices have degree 3 and the remaining five vertices have degree 2? Explain.

[7] 8. Prove **by induction on** n that $6 \mid (7^n + 11)$ for all integers $n \geq 0$.

$\overline{[80]}$