THE UNIVERSITY OF CALGARY
FACULTY OF SCIENCE
MATHEMATICS 271 (L01, L02)
FINAL EXAMINATION, WINTER 2012
TIME: 3 HOURS

NAME______________________________________ID_______________Section________

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Total (max. 80)	

SHOW ALL WORK. NO CALCULATORS PLEASE.

THE MARKS FOR EACH PROBLEM ARE GIVEN TO THE LEFT OF THE PROBLEM NUMBER. TOTAL MARKS [80]. THIS EXAM HAS 9 PAGES INCLUDING THIS ONE.

[8] 1. (a) Use the Euclidean algorithm to find $\gcd(100, 57)$. Also use the algorithm to find integers x and y such that $\gcd(100, 57) = 100x + 57y$.

(b) Use part (a) to find an inverse a for 57 modulo 100 so that $0 \leq a \leq 99$; that is, find an integer $a \in \{0, 1, \ldots, 99\}$ so that $57a \equiv 1 \pmod{100}$.

[11] 2. In this question, you may assume that every integer is either even or odd but not both. But otherwise use no facts about even or odd integers except for the definition. Let S be the statement:

for all integers a and b, if a is odd and $b \mid a$ then b is odd.

(a) Prove that S is true. Use contradiction.

(b) Write out the *converse* of statement S. Is it true or false? Explain.

(c) Write out the *contrapositive* of statement S. Is it true or false? Explain.

[11] 3. Define the relation R on the set $\mathbf{Z}^+$ of all positive integers by:

$$\text{for all } a, b \in \mathbf{Z}^+,\ aRb \text{ if and only if } \gcd(a, b) > 1.$$

(a) Is R reflexive? Symmetric? Transitive? Give reasons.

(b) Find and simplify the *number* of integers $a \in \{1, 2, 3, \ldots, 100\}$ so that $aR4$. Explain.

(c) Find and simplify the *number* of integers $a \in \{1, 2, 3, \ldots, 100\}$ so that $aR10$. Explain.

[10] 4. Let $\mathcal{S}$ be the statement: for all sets A, B, C, if $B \cap C = \emptyset$ then $B \subseteq A$ and $A \cap C = \emptyset$.
(a) **Disprove** $\mathcal{S}$ by writing out and proving its negation.

(b) Write out the *converse* of $\mathcal{S}$, and prove it using the element method and contradiction.

[14] 5. Let $\mathcal{X}$ be the set of all *nonempty* subsets of the set $\{1, 2, 3, \ldots, 10\}$. Define the relation $\mathscr{R}$ on $\mathcal{X}$ by: for all $A, B \in \mathcal{X}$, $A\mathscr{R}B$ if and only if the smallest element of A is equal to the smallest element of B. For example, $\{1, 2, 3\}\mathscr{R}\{1, 3, 5, 8\}$ because the smallest element of $\{1, 2, 3\}$ is 1 which is also the smallest element of $\{1, 3, 5, 8\}$.

(a) Prove that $\mathscr{R}$ is an equivalence relation on $\mathcal{X}$.

(b) Find and simplify the number of equivalence classes of $\mathscr{R}$. Explain.

(c) Find and simplify the number of elements in the equivalence class $[\{2, 6, 7\}]$. Explain.

(d) Find and simplify the number of *four-element* sets which are elements of the equivalence class $[\{2, 6, 7\}]$. Explain.

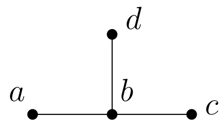
[12] 6. Let $\mathscr{F}$ denote the set of all functions from $\{1,2,3\}$ to $\{1,2,3\}$.

(a) One of statements (i) and (ii) is true and one is false. Prove the true statement. Write out the *negation* of the false statement and prove it.

(i) $\forall f \in \mathscr{F}\ \exists g \in \mathscr{F}$ so that $g(f(1)) = 2$.

(ii) $\forall f \in \mathscr{F}\ \exists g \in \mathscr{F}$ so that $f(g(1)) = 2$.

(b) Let $f \in \mathscr{F}$ be defined by: $f(1) = 2$, $f(2) = 3$, $f(3) = 2$. Find and simplify the *number* of functions $g \in \mathscr{F}$ so that $f(g(f(1))) = 2$. Explain.

[8] 7. Let G be the graph 

(a) Give the adjacency matrix of G.

(b) Draw a simple graph H with exactly six vertices a, b, c, d, e, f and exactly seven edges, and so that G is a subgraph of H.

(c) Draw a **tree** T with exactly six vertices a, b, c, d, e, f, each with degree either 1 or 3, and so that G is a subgraph of T.

[6] 8. Prove **by induction on** n that

$$1 \cdot 1 + 2 \cdot 3 + 3 \cdot 5 + \cdots + n(2n-1) > \frac{n^3}{2}$$

for all integers $n \geq 1$.

$\overline{[80]}$